GENERAL PHYSICS

FOR STUDENTS

EDWIN EDSER

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GENERAL PHYSICS FOR STUDENTS

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GENERAL PHYSICS FOR STUDENTS

A TEXT-BOOK ON THE FUNDAMENTAL PROPERTIES OF MATTER

BY

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PREFACE

In writing this book, my object has been to discuss the fundamental properties of matter in the simplest manner consistent with accuracy. The student who possesses a sound knowledge of the elements of algebra, geometry, and trigonometry, will find that his acquirements are sufficient to enable him to read the book, and to solve the problems set at the ends of the chapters, without referring to mathematical treatises. In many instances, the performance of integrations, commonly effected by the aid of the integral calculus, has been found to be unavoidable; in such cases I have used a method which is neither difficult nor lengthy, while it can be learnt without much trouble. This method is explained fully in the text, and a number of its applications are given sufficient to enable the student to become perfectly familiar with it. It must be understood that this method has not been devised in order to give the student an excuse for neglecting the study of the calculus; on the contrary, I consider that it forms a most valuable introduction to the calculus, as it presents a definite integral under its true form—the sum of a series; and the procedure by which the summation is effected is just as legitimate as that commonly used in books on mathematics. Geometrical properties, with which the student is likely to be unacquainted, are also investigated fully in the text.

The first five chapters are devoted to a development of mechanical principles, especial attention being paid to the rotational motion of solids (including the gyrostat) and to oscillatory motion. Subsequently the properties of gravitation are studied, and the phenomena attending the straining or elastic bodies are analysed. The surface tension of liquids forms a connecting link between the mechanical properties of solids and fluids. In Chapters X–XV a number of the most important properties of moving fluids are investigated. Finally, Chapter XVI is devoted to a consideration of those phenomena which can be explained only on the assumption that fluids consist of ultimate particles, or molecules, moving with great velocities.

A large number of experiments, some of which present novel features, are described in the text. Questions are set at the end of each chapter, and answers to these are given at the end of the book. In many cases, assistance has been afforded to the student in the solution of problems of a difficult character.

The illustrations to the text have been reproduced from original drawings; the wood-cuts have been executed by Messrs. Butterworth, and the line diagrams by Mr. Emery Walker; to these firms, as well as to the printers, Messrs. R. Clay & Sons, Ltd., my best thanks are due for their care and skill. During the passage of this book through the press, I have received assistance and advice from my friends, Prof. R. A. Gregory and Mr. A. T. Simmons, which has proved most helpful, and for which this brief expression of my gratitude is an inadequate return.

EDWIN EDSER.

UNIVERSITY OF LONDON, GOLDSMITHS' COLLEGE.
July, 1911.

CONTENTS

CHAPTER I

PAGE
INTRODUCTORY 1

Fundamental units of measurement. Force, torque, work, energy. Stable and unstable equilibrium.

CHAPTER II

ROTATIONAL MOTION 46

Rotating bodies. Moment of inertia and moment of momentum. The gyrostat. Motion in spiral orbit.

CHAPTER III

SIMPLE HARMONIC MOTION 84

Characterisation of simple harmonic motion. Simple and compound pendulums. Bifilar suspension.

CHAPTER IV

THE GYROSTATIC PENDULUM AND THE SPINNING TOP 122

The gyrostatic pendulum. The spinning top. Disc rolling on a plane.

CHAPTER V

GENERAL THEORY OF OSCILLATIONS 143

Pendulum swinging over finite arc. Corrections for buoyancy and inertia of air, curvature of knife-edges, elasticity of support. Coupled systems. Damped oscillations. Forced oscillations.

CHAPTER VI

GRAVITATION 191

Newton's law of gravitation. Gravitational potential. Determination of the constant of gravitation. Theories of gravitation.

CHAPTER VII

PAGE

ELASTICITY 217

Stresses and strains. The moduli of elasticity, and the relations between them. Torsion of a cylinder. Bending of a beam. Flat spiral springs.

CHAPTER VIII

ELASTICITY (*continued*) 263

Deformation of cross-section of bent beam. Twisted strip. Compressibility of liquids. Tensile strength of liquids.

CHAPTER IX

SURFACE TENSION 285

Surface tension produced by inter-molecular forces. Surface energy. Liquid drops. Pressure due to spherical and cylindrical curvature of a liquid surface.

CHAPTER X

SURFACE TENSION (*continued*) 322

Curvature of surfaces. Forms of soap bubbles. Stability of cylindrical film. Ripples on the surface of a liquid. Molecular theory of surface tension.

CHAPTER XI

THE MOTION OF FLUIDS 367

Linear and rotational motion of a fluid as a whole. Perfect fluid. Vortex filaments. Sources and sinks. Lines of flow, and velocity-potential. In a perfect fluid, vortex filaments would be uncreatable and indestructible.

CHAPTER XII

TRANSFER OF ENERGY AND MOMENTUM BY THE STEADY MOTION OF A FLUID 399

Law of transfer of energy. Relation between pressure and velocity of a fluid. Transfer of momentum, and the forces produced thereby.

CHAPTER XIII

EXAMPLES TO ILLUSTRATE THE LAWS OF MOTION OF FLUIDS . . 420

Mutual attraction of two sources or two sinks. Vortex motion. Vortex rings, and the conditions necessary to their stability. Flow of a gas from a small orifice in its containing vessel.

CHAPTER XIV

PAGE
WAVES ON THE SURFACE OF A LIQUID 449

Waves on a deep liquid: oscillations of particles of the liquid, and velocity of wave propagation. Effects of surface tension. Energy of waves, half potential and half kinetic. Wave velocity and group velocity. Waves on a shallow liquid.

CHAPTER XV

THE FLOW OF A VISCOUS LIQUID 487

Coefficient of viscosity. Turbulent motion, and the critical velocity. Methods of measuring the viscosity of a liquid. Lubrication. Viscosity of gases.

CHAPTER XVI

THE MOLECULAR STRUCTURE OF FLUIDS 531

Kinetic theory of gases, on the assumption that molecules are infinitely small. Molecules of finite size: mean free path. Speed of transmission of momentum, as influenced by size of molecules: van der Waals's theory. Counting molecules. Viscosity and allied properties of gases. The radiometer: thermal transpiration of a gas. Osmosis and osmotic pressure. Boiling and freezing points of solutions.

ANSWERS TO QUESTIONS 599

INDEX 635

GENERAL PHYSICS FOR STUDENTS

GENERAL PHYSICS FOR STUDENTS

CHAPTER I

INTRODUCTORY

What is matter?—Everyone has a more or less definite idea of the meaning to be attached to the word "matter"; it is not easy, however, to frame a rigid definition of matter which will prove useful in our subsequent investigations. **The property which distinguishes matter from any other substance is that it has weight;** in other words, when matter is near to the earth (where alone we can *directly* investigate its properties) it is pulled downward towards the ground. The word *weight* is applied to the pull which the earth exerts on matter near its surface.

In order to explain the phenomena of Light and Magnetism and Electricity, we are forced to assume the existence of a medium which pervades all space, and is called the *luminiferous ether.* This medium has many properties in common with matter; but it is imponderable, that is, it has no weight; and this characteristic distinguishes the ether from matter.

States of matter.—Matter can exist in several states. If it can maintain its shape at the surface of the earth for an indefinite time without lateral support, it is said to be in the **solid state.** If it cannot maintain its shape without lateral support (that is, if there is no permanent cohesion between its parts, so that it flows if not prevented from doing so by the walls of a containing vessel) it is said to be in the **fluid**[1]

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[1] From Latin *fluere*, to flow.

state. Fluids are divided into **liquids** and **gases.** The volume of a liquid is constant[1] at a given temperature; if the volume of a liquid is less than that of a vessel into which it is introduced, it will not fill the vessel, but will possess a free surface. When a gas is introduced into a closed vessel which was previously empty, it entirely fills the vessel, and consequently has no free surface. A **vapour** is distinguished from a gas by the property that it can be liquefied merely by compressing it, while a gas cannot be liquefied without cooling it below a certain critical temperature.[2]

Viscosity.—Some fluids are capable of flowing much more readily than others: for instance, water can be poured out of a vessel much more readily than treacle. The property of a fluid which retards its rate of flow is called its **viscosity**: thus treacle has a greater viscosity, or is more viscous, than water.

Many substances which, at first sight, appear to be solids, are really very viscous fluids. For instance, a large piece of pitch will gradually flow over the ground if it is not prevented from doing so by the walls of a containing vessel: a stick of sealing-wax, if supported only at its ends while in a horizontal position, will gradually sag under its own weight: a glacier is a stream of ice slowly flowing down the side of a mountain. Hence pitch, sealing-wax, and ice are fluids, and differ from treacle only in possessing much greater viscosities. Many fairly soft substances, such as tallow, can maintain their shapes for an indefinite time at ordinary temperatures: such substances are *soft solids.*

Elasticity.—If the volume and shape of a substance could not be altered by the application of any force, however great, that substance would be said to be **perfectly rigid**: no such substance is known, but certain solids require so great a force to modify their shapes that, occasionally for convenience, we may speak of them as perfectly rigid.

When the volume or shape of a substance can be altered appreciably by the application of forces of finite magnitude, and the original volume or shape of the substance is regained when the forces cease to act, the substance is said to be elastic, or to possess elasticity. There are two principal kinds of elasticity; **volume elasticity** and **shape elasticity.** Solids alone possess a definite shape, and consequently they alone possess shape elasticity. A gas can be compressed by the application of increased pressure, and it regains its original volume when the pressure regains its original value : hence gases are fluids possessing volume elasticity. Liquids and solids can be compressed appreciably only by the application of enormous pressure ; therefore they are often spoken of as incompressible : in reality, however, they possess volume elasticity.

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[1] Under very high pressure the volume of a liquid may be slightly diminished.

[2] See the Author's "Heat for Advanced Students," pp. 183 and 207.

When the shape of a solid can be changed by the application of forces of sufficient magnitude, and the original shape of the solid is not regained when these forces cease to act, the solid is said to be **plastic.** Putty and lead may serve as instances of plastic solids.

Fundamental units of measurement.—It has been said that "science is measurement." This statement means, that we can never understand a phenomenon fully unless we can describe it in terms of magnitudes which have been measured exactly. But an exact measurement can only be made in terms of some unit previously agreed upon ; thus, to measure the distance from one point to another we must have a unit of length, such, for instance, as the inch or the foot, and then determine how many of these units of length there are between the two points.

In early times all units of measurement were arbitrary and, for the most part, unrelated one to another. Even at the present time experimenters are forced to use certain arbitrary units which have no definite relation to other units ; as an example, the "standard candle" used in photometric researches may be cited. But it is recognised clearly that all units of measurement should be related, so far as possible, to the three fundamental units of length, mass, and time. These units will now be defined.

Units of length.—The British unit of length is the foot, which is one-third of the distance between two marks on a bar of metal preserved at the Standards Office, and called the "Standard Yard."

The unit of length used in nearly all scientific investigations is the centimetre (written for short cm.) ; this is the one-hundredth part of the distance between two marks on a bar of platinum, called the "Standard Metre," which is preserved at the International Bureau of Metric Standards, Saint-Cloud, near Paris. It is a copy of a standard originally made to be as nearly as possible equal to one ten-millionth part of a quadrant of the earth, measured from the equator to the pole; but it was decided that the standard, once constructed, should not be altered if subsequent more exact measurements of the earth showed that this relation does not hold exactly. It is now known that the length of the quadrant passing through Paris is equal to 10,002,100 metres.

The following multiples and sub-multiples of the metre are often used in measurements; but it must be remembered that in theoretical investigations all lengths must be expressed in centimetres:—

1 kilometre (km.) = 1,000 metres = 0·6214 mile.
1 decimetre (dm.) = 1/10 metre = 10 cm.
1 millimetre (mm.) = 1/1,000 metre = 1/10 cm.
1 micron (represented by the symbol μ) = 1/1,000 mm.
1 micro-millimetre (represented by the symbol $\mu\mu$) = (1/1,000) μ = 1/1,000,000 mm.

1 inch is roughly equal to 2·54 cm. 1 foot is equal to 30·5 cm. (nearly). More accurately, a metre is equal to 39·370113 inches.

Units of area and volume.—The unit of area is the area of a square, each side of which is of unit length. Hence, the British unit of area is the square foot, and the scientific unit is the square centimetre.

The unit of volume is the volume of a cube, each edge of which is of unit length. Hence, the British unit of volume is the cubic foot, and the scientific unit is the cubic centimetre (c.c.).

1 litre = 1,000 c.c.

The gallon is the volume of 10 lb. of fresh water at 62° F.; it is equal to 4·546 litres. Hence 1 pint = 568·23 c.c.

Units of mass.—The word **mass** is used to denote "quantity of matter." Hence it follows directly, that two cubic centimetres of a homogeneous substance, such as water, have twice the mass of one cubic centimetre of the same substance, the temperature being the same in both cases. To compare the masses of quantities of different substances, such, for instance, as lead and water, we may reason as follows. The characteristic property of matter is its weight; if a piece of lead is pulled downwards by the earth with a force equal to that exerted, at the same place, on a cubic centimetre of water, the two obviously have equal weights, and we may define them as having equal masses. This, however, does not necessarily imply that the weight of a body is directly proportional to its mass; we cannot find out whether two grams of water are pulled downwards by the earth with a force twice as great as that exerted on one gram, until we have defined accurately the word "force," and obtained a method of measuring forces. But without being in a position to measure forces, we may yet test the equality of weight of two masses of matter.

Let a beaker containing a known volume of water be placed on one scale pan of a balance, while matter of any kind is added to the other pan until equilibrium is obtained. The water is pulled downwards by the earth with a certain force, and if we remove the water and replace it by (say) a piece of brass, the equilibrium of the balance can only be preserved if the force exerted by gravity on the brass is equal to that previously exerted on the water, *i.e.*, unless the water and brass are equal in weight. Thus, we could construct a series of standard masses, say of brass, respectively equal to the masses of one c.c., two, three, four, etc., c.c. of water, and we could then balance other unknown masses against the standards so constructed.

In the scientific system, the unit of mass is the gram (gm.); this was originally defined as the mass of one cubic centimetre of water at 4° C., at which temperature the density of water has its maximum value. A standard kilogram (1,000 grams) was constructed, and is preserved at the Bureau of Metric Standards; it was decided that this should continue to be the standard of mass, even if subsequent researches showed that its mass was not exactly equal to that of 1,000 c.c. of water at 4° C. According to Guillaume, a cubic centimetre of water at 4° C. actually has a mass of 0·99995 gm.

The British standard of mass is the pound avoirdupois; this is the mass of a standard block of metal preserved at the Standards Office. It has no direct relation to the unit of volume.

1 lb. = 7,000 grains = 453·593 gm. (454 gm. nearly).
1 ton = 2,240 lb. = 1,016,050 gm. (1,000,000 gm. nearly).

The mass of unit volume of a substance is called the **density** of that substance. Density is measured in grams per c.c, or in pounds per cubic foot. The density of water at 4° C is equal to one gram per c.c., or 62·415 lb. per cubic foot.

Unit of time.—The **solar day** is defined as the time which elapses between two successive passages of the sun across the meridian; the meridian being a plane passing through a given point on the earth's surface, and also through the geographical poles of the earth. The earth rotates at a uniform rate about its axis, and also revolves in an orbit about the sun with its axis pointing always in one direction. Consequently, the solar day will have slightly different values at different times of the year; the **mean solar day** is the mean value of the solar day throughout the year. The mean solar day is divided into 24 hours, each hour into 60 minutes, and each minute into 60 seconds. The second is the unit of time used in all scientific researches.

We have now defined the three fundamental units of length, mass, and time. **The centimetre, gram, and second are called the c.g.s. units of measurement. The pound, foot, and second are called the British units of measurement.**

Dimensions of physical quantities.—A "physical quantity" may be defined as a measurement of some physical property: a length, a mass, or a time is a fundamental physical quantity, and most other physical quantities are related to these. The "dimensions of a physical quantity" denote an expression which shows how the quantity in question is related to the three fundamental units of length L, mass M, and time T.

For instance, the area of a rectangle is found by multiplying its length by its breadth, *i.e.*, by multiplying together two magnitudes each of which represents a certain number of units of length; the area of any other figure involves the multiplication together of two lengths, and hence the dimensions of an area are given by $L \times L$, or L^2. Similar reasoning shows that the dimensions of a volume are given by L^3. To find the density of a substance, we must determine the volume of a weighed mass of that substance, and then divide the mass by the volume; hence the dimensions of a density are given by $M \div L^3$, or ML^{-3}. A discussion of the dimensions of other physical quantities must be postponed until those quantities have been defined.

Displacements.—When a body is moved from one position to another, the distance, measured in a straight line from the initial to the final position of the body, is termed its displacement. It follows that **a displacement is a distance measured in a definite direction**; its description involves a numerical magnitude, together with a specification of the direction of measurement. Any measurement which involves, not only a magnitude, but a direction, is termed a **vector**, or directed quantity. A measurement which does not involve a direction (for example, a measurement of mass or time) is called a **scalar** quantity.

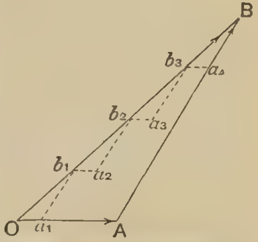

Fig. 1.—The composition of displacements.

Let the vectors OA, AB (Fig. 1) indicate, in magnitude and direction, two successive displacements of a body initially at O. Then the resultant displacement of the body is equal to the vector OB. Thus, the two successive displacements $\overrightarrow{OA}$ and $\overrightarrow{AB}$ are equivalent to the single displacement $\overrightarrow{OB}$. Hence we may write $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$.

We may, however, proceed in a different manner. Let us displace the body from O, first along OA, through the distance Oa_1 equal to OA/n, where n is any number; and then through a_1b_1, which is equal to AB/n, and is measured parallel to AB. These two successive displacements are equivalent to the single displacement Ob_1. If we next displace the body through the distance b_1a_2, parallel to OA and equal to OA/n; and then through the distance a_2b_2, parallel to AB and equal to AB/n, the body will have arrived at b_2. Repeating this procedure until n displacements, each equal to OA/n, have been effected parallel to OA, and an equal number of displacements, each equal to AB/n, have been effected parallel to AB, it is evident that the body finally arrives at B. Its actual path has been along the zig-zag line $Oa_1b_1a_2b_2a_3b_3$ . . . . B, and the resultant displacement is equal to OB. If we imagine that n, the number of steps in the zig-zag path, is increased indefinitely, then this path approximates to the straight line OB. The body still suffers the displacements OA and AB, but an infinitesimal displacement parallel to OA is followed immediately by an infinitesimal displacement parallel to AB, and so on. Thus, the conditions are practically the same as if the displacements OA and AB were performed simultaneously, and we see that the result is the single displacement OB.

An instance of the superposition of simultaneous displacements occurs when a person shifts his position in a moving railway carriage. The resultant displacement of the person is equivalent to the displacement due to the motion of the carriage, together with the motion of the person relative to the carriage.

Any two displacements OA and AB are equivalent to a displacement OB. The process of finding a single displacement which is equivalent to two or more displacements, is called the **composition** of those displacements. Conversely, any displacement OB can be replaced by the component displacements OA and AB, in which case the displacement OB is said to have been **resolved** into its components OA and OB. Any magnitude and direction may be chosen for one of the components, such as OA; but when this is given, the remaining component AB becomes known. If the directions of the components are given, it is easily seen that the magnitude of the components becomes known.

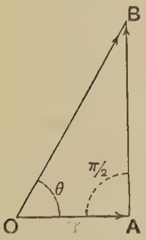

FIG. 2.—The resolution of displacements.

The most useful method of resolving a displacement is to choose its components so that the angle $OAB = \frac{\pi}{2}$ (Fig. 2).

Let $OA = x$, while $AB = y$ and $OB = r$. Let the angle $AOB = \theta$.

Then, $x/r = \cos\theta$, and $x = r\cos\theta$.

$y/r = \sin\theta$, and $y = r\sin\theta$.

A displacement obviously has the dimensions of a length, L.

Velocity.—When a body moves in a straight line, and travels over equal distances in equal intervals of time, the motion of the body is said to be **uniform and rectilinear.** The velocity of the body is defined as the distance over which it travels in unit time; to determine the velocity, we must measure the distance D travelled by the body in t seconds, and divide D by t.

If a body moves in a straight line, but does not travel through equal distances in equal intervals of time, the motion of the body is constant in direction but **variable** in magnitude. To measure the velocity of the body, we must mark off very short elements of length along its path, and divide the length of each element by the time which elapses while the body is passing over that element; each element must be so short that, while the body passes over it, the motion of the body may be considered to be uniform. Thus, the velocity of the body at any point in its path is equal to the length of a very short element of path which is bisected by the point in question, divided by the time which elapses while the body traverses that element; the same result will give the velocity of the body at the time when it passes through the point in question.

When a body travels along a curved path, the direction of its motion changes from instant to instant. Its velocity will then vary in direction from instant to instant, but the magnitude of its velocity will be uniform if equal distances, measured along the curved path, are traversed in equal times; if equal distances are not traversed in equal times, the magnitude as well as the direction of the motion varies.

A velocity has the dimensions of a length divided by a time, or LT^{-1}.

A velocity is obviously a vector or directed quantity, being equal to a displacement divided by a time.

Composition and resolution of velocities.—Let the velocity of a body at a given instant be represented in magnitude and direction by the vector OA (Fig. 3); then if the velocity remained constant, the body would travel a distance OA in one second, or the displacement of the body in one second would be equal to OA. At the given instant let an additional velocity equal to OC be communicated to the body; then in virtue of this velocity alone the body would suffer a displacement equal to OC per second, and therefore as a matter of fact the displacement of the body per second will be equal to the resultant of OA and OC. Draw AB equal and

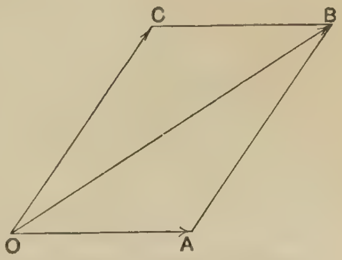

Fig. 3.—The composition of velocities.

parallel to OC; then the resultant of the two displacements OA and AB is equal to OB, and therefore the resultant of the two velocities, represented by OA per second and AB = OC per second, will be equal to OB per second. Hence, **to find the resultant of two velocities communicated simultaneously to a body, draw two vectors from a point so that they represent the two velocities in magnitude and direction; complete the parallelogram, and draw the diagonal from the starting point; this diagonal represents the resultant velocity of the body in magnitude and direction.**

Thus velocities, being vectors, can be compounded like displacements; also, a single velocity can always be resolved into two components in any given directions.

Further, if the velocity of a body at a given instant is equal to OA, (Fig. 3) and after a certain interval of time the velocity is found to be equal to OB, then we know that in this interval of time the body has acquired an additional velocity equal to AB or OC; thus, AB represents the change of velocity in the given interval of time.

Acceleration.—When the motion of a body is variable, the velocity changes from instant to instant; the rate of change of velocity is called the **acceleration** of the body. Thus, to determine the acceleration of a body, we must observe its velocity at a given instant and again after a very short interval of time, find the change of velocity which has occurred, and divide this by the magnitude of the interval of time. Since an acceleration is the difference between two velocities (each of which has the dimensions LT^{-1}) divided by a time, it follows that the dimensions of an acceleration are represented by LT^{-2}. Accelerations are measured in cm./sec.^{2} in the c.g.s. system, and in ft./sec.^{2} in the British system. An acceleration is obviously a vector quantity.

If a body is moving in a straight line, the acceleration of the body will be constant in direction; if the velocity of the body is increasing the acceleration will be positive (+), while if the velocity is decreasing the acceleration will be negative (−). The acceleration will be uniform if the velocity of the body increases uniformly with the time, otherwise it will be variable.

If a body moves in a curved path, its acceleration must be determined by finding the vector change in the velocity which occurs in a small interval of time (see preceding section) and dividing this by the magnitude of the interval of time.

Force.—Our idea of force is derived, in the first place, from the sensation of muscular effort experienced when we move matter, as, for instance, when we lift a heavy object. This sensation, however, affords us no trustworthy means of *measuring* forces, for a person may find it necessary to make quite different efforts at different times in order to lift the same object; therefore some other method must be devised, depending on a property of matter independent of our individual strengths.

We might measure forces in terms of their capacities to extend a spiral spring; making the arbitrary definition, that a force which can extend a standard spring by two centimetres is twice as great as that which can extend the same spring by one centimetre, and so on. We might further stipulate that a force which could extend the standard spring by some definite amount (say one centimetre) should be called the *unit force*, other forces being measured in terms of this unit. In this case the temperature at which the measurement is made would have to be specified, and the standard spring would have to be carefully protected from ill usage; even if these precautions were taken, difficulties, which will be discussed in the chapter devoted to elasticity, would occur. Further, the unit force, measured in this manner, would be arbitrary, and unrelated to the units of length, mass, and time.

Galileo introduced a method of measuring forces which depends on their capacities to set matter in motion; this method, which was defined and applied extensively by Newton, is now accepted universally.

If a body is set in motion so that it slides along a horizontal surface, such as that of a good pavement, the path traversed will be approximately straight, and the body will come to rest in a time which will be great or small according as the surface is smooth or rough. If a similar experiment is made on the surface of a sheet of ice, the path traversed will be still straighter, and a longer time will elapse before the body comes to rest. Now, we know that a greater effort is required to drag a body over a rough surface than over a smooth one; hence we may infer that the retardation of the body's motion is due to a frictional force dependent on the nature of the surface ; and that if a perfectly smooth surface could be obtained, the frictional force would be absent, and the motion of the body would be unimpeded. This idea gives rise to **Newton's first law of motion**, which is equivalent to the following statement :—

Every body continues in its state of rest, or of uniform motion in a straight line, except so far as it may be compelled by impressed forces to change that state.

This law cannot be proved directly, for it is impossible to arrange that a body shall be acted upon by no forces ; to do this we should have to experiment on a body at an infinite distance, not only from the earth, moon, sun, and stars, but also from ourselves, so that none of these bodies could exert forces on it. It is more logical to consider Newton's first law as giving a definition of force—that which modifies, or tends to modify, the state of rest or uniform motion of a body; in this case a body which continues at rest or in uniform motion is, by definition, either acted upon by no force, or if forces act upon it they must neutralise each other's effects. For instance, if the ice in the experiment described above were perfectly smooth, so that no frictional force acted on the body, there would still be the downward pull exerted on it by gravity, and this would cause the body to descend were it not that the ice exerts an equal upward force on the body.

Newton's second law of motion indicates the manner in which forces are to be measured.

Change of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.

In applying this law, it must be understood that the conditions under which the forces act must be similar : that is, they must set equal masses of matter in motion, and must act for equal times ; in this case the forces are defined as being proportional to the changes of motion (more strictly the changes of velocity) produced. Thus, if a number of different forces successively act on the same mass of matter, which in each case is at rest to start with, then the velocities produced after equal times serve to measure the magnitudes of the forces. Hence, if f indicates the magnitude of a force, and v the velocity which it confers on a given mass of matter after acting for some definite time, then

$$f \propto v, \text{ and conversely } v \propto f.$$

Attwood's machine.—It now becomes necessary to investigate the relation between the velocity produced by a given force, and (*a*) the time during which the force acts, and (*b*) the mass of matter set in motion. A piece of apparatus invented by George Attwood, and known as Attwood's machine, may conveniently be used for this purpose.

Two equal masses of matter A and B are attached, one at each end of a long cord of fine silk (Fig. 4). This cord passes round the grooved edge of a wheel W, the axle of which is supported so that rotation can take place with as little friction as possible. The two masses A and B being equal, gravity exerts equal forces on them; and as one cannot move downwards without pulling the other upwards, the effect of the pull of gravity on one is neutralised by the equal pull of gravity on the other. In consonance with Newton's first law of motion, the two masses, if originally at rest, remain in that condition; when they are set in motion by any force, they continue to move with uniform velocity after that force ceases to act. The silk cord should be so thin that the pull of gravity on it is negligibly small in comparison with the pull of gravity on the masses A and B. The friction between the axle and its bearings can be made so small that it does not perceptibly retard the motion of the suspended masses.

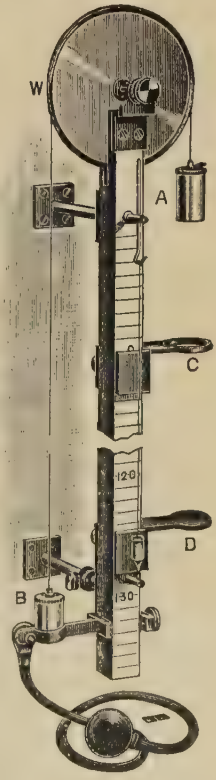

FIG. 4.—Attwood's machine.

In order to obtain a constant force to set the masses in motion, a metal rider is placed on the mass A, and is removed after the system has been free to move for a definite time. At the commencement of an experiment, the mass B is held by a pneumatic clutch, which can be released on one tick of a metronome timed to tick seconds. The mass A then commences to descend and B to rise. A horizontal metal ring C is adjusted so that the mass A can pass through it without contact, while the rider is caught and lifted off; by moving C up or down it can be arranged that the click due to the removal of the rider coincides with the first, second, third, or fourth tick of the metronome after that at which the masses were set in motion. The mass A can be stopped by means of a flat platform D placed at a suitable distance below C; the distance through which A or B has travelled, subsequent to the instant when the rider was removed, is then equal to the distance between the upper surface of the ring C and the upper surface of the mass A when it rests on the platform D.

EXPT. 1.—Arrange the position of the ring C so that the rider is removed one second after the system commences to move. Adjust the platform D so that the mass A is stopped one second after the rider is removed, and measure the distance through which A has travelled in that second. Then arrange the position of D so that A is stopped 2, 3, and 4 seconds after the rider is removed, and measure the distance traversed by A in each case. If the distance traversed is proportional to the time which has elapsed after the removal of the rider, the velocity of the system is uniform, and friction does not vitiate the results.

EXPT. 2.—Arrange the position of the ring C so that the rider is removed 2, 3, and 4 seconds after the system commences to move, and measure the velocity produced in each case. What relation is there between the velocity produced and the time of action of the force?

Experiment shows that **if a given force sets a given mass of matter in motion, the velocity v produced is directly proportional to the time t during which the force acts, or—**

$$v \propto t.$$

The same result may be expressed by the relation—

$$\frac{v}{t} = \text{a constant quantity};$$

that is, the velocity increases uniformly with the time, or the *acceleration is constant*, when a constant force acts on a body which moves freely.

In order to determine the relation between the velocity produced by a given force acting for a given time, and the mass of matter set in motion, we must be able to alter the masses of A and B. To this end, A and B can each be made in the form of a number of metal cylinders which can be screwed together; the mass set in motion can be varied by removing equal masses from A and B, or adding equal masses to these.

Expt. 3.—Arrange A and B so that their masses are as great as possible, and adjust the position of the ring C so that the rider is removed (say) after two seconds from starting. Determine the velocity produced. Then remove equal masses from A and B, and again determine the velocity produced when the same rider is removed after the same time. Repeat this procedure, varying the masses as much as possible. Finally tabulate your results, placing the velocity generated in each case opposite to the combined mass of A and B together with the rider.

The following table gives the result of a series of experiments such as those just described. In each case the rider was removed two seconds after the system commenced to move.

Mass of A and B. gm.	Mass of Rider. gm.	Combined mass of A, B, and rider. gm.	Velocity acquired, cm. per sec.	$\frac{1}{\text{velocity}}$.
400	10	410	39·5	0·0253
300	10	310	48·5	0·0206
250	10	260	54·0	0·0185
150	10	160	73·8	0·0135

On examining this table cursorily, no very exact relation can be traced between the velocity acquired and the combined mass of A, B, and the rider. We may notice, however, that the mass 310 is nearly twice as great as the mass 160, and the velocity 48·5 corresponding to the mass 310 is something like half as great as the velocity 73·8 corresponding to the mass 160. This suggests that the velocity generated is doubled when we halve the mass of matter set in motion; but the errors to be expected in the above experiments would not account for the discrepancy between the results and this law. It must be noticed, however, that A, B, and the rider are not the only masses set in motion; the wheel also moves, and we must make allowance for this. It is not enough for this purpose to weigh the wheel; for parts of the wheel at different distances from the axis of rotation move with different velocities. Let us, however, assume for the moment that, if all the matter set in motion acquired one uniform velocity, then the law connecting the velocity v and the mass m set in motion would be expressed by—

$$v \propto \frac{1}{m};$$

then we should have—

$$\frac{1}{v} \propto m.$$

In this case, if we plotted values of $1/v$ against m, we should obtain a straight line, and equal increments of m would correspond to equal increments of $1/v$.

Now, whatever allowance must be made for the wheel, the experimental results tabulated above suffice to show the *change* in the value of $1/v$ due to a *known increase* in the mass set in motion; thus, if the law assumed above is true, we ought to obtain a straight line on plotting the values of $1/v$, obtained from the above table, against the mass of A, B, and the rider. On referring to Fig. 5 we see that this is the case

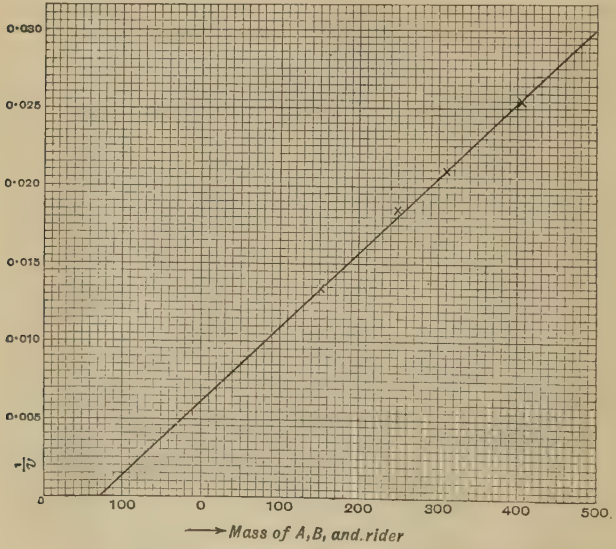

FIG. 5.—Graph of the results obtained in experiment 3.

within the limits of error incidental to the experiments. It will be noticed that, starting with any mass of matter set in motion, the addition of 100 grams always increases the value of $1/v$ by about 0·0048, that is, $1/v$ increases proportionately with the mass of matter set in motion.

Produce the straight line passing through the experimental points until it cuts the horizontal axis; then, if we read off the masses from this point of intersection, we find that the corresponding values of $1/v$ are proportional to these readings. Thus, we must increase the mass of A, B, and the rider by about 125 grams in order to allow for the action of the wheel; hence, if we could obtain a wheel with a mass of 125 grams concentrated along the bottom of the groove in which the silk cord rests, so that this mass must move with the same velocity as the cord, then we should obtain the same results as with the wheel actually used.

It must be noticed that a very high order of accuracy cannot be expected in experiments such as those just described; if two sounds occur within an interval of 0·1 second they are not heard separately, but merge into a single sensation; hence, we can scarcely hope to adjust the platforms so that the various clicks are within 0·1 second of the corresponding ticks of the metronome. The above experiments, however, indicate the probable connection between the various quantities measured; the best confirmation of the laws obtained is, that these laws enable us to predict other phenomena, which can be observed with extreme accuracy, and these predictions prove to be correct.

Experiment, therefore, shows that **when a given force acts for a given time, the velocity v produced is inversely proportional to the mass m of matter set in motion,** or

$$v \propto \frac{1}{m};$$

provided that the whole of the mass m acquires the velocity v.

In the expression—

$$\frac{1}{v} \propto m$$

let us call the quantity $1/v$ the *sluggishness* of motion; then we see that the greater the mass of matter set in motion by a given force acting for a given time, the greater will be the sluggishness of the motion produced.

We now have the following laws connecting the velocity v, the force f, the time t during which the force acts, and the mass m of matter set in motion :—

$$v \propto f \qquad \text{(By definition).}$$

$$\left.\begin{array}{l} v \propto t \\ v \propto \dfrac{1}{m} \end{array}\right\} \quad \text{(Experimental relations).}$$

We may combine these in the single expression—

$$v \propto \frac{ft}{m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which expresses the fundamental quantitative laws of dynamics.

The unit force.—In (1) above, the quantities v, t, and m must be measured in terms of units already defined, and therefore their magnitudes are directly ascertainable. As yet, however, we have not chosen the unit of force; by a suitable choice of this unit we may give any value we please to the right hand side of (1); in particular, we may choose the unit of force so that, if f is measured in terms of this unit, the two sides of (1) become equal. In this case we have the relation—

$$v = \frac{ft}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The unit force must now be defined so that equation (2) is satisfied. If v, t, and m are each equal to unity, then f must be equal to unity; hence we have the following definition :—

If unit force acts for unit time and sets unit mass of matter in motion, then unit velocity will be generated. To measure a force directly in terms of this unit, we must allow it to act for a known time and set a known mass of matter in motion, and measure the velocity generated; on substituting the values of the velocity, mass, and time in equation (2), the value of the force f can be calculated.

The value of the unit force will vary with the units of length, mass, and time adopted. Two units are in use.

The **dyne** is the c.g.s. unit of force. **A force of one dyne, acting for one second and setting one gram of matter in motion, produces a velocity of one centimetre per second.** If v, t, and m in (2) are measured in the c.g.s. (centimetre, gram, second) system, f is obtained in dynes.

The **poundal** is the British unit of force. **A force of one poundal, acting for one second and setting one pound (avoirdupois)**

of matter in motion, produces a velocity of one foot per second. If v, t, and m in (2) are measured in the f.lb.s. (foot, pound, second) system, f is obtained in poundals.

Problem.—*How many dynes are equivalent to one poundal?*

A poundal, acting for one second and setting 454 gm. of matter in motion, produces a velocity of 30·5 cm. per second. Hence, substituting $v=30{\cdot}5$, together with $m=454$ and $t=1$ in equation (2), we obtain—

$$30{\cdot}5 = \frac{f}{454}. \quad \therefore f = 30{\cdot}5 \times 454 = 13{,}850 \text{ dynes (nearly).}$$

Problem.—*From the data tabulated on p.* 15, *combined with the result that the wheel is equivalent to an additional* 125 *gm. of matter set in motion with the same velocity as* A, B, *and the rider, find the force in dynes exerted by the earth on the rider.*

Mass set in motion $= m = 410 + 125 = 535$ gm.
Time of action $= t = 2$ sec.
Velocity generated $= v = 39{\cdot}5$ cm. per sec.

Hence $$f = \frac{39{\cdot}5 \times 535}{2} = 10{,}600 \text{ dynes (nearly).}$$

Equation (2) may be thrown into the form—

$$\frac{v}{t} = \frac{f}{m};$$

and since v/t is equal to the acceleration (increase of velocity per unit time) we see that **the acceleration is equal to the force divided by the mass set in motion, or to the force per unit mass.**

Also—

$$f = \frac{mv}{t}.$$

The product mv is called the **momentum.** Thus we see that the force **acting on a body is equal to the rate at which its momentum is increasing.**

Momentum is a vector or directed quantity, since it involves the measurement of a velocity, which is a vector.

Substituting the dimensions of m, v, and t in the above equation, we see that the dimensions F of a force are given by the equation—

$$\mathrm{F} = \frac{\mathrm{M}\dfrac{\mathrm{L}}{\mathrm{T}}}{\mathrm{T}} = \frac{\mathrm{ML}}{\mathrm{T}^2} \text{ or } \mathrm{MLT}^{-2}.$$

The dimensions of momentum are given by MLT^{-1}.

Inertia.—It frequently happens that a force f acting for a time t, generates a velocity v in a body, and in addition produces motion in matter connected with the body, the velocity generated varying from particle to particle of this matter. The action of Attwood's machine affords an instance in point. In such cases, a value M can be found, such that—

$$v = \frac{ft}{\mathrm{M}}, \text{ and therefore } \mathrm{M} = \frac{1}{v} \cdot ft.$$

M is called the **inertia** of the body; thus the **inertia of a body is equal to the reciprocal of the velocity v that would be generated in it by the application of unit force for unit time.**

Let a bubble of air, having a volume of 1 c.c., be surrounded by water; it is known that if a force of 1 dyne acts on this bubble for a second, the velocity generated will be equal to 2 cm. per sec.; that is, a velocity equal to that which would be generated by unit force acting for a second on half a gram of matter, if this mass alone were set in motion. Thus, the inertia of the bubble is equal to half a gram. On the other hand, the mass of the bubble is merely the mass of the air comprised in it, and this mass would be of the order of magnitude of 0·001 gram. The inertia of the bubble is due to the circumstance that the bubble cannot move without setting the surrounding water in motion. In this and similar cases, the momentum of the body is equal to the product of the inertia and the velocity; as before, the applied force is equal to the rate of change of momentum. Also, the acceleration of the body is equal to the applied force divided by the inertia of the body.

Problem.—*A body of mass m, initially at rest, is acted upon by a constant force f; how far will the body move in t seconds?*

The velocity at any instant is proportional to the time during which the force has acted; thus the velocity increases uniformly from zero at the start, to the final value $(ft)/m$, and the average velocity of the body during these t seconds is $(ft)/2m$. If s represents the distance through which the body travels in the t seconds,

$$s = \textit{average velocity} \times \textit{time} = \frac{ft}{2m} \times t = \tfrac{1}{2}\frac{f}{m}t^2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The fall of bodies towards the earth.—If a sheet of paper and a large piece of metal are dropped from points at the same distance above the earth, the paper descends slowly, while the piece of metal falls much more quickly. Experiences such as this gave rise to the belief, entertained in early times, that heavy bodies always fall towards the earth more quickly than light ones. A simple variation of the experiment shows, however, that this is not the case; for if the paper is rolled tightly into a ball, and the experiment is repeated, it will be found that now the paper and the piece of metal fall to the earth in approximately equal times. Hence it may be inferred that the air opposes the motion of the flat sheet of paper more than it does that of the compact mass of metal.

EXPT. 4.—Obtain a glass tube, two or three inches in diameter, and about four feet long. Place a small piece of fluffy feather and a disc of lead in the tube, and then close its ends with rubber stoppers one of which is provided with a glass tube and stop-cock. Thoroughly exhaust the tube by means of a good air pump (such as a Fleuss pump) and close the stop-cock. Hold the tube in an upright position, and then suddenly invert it: the feather and lead remain at the end of the tube during the inversion, then commence to fall together, and both reach the other end of the tube at the same instant. Careful observation will often show that the feather rebounds from the end of the tube like an elastic ball.

This experiment shows that, **in the absence of the opposing force which the air exerts on bodies moving through it, all bodies that start from rest fall towards the earth through equal distances in equal times.**

Let f_1 be the force exerted by the earth on the feather, while m_1 is the mass of the feather. Further, let f_2 be the force exerted on the disc of lead, of which the mass is m_2. Then, since both bodies fall through the same distance in the same time t, we have from (3), p. 20—

$$\tfrac{1}{2}\frac{f_1}{m_1}t^2 = \tfrac{1}{2}\frac{f_2}{m_2}t^2$$

$$\therefore \frac{f_1}{m_1} = \frac{f_2}{m_2}.$$

Hence, we conclude that **the force exerted by the earth** (or **the force of gravity**) **is directly proportional to the mass of the body on which this force is exerted.** Thus, ten grams of matter are pulled toward the earth with a force exactly ten times as great as that exerted on one gram of matter. The force exerted by gravity on unit mass of matter is thus a constant; and this constant, which is denoted by g, also gives the acceleration of matter falling towards the earth (p. 19).

It is found that, in England, a mass of metal falls from rest through 490·5 cm. in the first second. Hence from (3)

$$490{\cdot}5 = \tfrac{1}{2}\frac{f}{m} \times 1^2, \text{ and } g = \frac{f}{m} = 981.$$

Hence we may say that in England **gravity exerts a force of 981 dynes per gram of matter attracted,** or that **the acceleration due to gravity is equal to** 981 **cm./sec.**^{2}

Problem.—*Find the value of g, in the British system.*

$$981 \text{ cm.} = \frac{981}{30{\cdot}48} \text{ ft.} = 32{\cdot}18 \text{ ft.}$$

$$981 \text{ cm. per sec. per sec.} = 32{\cdot}18 \text{ ft. per sec. per sec.}$$

$$\therefore g = 32{\cdot}2 \text{ poundals per pound, or } 32{\cdot}2 \text{ ft./sec.}^2 \text{ (nearly).}$$

Composition and resolution of forces.—Let OA and OC (Fig. 3, p. 9) represent the magnitudes and directions of two forces acting at the point O. Then if OA were to act for one second on one gram of matter, it would, in that second, generate a velocity equal to OA per second. Similarly, if the force OC were to act for one second on a gram of matter, it would, in that second, generate a velocity equal to OC per second. If both forces acted simultaneously on a gram of matter for one second, each would generate the same velocity as if it acted alone (Newton's second law of motion, p. 12), and therefore the actual velocity generated would be the resultant of the two velocities OA per second and OC per second, and this resultant is given, in magnitude and direction, by the diagonal OB of the parallelogram OCBA. But a single force equal to OB would generate a velocity equal to OB per second if it acted on one gram of matter for one second; hence, a single force OB is equivalent to the two forces OA and OC acting simultaneously at the point O, and OB thus represents the resultant of the two forces OA and OC. This construction for the resultant of two forces is known as the **parallelogram of forces.**

Thus, two forces acting at a point are equivalent to a single force called their **resultant.** Conversely, a single force represented by OB (Fig. 3) is equivalent to the two components OA and OC, so that a force may always be resolved into two components acting in any assigned directions. In particular we may, when we find it convenient to do so, resolve a force into two rectangular components by the method explained on p. 8.

We can now readily find a single force which is equivalent to any number of forces OA, OC, OD, OE, OH (Fig. 6), acting at a point O. Find the resultant OB of the forces OA and OC; this can be done either by completing the parallelogram OCBA and drawing the diagonal OB, or, more simply, by drawing AB equal and parallel to, and in the same direction as, OC, and then joining OB. From B draw BF equal and parallel to, and in the same direction as, OD, and join OF; then OF represents the resultant of the forces OB and OD, that is, it represents the resultant of the forces OA, OC, and OD. In a similar manner, we may find the resultant OG of the forces OF and OE, that is, the resultant of the forces OA, OC, OD, and OE; and so on. Hence, *to find the resultant of any number of forces acting at a point, we construct a polygon of which the sides are respectively equal and parallel to, and are drawn in the same direction as, the forces in question; a line drawn from the starting point to the termination of the last side of the polygon gives a single force which is the resultant of the given forces.* This construction for finding the resultant of any number of forces is called the **polygon of forces.**

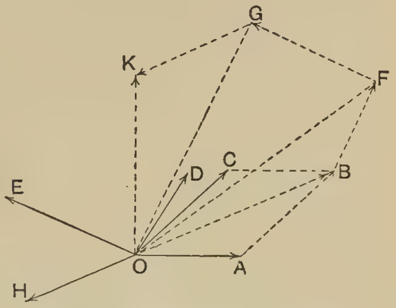

FIG. 6.—The composition of a number of forces.

Equilibrium.—A body which remains at rest or in uniform rectilinear motion is either acted upon by no force, or by several forces which together are equivalent to no force (Newton's first law of motion, p. 12). In the latter case the forces are said to be in equilibrium.

If several forces act at a point, their resultant must have zero value in order that equilibrium may be produced. Hence, if we construct the polygon of forces in the manner already described, the last side will terminate at the starting point, or the polygon will be closed, when the forces are in equilibrium. In particular if AB, BC, CA are the sides of any triangle, then three forces, which may be represented in magnitude and direction by the vectors AB, BC, and CA, will produce equilibrium.

If several forces in equilibrium act at a point, and these forces are resolved in two rectangular directions, then the components in either direction must be in equilibrium among themselves. For, if the components in either direction were equivalent to a resultant of finite magnitude, the body would commence to move in the direction of this resultant; it is obvious that the effect of the resultant in question could not be neutralised by the components in a perpendicular direction, since if these possessed a resultant its effect would be to make the body move in its own direction.

Action and reaction.—If two persons A and B stand on smooth ice, and A pushes B, not only does B move in the direction of the push, but A moves in the opposite direction. If the ice were perfectly smooth, and the two persons were equal in mass, the result of the push would be that A and B would move away from each other at equal velocities. Hence, the force acting on A is equal in magnitude and opposite in direction to that exerted on B. Similarly, if a magnet attracts a piece of iron with a certain force, the iron attracts the magnet with an equal and opposite force, so that the result is the same as if a stretched elastic filament extended from one to the other, and each moved, or tended to move, toward the other under the tension of the filament.

Newton's third law of motion is a generalisation derived from the observation of phenomena such as the above. It may be stated as follows :—

To every action there is always an equal and opposite reaction; or the mutual actions of any two bodies are always equal and oppositely directed.

Problem.—*A stationary body is surrounded by a fluid which is also stationary. Prove that the fluid can exert no tangential force on any element of area of the solid.*

A tangential force is a force parallel to the element of area on which it acts. If the fluid in contact with the solid were to exert a tangential force on any element of area of the solid, then the element of area would exert an equal but oppositely directed force on the fluid. Such a force would cause the fluid to flow continuously along the surface of the solid; for it is a characteristic property of a fluid that one layer can move relatively to contiguous layers, although the viscosity of the fluid may retard the rate of movement. But the layer of fluid in contact with the solid is at rest. Therefore no tangential force can be exerted by the fluid on the solid. Thus it follows that the only kind of force which a fluid can exert on a solid, when both are stationary, is a normal force ; that is, a force perpendicular to the element of area on which it acts. The normal force per unit area is called the **pressure** ; to obtain the value of the pressure acting on any element of area, the normal force acting on the element must be divided by the area of the element.

Conservation of momentum.—Let two bodies A and B be moving in any manner, subject to the condition that no force, exerted by any other body, acts on either of them. Then any change in the motion of one of these bodies must be due to a force exerted by the other body. Further, since action and reaction are equal and opposite, if A exerts any force on B, then B must exert an equal force in the opposite direction on A. Since force is measured by rate of change of momentum, it follows that if the momentum of A is increasing at a finite rate in one direction, then the momentum of B must be increasing at an equal rate in the opposite direction, so that if we prefix a positive (+) sign to the rate of increase of A's momentum, we must prefix a negative (−) sign to the rate of increase of B's momentum, and on adding these two quantities together we obtain zero as a result. Similar reasoning applies to any system of bodies which is unacted upon by external forces ; hence we obtain the law of **conservation**[1] **of momentum,** viz., that **when any system of bodies is unacted upon by external forces, the vector sum of the momenta of the bodies remains constant.** As a direct corollary, we have the law that any change in the vector sum of the momenta of a system of bodies, must be due to a force or forces exerted on the system by external bodies.

Motion in a curved path—When a body moves in a curved path, the direction of motion continually changes, and therefore a force must act upon the body. The following example will serve to explain how the magnitude and direction of this force are calculated.

---

[1] Latin *conservare*, *con*=together, and *servare*, to guard.

Problem.—*A body of mass m revolves uniformly with a velocity equal to v, in a circular orbit of radius* OA $= r$ (Fig. 7). *Determine the magnitude and direction of the force that must act on the body.*

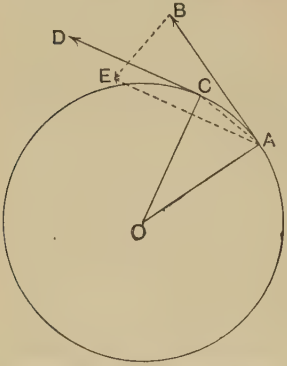

FIG. 7.—Uniform circular motion.

If we select two points very near to each other in the circular orbit, the very short arc joining these points will approximate to a straight line, and during the very small time which elapses whilst the body is passing over this element of path, the motion of the body will be in this straight line, that is, it will be tangential[1] to the circle.

From A and C, neighbouring points on the circle, draw tangents AB and CD, each numerically equal to v, the velocity of the body.

Let the body move from A to C in a time t; then its velocity in this time changes from AB to CD. From A draw AE equal and parallel to CD. Then, to change the velocity represented by AB into that represented by AE, we must add to it a velocity represented by BE (p. 10). But the change of velocity per second is equal to the force acting on each unit of mass of the body (p. 19). Thus, BE/t is equal to the force per unit mass acting on the body. As t is diminished indefinitely, the lines AB and AE become more and more nearly parallel, and BE becomes more and more nearly perpendicular to AB, or parallel to AO. Thus, **the force acting on the body when at A must be in the direction AO, that is, it must be directed toward the centre of the circular path.**

The magnitude of the force will be equal to mBE/t, when t is diminished indefinitely. To evaluate this ratio, notice that the triangles AOC and BAE are similar, both being isosceles, while $\angle$ AOC $= \angle$ BAE. Also, in the limit, the arc AC and its chord will be equal: the body traverses the arc AC in t seconds with a velocity v, and the arc AC will be equal to vt. Thus, from the similarity of the triangles AOC and BAE—

$$\frac{\mathrm{BE}}{\mathrm{AB}} = \frac{\mathrm{AC}}{\mathrm{OA}}. \quad \therefore \quad \frac{\mathrm{BE}}{v} = \frac{vt}{r}, \text{ and } \frac{\mathrm{BE}}{t} = \frac{v^2}{r}.$$

---

[1] A tangent to a curve is a straight line drawn through two points infinitely near to each other on the curve.

Consequently, the force acting along AO $= f = m\frac{\text{BE}}{t} = m \cdot \frac{v^2}{r}$.

Thus, a force, equal in magnitude to mv^2/r, must constantly pull the body toward the centre of the circle; this force is called the **centripetal force**[1] acting on the body. Now, this force must be due to some agent which pulls the body toward the centre; and since action and reaction are equal and opposite, an equal force, called the **centrifugal**[2] **force**, must act on this agent, tending to pull it away from the centre.

Let a body move in a curved path of any shape; then a circle, called the circle of curvature, can be drawn through three points infinitely near to one another on the curve; and while the body is passing over the part of the curve common to this and the circle, it is virtually moving round the circle. Hence, at the instant considered, the body will be acted upon by a force directed toward the centre of curvature, and equal to mv^2/r, where v is the instantaneous velocity of the body and r is the radius of curvature of the element of path over which the body is passing.

Problem.—*A projectile is fired, from the level of the ground, with an initial velocity* V, *in a direction making an angle* θ *with the horizontal plane. Determine the point where it will strike the ground. Neglect air friction.*

From the instant when the projectile leaves the muzzle of the gun the force of gravity acts upon it, conferring on it a downward velocity which increases at the rate of g units of length per second during each second. Let the initial velocity V of the projectile be resolved into its horizontal and vertical components, *viz.*, $\text{V} \cos \theta$ and $\text{V} \sin \theta$ (p. 8). Then the projectile will continue to rise for a time t_1, in which the downward velocity gt_1 conferred by gravity becomes equal to $\text{V} \sin \theta$; *i.e.* it will rise for $(\text{V} \sin \theta)/g$ seconds. During this time its upward velocity has diminished uniformly from $\text{V} \sin \theta$ to zero, so that its average upward velocity has been equal to $(\text{V} \sin \theta)/2$; hence it has risen through the height

$$\text{average velocity} \times \text{time} = \frac{\text{V} \sin \theta}{2} \times \frac{\text{V} \sin \theta}{g} = \frac{\text{V}^2 \sin^2 \theta}{2g}.$$

After reaching the highest point in its path the projectile commences to fall toward the earth, and it will reach the level of the ground after an additional time t_2, given by the equation

---

[1] Latin *centrum*, centre, and *petere*, to move towards.

[2] Latin *centrum*, and *fugere*, to flee.

$$s = \frac{V^2 \sin^2 \theta}{2g} = \tfrac{1}{2} g t_2^2$$

$$\therefore t_2 = \frac{V \sin \theta}{g}.$$

Hence, the time which elapses, between the instant when the projectile leaves the muzzle of the gun and that at which it strikes the ground, is equal to $(2V \sin \theta)/g$. During this time the horizontal component of the velocity has remained equal to $V \cos \theta$, and therefore the horizontal distance, through which the projectile has moved, or the *range* of the projectile, is equal to

$$V \cos \theta \times \frac{2V \sin \theta}{g} = \frac{V^2}{g} \cdot 2 \sin \theta \cos \theta = \frac{V^2}{g} \sin 2\theta.$$

Hence, for a given velocity of projection the range will be greatest when $\sin 2\theta$ has its maximum value, *i.e.*, when $2\theta = \pi/2$, and therefore $\theta = \pi/4$. In this case the range is equal to V^2/g.

For any range less than the maximum, there are two possible elevations, one less and the other greater than 45°. When a cannon is used to fire a shot at a given object, the elevation chosen is that which is less than 45°; if a shot were fired to the same place from a howitzer, the elevation chosen would be that which is greater than 45°. It must be remembered, however, that we have neglected air friction in the above calculations, and the effect of this on high speed projectiles is very considerable, so that the range obtained in practice is much less than that obtained above.

Work.—When a body moves in any direction against an opposing force, **work** is said to be done. The work done is measured by the product of the opposing force and the displacement of the body, or

work = force × displacement.

Thus, unit work will be performed when a body is displaced through unit distance against unit opposing force.

When a force of one dyne is overcome through a distance of one centimetre, the work done is called an erg. This unit of work is inconveniently small, so that for practical purposes a larger unit is used; this unit is equal to ten million (or 10^7) ergs, and is called a **joule.**

When a force of one poundal is overcome though a distance of one

foot, the work done is called a foot-poundal. British engineers use another unit called the **foot-pound. One foot-pound of work is done when a pound of matter is raised against the force of gravity through a distance of one foot.** Since, in England, a pound of matter is attracted toward the earth with a force of 32·2 poundals (p. 22), it follows that in England a foot-pound is equivalent to 32·2 foot-poundals.

The disadvantage incurred by using the ft.-lb., or any other gravitational unit of work, is that a given mass of matter is attracted toward the earth with different forces in different places. Thus at Greenwich the force of gravity is equal to 981·17 dynes per gram, which is equivalent to 32·191 poundals per lb. At Aberdeen the force of gravity is equal to 981·64 dynes per gram, which is equivalent to 32·206 poundals per lb. At the equator the force of gravity is equal to 978·10 dynes per gram, which is equivalent to 32·090 poundals per lb. Hence, the magnitude of the foot-pound varies from place to place on the earth's surface, and therefore the foot-pound is an arbitrary unit; on the other hand, the erg and the foot-poundal are units of work which depend only on the fundamental units of length, mass, and time, and therefore the value of either of these units is independent of local peculiarities of the earth.

When a body is moved at right angles to a force, no work is done. Let a force f make an angle θ with a line drawn in a direction opposite to that in which a body is moved; then this force is equivalent to a component $f \cos \theta$ directly opposed to the motion of the body, and a component $f \sin \theta$ perpendicular to the direction of motion. Since no work is performed against the latter component, the work done is equal to the product of $f \cos \theta$ and the displacement of the body.

When a body moves along the direction in which a force acts, work is said to be done *by* the force: when the direction of motion is opposite to that in which the force acts, work is said to be done *against* the force. Thus, when a body is raised from the earth, work is done *against* gravity; when a body falls towards the earth, work is done *by* gravity. In any case, the value of the work is equal to the displacement multiplied by the component force acting in the line of the displacement, and will be obtained in ergs if the displacement is measured in centimetres and the force in dynes, or in foot-poundals, if the displacement is measured in feet and the force in poundals.

The dimensions of work are obtained by multiplying the dimensions of a force by a length, and are therefore given by $MLT^{-2} \times L$, or by ML^2T^{-2}.

Power or activity is defined as the rate at which work is done. The c.g.s. unit of power or activity is one erg per second: no special name is given to this unit. The practical unit, called the **watt**, is equal to one joule (or ten million ergs) per second.

British engineers use a unit of power called the **horse-power**; this is equal to 33,000 ft.-lb. per minute, or 550 ft.-lb. per second.

It must be noticed that the work done in any circumstances does not depend on the time taken to move the body against the opposing force; thus, if a given body is raised through a given distance from the ground, the work done has the same value whether the displacement is completed in a second or in a year. It is only when we come to measure the power or activity that we need know the time which elapses during the performance of the work.

Problem.—*Find the equivalent of one horse-power in terms of the watt.*

In raising 1 lb. (454 gm.) through 1 ft. (30·5 cm.), a force of 454×981 dynes is overcome through 30·5 cm.; therefore

$$1 \text{ ft.-lb.} = 454 \times 981 \times 30{\cdot}5 = 13{,}580{,}000 \text{ ergs.}$$

$$\begin{aligned} \therefore\ 550 \text{ ft.-lb. per sec.} &= 550 \times 13{,}580{,}000 = 7{,}460{,}000{,}000 \text{ ergs per sec.} \\ &= 746 \times 10^7 \text{ ergs per sec.} \\ &= 746 \text{ watts.} \end{aligned}$$

The Continental horse-power is equivalent to 735 watts. At the time when it first became necessary to measure power, the Continental horses were not so strong as the British horses, or possibly the strength of the British horses was over-rated; hence the difference in the values of the Continental and British horse-power.

The dimensions of power or activity are obtained by dividing the dimensions of work by a time, and are therefore given by $ML^2T^{-2} \div T$ or ML^2T^{-3}.

Energy.—A body which is capable of performing work is said to possess energy; the value of its energy is measured by the work that the body can perform, and is expressed as so many ergs or foot-poundals in theoretical investigations, or as so many joules or foot-pounds for practical purposes.

The dimensions of energy are obviously the same as those of work.

There are several different kinds of energy which must now be considered.

Kinetic energy is the energy which a body possesses in virtue of its motion. When a stone is thrown vertically upwards, it rises to a certain height which depends on its initial velocity, and in so doing work is done against the opposing force of gravity ; hence, at the instant when the stone leaves the hand it possesses an amount of kinetic energy, which is equal to the work which it can do against gravity in rising to the highest point in its course.

Let it be required to calculate the kinetic energy possessed by a body of mass m moving with a velocity of v units of length per second. Let the motion of the body be opposed by a uniform force f, and let the body be brought to rest after it has moved through a distance equal to s units of length against this force. Then the work done by the body against the opposing force is equal to fs ; and this, therefore, is the value of the kinetic energy possessed by the body. We must now find the value of fs in terms of the mass and velocity of the body.

Since the opposing force is uniform, it produces uniform acceleration, it may be considered to generate a uniformly increasing velocity in a direction opposite to the initial velocity of the body, and the body will be brought to rest when the two opposite velocities are numerically equal. Let the velocity generated by the force f become equal to the initial velocity v of the body at the instant when the force has acted for t seconds ; then (p. 18)—

$$v = \frac{ft}{m}.$$

During these t seconds, the resultant velocity of the body has decreased continuously from its initial value v to its final value 0, and therefore the average velocity—

$$= \frac{v}{2} = \tfrac{1}{2}\frac{ft}{m},$$

and the distance s through which the body has moved during these t seconds is given by the equation—

$$s = \tfrac{1}{2}\frac{ft}{m} \times t = \tfrac{1}{2}\frac{ft^2}{m};$$

$$\therefore fs = \tfrac{1}{2}\frac{f^2t^2}{m} = \tfrac{1}{2}m\left(\frac{ft}{m}\right)^2 = \tfrac{1}{2}mv^2.$$

Hence **the kinetic energy possessed by a body of mass m moving with a velocity v is equal to the product of half the mass and the square of the velocity of the body.**

If m is measured in grams, and v in centimetres per second, the energy will be obtained in ergs; if m is measured in pounds, and v in feet per second, the energy will be obtained in foot-poundals. To convert foot-poundals into foot-pounds, divide by 32·2.

Potential energy.—When a body is stationary and yet possesses the capacity for performing work under suitable conditions, it is said to possess potential energy. Potential energy may take a number of different forms.

Gravitational energy.—When a body is raised above the ground, it possesses energy in virtue of its position. Let two equal masses, A and B, be attached to the ends of a silk cord which passes over the rim of a frictionless wheel (compare Attwood's machine, p. 13); and, in the first place, let A be higher than B. On giving the smallest possible downward impulse to A, it commences to descend, and B commences to rise at an equal rate. The velocity communicated to the system remains constant (in the absence of friction) and therefore the kinetic energy of the system remains constant; but as B rises through a distance s, the force f exerted by gravity on B is overcome through that distance, and fs units of work are done against gravity. This work is done by the tension of the silk cord, which is due to the force f exerted by gravity on the mass A; and the work done by the force of gravity acting on A is equal to fs, that is, to the work done *against* the force of gravity acting on B. When A reaches the ground, B can rise no further unless some external force acts upon it; hence the possibility of work being done by A depends on its position, and the lower A is the less work can be done by it. Let A be at a height h above the ground; then in descending to the ground it can raise B through an equal height h, and thus perform fh units of work; hence A possesses the capacity of performing fh units of work when it is at a height h above the ground, and therefore, in these circumstances, it possesses fh units of potential energy. This energy is lost when A descends to the ground; and in general, when a body descends through a given distance toward the earth, it loses as much potential energy as it would gain if it were raised through the same distance from the earth.

When a body is projected vertically upwards with a velocity v, it possesses $mv^2/2$ units of kinetic energy. In the absence of air friction it will rise till the downward velocity gt, communicated to it by gravity, is numerically equal to v, and, in so doing, reach a height h, given by the equation—

$$h = \frac{gt}{2} \times t = \tfrac{1}{2}gt^2 \text{ (compare p. 31)},$$

where g is the force exerted by gravity on each gram of matter. In rising through the height h, against the force of gravity mg, the work done (or the potential energy gained)—

$$= mgh = \tfrac{1}{2}m(gt)^2 = \tfrac{1}{2}mv^2;$$

that is, the potential energy gained is equal to the kinetic energy that has disappeared. If the body now falls to the earth, it will lose mgh units of potential energy, and regain its original kinetic energy. If the body were perfectly elastic, and rebounded from the earth with a velocity numerically equal to that which it possessed just before the impact, then it would once more rise to the height h, once more losing its kinetic energy and gaining an equal amount of potential energy; and similar transformations of energy would succeed each other indefinitely.

When work is done against a force which partakes of the nature of friction, there is no corresponding gain of potential energy; for a frictional force tends to diminish the velocity of matter, but not to set matter in motion. Hence, work may be done *against* a frictional force but not *by* a frictional force.

When a body is raised from the earth in such a manner that no frictional forces are overcome, the work done, and the potential energy gained, depend only on the height to which it is raised; if it is moved in a direction inclined to the vertical, and finally reaches a point h units of length above the ground, the work done has just the same value as if the body had merely been raised vertically through h units of length. To prove this statement, let the actual path of the body be replaced by an infinite number of steps, each consisting of a vertical and a horizontal portion (compare p. 7); in traversing each horizontal portion of a step no work is done against gravity, since the motion is perpendicular to the force (p. 29); the sum of the vertical portions of the steps is equal to h, and in traversing these the work done is the same as if the body were raised vertically through a distance h. Hence we obtain the following general law: **in the absence of friction, the work done against gravity, during any rearrangement of matter, is independent of the precise manner in which that rearrangement is effected.**

Problem.—*Find an expression for the pressure p at a distance h below the surface of a liquid of density ρ, the atmospheric pressure on the surface of the liquid being equal to* P.

By the pressure at a distance h below the surface of a liquid, we mean the force acting normally on one side of an imaginary surface of unit area, at a distance h below the surface of the liquid. Let the liquid be contained in a vessel, the area of the free surface SS (Fig. 8) being A. Let a small cylinder C, provided with a frictionless piston of area a, be placed so that the piston is at a distance h below the surface. Let the interior of the cylinder be exhausted, the piston being prevented from being pushed inwards by a spring. If the piston is caused by some means to move outwards through a small distance d, the work done against the pressure p will be equal to $pa \times d$, since pa is the force tending to push the piston inwards, and this is overcome through the distance d. In moving outwards, the piston will displace a volume ad of the liquid, and as a consequence the surface SS will rise through a distance δ, such that $A\delta = ad$. The work done will have the same value as if a volume ad of the liquid were removed from a point at a distance h below the surface, and then raised to, and spread over, the surface. The liquid raised has a mass equal to ρad, and the work done in raising this mass through the distance h is equal to $g\rho ad \times h$. In raising the surface SS through a height δ against the force PA due to the atmospheric pressure, the work done is $PA\delta = Pad$. Hence, the total work performed against gravity and atmospheric pressure $= (g\rho h + P) \times ad$; and equating this expression to the work done by the piston against the pressure p, we have

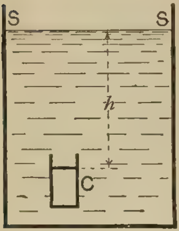

FIG. 8.—Method of determining the pressure at any point in a liquid.

$$pad = (g\rho h + P)ad;$$

$$\therefore p = g\rho h + P.$$

Exactly the same amount of work will be performed whatever may be the direction in which the cylinder points; and as the pressure, by definition, always gives rise to a force normal to the surface of the piston, it follows that the pressure is equal in all directions.

Buoyancy.—A body immersed in a material fluid medium is pulled downwards with a smaller resultant force than would be exerted on it if the medium were removed. The gravitational force exerted on a body is called the weight of the body, so we may say that a body apparently loses weight when it is immersed in a material fluid medium. Let W be the weight of a body in a vacuum, and W′ its apparent weight when immersed in a given medium; and let w be the weight of that part of the medium which is displaced by the body. If the body is raised through a distance d, then $W'd$ units of work are performed. Let us now consider the redistribution of matter; as the body is raised, an equal volume of the medium may be supposed to descend so as to occupy the space which the body has vacated; that is, Wd units of work are done *against* gravity and wd units of work *by* gravity. Hence—

$$W'd = Wd - wd;$$

$$\therefore\ W' = W - w.$$

Thus **the weight apparently lost by the body is equal to the weight of that part of the medium displaced by the body.**

The way in which the apparent loss of weight is brought about may be realised by the examination of a particular instance. Let the body have the form of a cylinder of length l and cross-sectional area a; and let the cylinder be suspended with its axis vertical, its upper plane end being at a distance h below the surface of the liquid. Then the pressure acting on the upper plane end produces a force equal to $(P+g\rho h)a$, acting downwards (p. 34), where P denotes the atmospheric pressure on the surface of the liquid and ρ is the density of the liquid. The pressure acting on the lower plane end produces a force equal to $\{P+g\rho(h+l)\}a$, acting upwards. The resultant of these two forces is equal to $g\rho la$, acting upwards; and this is the force that would be exerted by gravity on the volume al of the fluid displaced by the cylinder. The forces acting on the curved sides of the cylinder are horizontal, and therefore can have no vertical component.

The student should now realise that in air, a pound of feathers weighs less than a pound of lead. (Remember that the pound is a unit of *mass*.) When a body is weighed in air, a correction must be applied for the mass of air displaced by it and by the weights.

Energy of strain.—When the shape of an elastic solid is modified by the application of suitable forces, the solid is said to be strained. When the volume of a gas is changed it is also said to be strained. In producing a given **strain**, forces are called into play which tend to restore the substance to its original condition ; these forces are called **stresses**. In order to maintain the strain, it is obvious that the applied forces must be equal and opposite to the stresses.

In producing any strain the opposing stresses must be overcome, and hence work must be done. This work can be recovered by allowing the substance to regain its original condition ; thus, potential energy is stored in the strained substance.

Problem.—*When an elastic filament is stretched by the application of a force f_1 at each end, the length of the filament is increased by unity. How much work would be done in increasing the length of the filament by s units?*

Experiment shows that when the elongation produced is small in comparison with the original length of the filament, the stress is proportional to the elongation. Thus, since unit elongation produces a stress f_1, an elongation of s units produces a stress $f_1 s$, and the average stress during the elongation $= (f_1 s)/2$.

$$\therefore \text{ Work done} = \text{average stress} \times \text{strain} = \tfrac{1}{2} f_1 s \times s = \tfrac{1}{2} f_1 s^2.$$

Other kinds of energy.—**Heat.** When a body is moved against friction, work, of course, is done ; but this work is not represented by a gain of either potential or kinetic energy. Heat is, however, produced ; and the researches of Mayer, Joule, and other later experimenters have shown that the amount of heat produced is directly proportional to the work done. A steam engine is driven by steam which enters the cylinder, expands and drives the piston before it, and then escapes ; Hirn has shown that the steam contains more heat when it enters than it does when it leaves the cylinder, and that the heat which has disappeared is proportional to the work done by the engine. Hence we conclude that **heat is a form of energy.**

The unit quantity of heat, called the *gram-calorie* or *therm*, is defined as the amount of heat that will raise the temperature of one gram of water from 14·5° C to 15·5° C. It has been found that this quantity of heat is equivalent to 4·19 joules, or roughly to 4·2 joules, of energy. The quantity of energy equivalent to unit heat is called **Joule's equivalent**, and is denoted by J.

Light.—When light falls on a blackened surface it is absorbed, and the temperature of the surface rises: hence heat is produced, and experiment shows that the amount of heat is proportional to the time during which light of a given character and intensity falls on the surface. We may therefore conclude that light possesses energy; we are unable, however, to express the mechanical equivalent of light in any very useful form, since light is detected, and its intensity is measured, by the eye, instead of by some instrument not dependent on the peculiarities of the observer.

Sound.—Sound consists of compressions, or expansions, or both, transmitted through a gas, a liquid, or a solid. Compressions and expansions are particular forms of strain; hence we conclude that the transmission of sound is equivalent to the transmission of energy through matter.

Magnetic energy.—A magnet possesses the capacity of attracting pieces of iron toward itself, and in so doing, work is performed; hence a magnet possesses a store of energy. This store is limited; for when a horseshoe magnet has drawn its keeper up against its poles, it can do very little additional work in attracting other pieces of iron. A piece of steel may be magnetised by stroking it from one end to the other with the pole of a magnet; during this process work must be done, since the pole clings to the steel, and its movement is thus opposed; if we subtract the mechanical equivalent of the heat produced, by friction and otherwise, from the work done during the stroking, the residue represents the energy of the magnet.

Electric energy.—An electrified body can attract uncharged bodies toward itself; it can also attract bodies with charges dissimilar to its own, or repel bodies with charges similar to its own. In each of these cases work is done, and therefore an electric charge possesses a store of energy. The energy is not, however, merely proportional to the charge, but depends on the shape and size of the conductor which carries it, and also on the presence of other bodies, charged or uncharged, in its neighbourhood.

Chemical energy.—During the progress of many chemical reactions, such as the combination of hydrogen and oxygen to form water, heat is produced; and the amount of this heat is proportional to the amounts of the reacting substances that have combined ; thus, when two grams of hydrogen combine with oxygen, twice as much heat is produced as when only one gram combines. We conclude that an equal amount of work must be performed when the process is reversed ; that is, when water is decomposed into hydrogen and oxygen in the water voltameter. We are not in a position to state that unit mass of any particular element possesses an assignable store of energy ; but we know how much energy will be liberated, in the form of heat, when known masses of two or more elements combine to form a definite compound.

Conservation of energy.—We have now examined a number of forms which energy can take, and have noticed that one form of energy can only increase at the expense of some other form or forms of energy. This leads to the important generalisation that **energy can neither be created nor destroyed**, and therefore that the sum total of the energy in the universe is constant.

This is known as the law of **conservation of energy.**

We must not conclude that energy is a definite entity, like matter, for instance ; we have already seen that in certain circumstances the momentum of a system remains constant (p. 25), and this constancy does not imply that momentum is an entity. The laws of conservation of momentum and energy must be considered to be generalisations derived from experience, which aid us in the task of explaining physical phenomena.

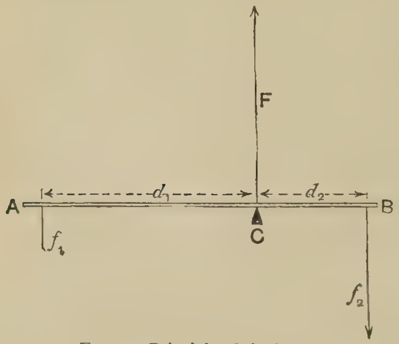

FIG. 9.—Principle of the lever.

The principle of virtual displacements may be stated as follows :—If a body is in equilibrium under the action of any forces, then during a small displacement of the body, the total work performed by these forces is equal to zero. The truth of this principle is self-evident. When a body is in equilibrium under the action of any forces, these forces have no tendency to set the body in motion, that is, to communicate kinetic energy to it; hence during a small displacement the body gains no kinetic energy, and therefore loses no potential energy; that is, no work is done during the displacement. It must be remembered that the displacement must be so small that no finite change is produced in the forces acting on the body.

Equilibrium of the lever.—Let AB (Fig. 9) represent a horizontal lever, resting on the fulcrum C, and in equilibrium under the action of the vertical downward forces f_1 and f_2 near its ends.

To find the force F *exerted by the fulcrum*, imagine the lever to be displaced vertically upwards through a small distance δ. Then the work done against the forces f_1, f_2, and F, is equal to zero, that is—

$$f_1\delta + f_2\delta + \mathrm{F}\delta = 0,$$

by the principle of virtual displacements;

$$\therefore\ \mathrm{F} = -(f_1 + f_2).$$

That is, the force exerted by the fulcrum is numerically equal to the sum of the downward forces acting on the lever, but its direction is vertically upward.

To find the relation between f_1 and f_2, imagine the lever to be rotated through a very small angle θ about the fulcrum; then, if the point of application of f_1 is at a distance d_1 from the fulcrum, and the end A of the lever sinks, the work done *by* f_1 is equal to $f_1d_1\theta$; similarly, if f_2 acts at a distance d_2 from C, the work done *against* f_2 will be $f_2d_2\theta$. The work done against f_2 increases the potential energy, and that done by f_1 diminishes the potential energy; therefore by the principle of virtual displacements—

$$f_1d_1\theta - f_2d_2\theta = 0;$$

$$\therefore\ f_1d_1 = f_2d_2.$$

We thus see that the tendency of f_1 to rotate the lever in an anti-clockwise direction about the fixed point C is measured by f_1d_1, and this tendency can be neutralised by the force f_2, provided that this force tends to rotate the lever in a clockwise direction, and that $f_2d_2 = f_1d_1$.

The product of a force, into the perpendicular let fall on it from any point in a body, is called the turning moment or torque exerted about that point. A positive sign may be given to torques which tend to produce clockwise rotations, and a negative sign to those which tend to produce anti-clockwise rotations. For a body to be in equilibrium, it is obvious that the sum of all the torques about any point in the body must be zero.

Two equal and oppositely directed forces, which act in a straight line, are in equilibrium. Two equal and oppositely directed forces which do not act in a straight line, have no tendency to impart translatory velocity in any direction to the body. To prove this, let the body be displaced through a small distance in any direction, without rotation. Then no work is done by the forces, and therefore no kinetic energy is gained by the body; in other words, the forces have imparted no translatory velocity to the body. It can be proved, without difficulty, that the torque exerted *about any point in the body*, is equal to the product of either force and the perpendicular distance from one force to the other.

In the c.g.s. system of units, a torque is measured in dyne-centimetres. In the British system it is measured in poundal-ft. If gravitational units (p. 29) are used, a torque may be measured in gram-centimetres, kilogram-centimetres, pound-feet, etc. (notice that the *foot-pound* is a unit of work, while the *pound-foot* is a unit of torque).

A torque is a vector quantity, since it cannot be defined without specifying the axis about which it tends to produce rotation, and the direction of this rotation. The following convention is used in representing torques graphically. Look at a body along the axis about which the torque tends to produce rotation; then if the rotation is clockwise, the vector is drawn away from the observer; if the rotation is anti-clockwise, the vector is drawn toward the observer. The length of the vector is equal to the product of the force into the perpendicular let fall on it from the axis of rotation.

Centre of Gravity.—The fact that the force exerted on a body by gravity is proportional to the mass of the body (p. 21) implies that gravity attracts each particle of a body with a force which is not modified by the action of neighbouring particles. Let us suppose that a body can be divided into particles which are severally attracted downwards with forces f_1, f_2, f_3, etc., and let us assume that the body as a whole may be prevented from falling by the application, at some point, of an upward force F. Then if the body is raised vertically through a small distance δ, we have, by the principle of virtual displacement,

$$F\delta = f_1\delta + f_2\delta + f_3\delta \; . \; . \; . \; . \; \alpha = \delta \, . \, \Sigma f,$$

where Σf denotes the sum of all the forces exerted by gravity on the body. Then—

$$F = \Sigma f.$$

Let the forces $f_1, f_2, f_3, \ldots$ etc. act at horizontal distances d_1, d_2, d_3 . . . etc. from some definite point of the body, and let it be required to find the distance D from this point to the line of action of the force F. Then the torque due to F about the specified point must be equal and opposite to the sum of the torques due to $f_1, f_2, f_3, \ldots$ etc., about the same point; therefore—

$$\mathrm{FD} = f_1d_1 + f_2d_2 + f_3d_3 + \ldots = \Sigma(fd);$$

$$\therefore\ \mathrm{D} = \frac{\Sigma(fd)}{\mathrm{F}} = \frac{\Sigma(fd)}{\Sigma f}.$$

By placing the body in three different positions, we can find the distances from three definite points to the line of action of the force F, and hence the position of the point at which F must act becomes definitely known. Since the force F neutralises the effects of the pull of gravity on all the particles comprised in the body, it follows that a downward force equal to F, and applied at the point whereat F must act, will produce the same effect on the body (supposing it to be rigid) as is actually produced by gravity pulling on all the particles of the body. The point at which F must act is consequently called the **centre of gravity** of the body.

Work performed in overcoming a torque.—Let it be supposed that a circular cylinder can turn about an axle which coincides with the axis of symmetry of the cylinder; and let the axle be mounted in a horizontal position, in frictionless bearings. Let one end of a cord be fastened to the surface of the cylinder, and let a turn or two of the cord be wound round the cylinder, the free end of the cord being attached to a heavy body which hangs freely. Let the pull of gravity on the body be equal to f; then if the cylinder is rotated so as to raise the body, the force f is overcome, and work is performed. If the radius of the cylinder is equal to r, a rotation of the cylinder through an angle θ winds an extra length of cord, equal to $r\theta$, on to the cylinder; and the heavy body is raised through a vertical distance equal to $r\theta$. Thus the work done during the rotation is equal to $f \times r\theta = fr\theta$. But the perpendicular distance from the axis of rotation to the vertical force f is equal to r, and therefore fr is equal to the torque exerted about the axis of rotation. Let the angle θ through which the cylinder has been rotated be called the *twist*; then **the work done is equal to the product of the torque and the twist.**

In the case considered above, the force f is constant, and therefore the torque is constant. Now let it be supposed that the free end of the cord is pulled vertically downwards by a force which varies during the rotation of the cylinder. Let the twist θ be divided into a very large number n of equal elements, and let the force have the value f_1 over the first element of twist, its value over succeeding elements being equal to $f_2, f_3, \ldots . f_n$. Then the work done during the first element of twist is equal to $f_1 \times (r\theta/n)$, and the total work performed during the twist θ is equal to—

$$f_1 \frac{r\theta}{n} + f_2 \frac{r\theta}{n} + \ldots f_n \frac{r\theta}{n}$$

$$= \frac{f_1 r + f_2 r + \ldots + f_n r}{n} . \theta.$$

But $(f_1 r + f_2 r + \ldots . + f_n r)/n$ is the average value of the torque during the complete twist; hence **the work done is equal to the average value of the torque overcome, multiplied by the twist.**

Stable, unstable, and neutral equilibrium.—A body is in equilibrium when the forces acting on it do not tend to set it in motion. Now let a small force act on a body that, in the absence of this force, would be in equilibrium; the force will move the body, and as a consequence of this motion the other forces acting on the body will be slightly changed. If the change is such that the body tends to return to its original position of equilibrium when the extra force is removed, the equilibrium of the body is said to be **stable.** If the body does not tend to return to its original position of equilibrium, but to move away from that position, the equilibrium is said to be **unstable.** If the body does not tend to return to its original position of equilibrium, but also does not tend to move farther away from that position, the equilibrium is said to be **neutral.**

A cone standing on its base is in stable equilibrium. If the cone were placed with its centre of gravity exactly over its apex, and the latter were supported on a table, the cone would be in unstable equilibrium; for in the position described the downward force exerted by the earth on the centre of gravity would act in the same straight line, and have the same numerical value as the upward force exerted by the table on the apex; but the slightest displacement would shift the position of the centre of gravity, and then the force of gravity would produce an uncompensated torque about the apex, and the cone would rotate about that point until one side came into contact with the table. A cone lying on its side on a horizontal table is in neutral equilibrium.

The movement of a body from one position to another is generally accompanied by a change of potential energy. Let a cylinder of elliptical section rest with its curved surface on a horizontal table; its position of stable equilibrium is that in which the minor axis of the elliptic section is vertical. If we roll it through a very small angle from this position, its centre of gravity moves in an *almost* horizontal direction; hence, to a first degree of approximation, no work is done during this displacement, and the potential energy of the cylinder remains unchanged. But owing to the motion of the centre of gravity, a torque is produced which tends to bring the cylinder back to its original position. As we continue to rotate the cylinder, its centre of gravity commences to rise perceptibly, so that its potential energy increases; its potential energy will acquire its maximum value when the major axis of the elliptic section is vertical, for in that position the centre of gravity is at its maximum height above the table. A small angular displacement about this position will produce no appreciable change in the potential energy of the cylinder, since the centre of gravity moves in an *almost* horizontal direction; but a torque is called into play which tends to rotate the cylinder away from the position in which its major axis is vertical. Hence we see that, in the position where the major axis is vertical, the equilibrium is unstable, and the potential energy of the cylinder has a maximum value; when the minor axis is vertical, the equilibrium is stable, and the potential energy has a minimum value. If we plot the potential energy of the cylinder for various inclinations of the minor axis to the vertical, we obtain a curve of the general form shown in Fig. 10. The curve is horizontal for a small distance on either side of the points which correspond to equilibrium, whether stable or unstable; hence the principle of virtual displacements, which has been discussed previously. **Unstable equilibrium corresponds to maximum potential energy, stable equilibrium to minimum potential energy.**

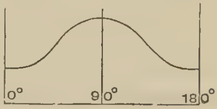

FIG. 10. — Curve showing the potential energy of an elliptical cylinder in various positions.

The law thus derived applies, not only to the particular case considered, but to the equilibrium of bodies in general. It must be remembered, however, that the words *maximum* and *minimum* must be understood in their mathematical sense. A point on a curve has a maximum position if the curve commences to slope downwards on either side of it, and it has a minimum position if the curve commences to slope upwards on either side of it; thus a maximum position need not necessarily be the highest possible position on a curve, nor need a minimum position be the lowest possible position. When we state that **matter tends to arrange itself so that its potential energy is a minimum**, in accordance with the above law, we do not mean that its potential energy will acquire its smallest possible value; water tends to flow downwards, but it may collect in a lake among the mountains at a great altitude above the sea-level.

QUESTIONS ON CHAPTER I.

1. A locomotive engine weighing 12 tons is moving with a uniform velocity of 20 miles per hour, when the steam is cut off, and a constant opposing force is applied which brings the engine to rest in 20 seconds. Determine the value of the opposing force, expressed in dynes, in poundals, and in pounds. ($g=32{\cdot}2$ ft./sec.^{2})

2. A locomotive engine weighing 12 tons is moving with a uniform velocity of 20 miles per hour, when the steam is cut off, and a constant opposing force is applied; the locomotive subsequently moves over a distance of 100 yards before it comes to rest. Determine the value of the opposing force, expressed in dynes, poundals, and pounds.

3. An engine, working at 600 horse-power, keeps a train moving on the level with a uniform velocity of 20 miles per hour. Determine the value of the frictional force which opposes the motion of the engine and train; express the result obtained in dynes, poundals, and pounds.

4. An engine, working at a constant horse-power, can draw a train of 200 tons mass up an incline of 1 in 200 at a speed of 30 miles per hour, or up an incline of 1 in 400 at a speed of 40 miles per hour. Determine the value of the frictional opposing force exerted.

5. A heavy body is supported by an elastic filament attached to the ceiling of a lift chamber. If the lift chamber were suddenly set free and allowed to fall freely under the action of gravity, how would the heavy body move relatively to the chamber?

6. A heavy body is supported by an inextensible filament attached to the ceiling of a railway carriage. Prove that the filament will be inclined to the vertical when the acceleration of the carriage is constant;

and determine the angle θ which the filament makes with the vertical, when the constant acceleration of the carriage is equal to α.

7. A body is hung from a spring balance, which is supported in a lift chamber. When the chamber is stationary, the reading of the balance is 10 lb. When the lift chamber is rising with a uniform acceleration, the reading of the balance is 11 lb. Calculate the value of the acceleration of the lift chamber.

8. A person drops a stone into a well, and hears it strike the water after a lapse of 3 seconds. Calculate the depth of the well, if $g = 981$ cm./sec.^{2}, and the velocity of sound is equal to 340 metres per sec.

9. A pail containing water is allowed to fall freely from rest, under the action of gravity. Prove that the pressure at any point in the water is equal to the atmospheric pressure exerted on the surface of the water.

10. The mean velocity of the earth in its orbit is equal to 18·47 miles per second. Assuming that the earth's orbit is circular, its radius being 92·8 million miles, calculate the value of the attraction exerted by the sun on each unit of mass of the earth. Could this attraction be observed by the aid of a spring balance if it were sufficiently sensitive?

11. A coil of perfectly flexible, heavy cord rests on a table, and one end of the cord hangs over the edge of the table and extends as far as the ground. Prove that, in the absence of friction, the cord which hangs over the edge of the table will continue to descend at a constant speed, and determine this speed.

12. A body is attached to the end of a wire, of which the other end is fixed; and it is found that when the body is twisted about the wire as axis, through an angle of one radian, the wire exerts an opposing torque equal to one pound-foot. Assuming that the opposing torque called into play is proportional to the twist, calculate the work done in twisting the body through 30°; express the result in foot-pounds, and in ergs.

13. The extremities of a long thin cord are attached to the ends of a heavy uniform rod, and the rod is supported by passing the cord over a small frictionless pulley of which the axis is horizontal. Determine the position of stable equilibrium of the rod.

14. In order to determine whether air has weight, Voltaire weighed a flexible bladder, first when it was inflated with air, and afterwards when it was deflated. He found both weighings to be equal, and concluded that air has no weight. Criticise this conclusion.

CHAPTER II

ROTATIONAL MOTION

Energy of a rotating body.—When a body is merely rotating, such of its particles as lie on a particular line will not be moving through space; this line is called the **axis of rotation.** If the body is rigid, the relative positions of its particles must remain unchanged; hence the distance of any particle from the axis of rotation must remain constant, and therefore all particles which do not lie on the axis of rotation must revolve about that line in circles. Further, the velocity of any particle will be proportional to its distance from the axis; for all particles must complete a revolution in the same time, and therefore if the body makes n complete rotations per second, a particle at a distance r from the axis must move through $2\pi r \times n$ units of length in a second, and this gives the linear velocity of the particle.

Now, the value of r will vary from particle to particle, but $2\pi n$ will have the same value for all particles. $2\pi n$ is called the **angular velocity** of the body; it measures the angle swept out in a second by any line which is perpendicular to the axis. If the body completes a rotation in T seconds, it is clear that $n = 1/\mathrm{T}$. The angular velocity of a body is generally denoted by ω; the linear velocity of a particle $= v = \omega r$.

Since the particles comprised in the body move with various velocities, they must possess various amounts of kinetic energy; the kinetic energy of the body as a whole is equal to the sum of the kinetic energies of its particles. Let the body consist of a number of particles of which the masses are equal to

$$m_1, m_2, m_3, \ldots \text{\&c.},$$

while the distances of these particles from the axis are

$$r_1, r_2, r_3, \ldots \text{\&c.}$$

Then the kinetic energy of the particle of mass m_1 will be equal to $\frac{1}{2}m_1v_1^2=\frac{1}{2}m_1r_1^2\omega^2$, and therefore the kinetic energy of the body as a whole will be equal to—

$$\tfrac{1}{2}\omega^2\{m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\} = \tfrac{1}{2}\omega^2\Sigma(mr^2).$$

Moment of inertia.—The quantity $\Sigma(mr^2)$, which will be denoted by I, is called the moment of inertia of the body: its value is found by dividing the body (in imagination) into minute particles, multiplying the mass of each particle by the square of its distance from the axis, and then adding together the results obtained for all the particles. A moment of inertia will be measured in gram-(cm.)^2 or lb.-(ft.)^2 according as the c.g.s. or the British system of units is used. The dimensions of a moment of inertia are ML^2. The kinetic energy of a body of moment of inertia I, rotating with an angular velocity ω, is equal to $\frac{1}{2}I\omega^2$.

We can calculate the values of the moments of inertia of bodies possessing certain simple geometrical shapes, and the moments of inertia of bodies of irregular shapes can be compared with these by an experimental method which will be described later. Hence it is a matter of some importance to be able to calculate the moments of inertia of bodies which may be conveniently used as standards for comparison. The mathematical determination of moments of inertia can be effected most expeditiously by the use of the Integral Calculus; the method which will be used here is fundamentally similar to that used in the calculus, but it can be understood and applied by students possessing only a sound knowledge of algebra.

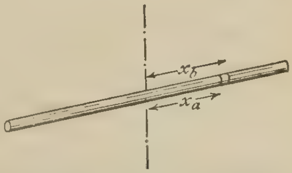

FIG. 11.—Method of calculating the moment of inertia of a thin rod.

Moment of inertia of a thin rod, about an axis through its middle point and perpendicular to its length. Let l be the length, and m the mass of the rod. It is clear that the moment of inertia of the whole rod about the axis specified must be twice as great as that of either half of the rod about the same axis; hence we may, in the first place, confine our attention to one half of the rod, which may be imagined to be divided into very short elements of length. Let the mass per unit length of the rod (that is, m/l) be denoted by m_1. Let the two ends of a particular element be at distances x_a and x_b from the axis, where $x_b > x_a$ (Fig. 11); then the length of the element is $(x_b - x_a)$, and its mass is $m_1(x_b - x_a)$. This mass must be multiplied by the average value of the square of the distance of the matter in the element from the axis.

Now, the greatest distance from the axis to any particle of matter in the element is equal to x_b, and the smallest distance is equal to x_a. Therefore the mean value of the square of the distance must be less than $x^2{}_b$. and greater than $x^2{}_a$. If the length $(x_b - x_a)$ of the element is infinitely small, *any* value which is intermediate between $x^2{}_b$ and $x^2{}_a$ will differ but infinitesimally from the mean value of the square of the distance. Therefore, in the first place, we may write down a number of expressions, each of which is intermediate in value between $x^2{}_b$ and $x^2{}_a$; any one of these expressions will give a sufficiently close approximation to the mean value of the square of the distance, when the element is infinitely small. The following expressions, among others, suggest themselves :—

$$(a)\quad \frac{x_b^2 + x_a^2}{2}; \qquad (b)\quad \left(\frac{x_b + x_a}{2}\right)^2; \qquad (c)\quad \frac{x_b^2 + x_b x_a + x_a^2}{3}.$$

When x_b and x_a differ appreciably in magnitude, the values given by (*a*), (*b*), and (*c*) will differ considerably; but each of these values is less than that of x_b^2, and greater than that of x_a^2, and therefore if the difference between x_b and x_a is infinitely small, that is, if the length of the element is infinitely small, we can use either (*a*), (*b*), or (*c*) to give the square of the length by which the mass of the element is to be multiplied. Now, it must be remembered that we must subsequently treat each element of the rod in a similar manner, and then add together the results so obtained; we therefore choose that value of the mean square of the distance which will lead to a simple expression when the results for all elements of the rod are added together, and trial will show that in the present case (*c*) fulfils this condition. Thus, that part of the moment of inertia due to the element lying between the distances x_b and x_a from the axis, is given by—

$$m_1(x_b - x_a)\left(\frac{x_b^2 + x_b x_a + x_a^2}{3}\right) = \frac{m_1}{3}(x_b^3 - x_a^3). \quad . \quad . \quad . \quad . \quad (1)$$

Now, let one half of the rod be divided into infinitely short elements of length, by imaginary cuts at distances $x_0, x_1, x_2, x_3, \ldots x_{n-2}, x_{n-1}, x_n$, from the axis; where finally x_0 will be equated to zero, and $x_1, x_2 \ldots$ are in increasing order of magnitude, x_n being equal to $l/2$. Then we may write down an expression similar to (1) for each of these elements, and add the results together. We thus obtain—

For element between x_0 and x_1, ... $\frac{m_1}{3}(x_1^3 - x_0^3)$,

" " " x_1 " x_2, ... $\frac{m_1}{3}(x_2^3 - x_1^3)$,

" " " x_2 " x_3, ... $\frac{m_1}{3}(x_3^3 - x_2^3)$,

. . . .

. . . .

" " " x_{n-2} " x_{n-1}, ... $\frac{m_1}{3}(x_{n-1}^3 - x_{n-2}^3)$,

" " " x_{n-1} " x_n, ... $\frac{m_1}{3}(x_n^3 - x_{n-1}^3)$.

Adding these together, we obtain that part of the moment of inertia due to one half of the rod—

$$= \frac{m_1}{3}(x_n^3 - x_0^3)$$

$$= \frac{m_1}{3}\left(\frac{l}{2}\right)^3 = \frac{m_1 l^3}{24},$$

since $x_n = l/2$, and $x_0 = 0$. ∴ Total moment of inertia of rod—

$$= \mathrm{I} = 2 \times \frac{m_1 l^3}{24}.$$

$$= (m_1 l)\frac{l^2}{12} = \frac{ml^2}{12}.$$

since $m_1 l$ is equal to the mass m of the whole rod.

The above result applies to a rod which is so thin that, if two imaginary cross cuts are made infinitely near to each other, then all points in the thin slice cut out must be at one and the same distance from the axis. If the sectional area of the rod is not exceedingly small, the edges of the slice will be at a greater distance from the axis than the middle point, and the above reasoning will not be strictly applicable. The moment of inertia of a rod of finite section will be dealt with later.

Moments of inertia of other bodies may be obtained in a similar manner; the general nature of the procedure has been explained in detail in the case just considered, so that no difficulty should be experienced in following the reasoning now to be used, although less detailed instruction is given.

Moment of inertia of a thin rectangular lamina, about an axis through its centre of gravity and parallel to its breadth.—Let m be the mass, and l the length of the lamina. Divide it into n equal narrow strips by imaginary cuts parallel to its length; then the mass of each strip is m/n, and the moment of inertia of a strip is equal to its mass multiplied by $(l^2/12)$. The moment of inertia of the lamina as a whole is equal to the sum of the moments of inertia of the n strips; hence

$$\mathrm{I} = n \cdot \frac{m}{n} \cdot \frac{l^2}{12} = m\frac{l^2}{12}.$$

Moment of inertia of a thin circular ring, about an axis through its centre and perpendicular to its plane.—Let m be the mass, and r the radius of the ring. Then since the ring is thin, the whole of its mass is at a uniform distance, r, from the axis, and therefore the moment of inertia is given by the equation—

$$\mathrm{I} = mr^2.$$

Moment of inertia of a circular disc about an axis through its centre and perpendicular to its plane.—Let m be the mass and r the radius of the disc. Let the mass per unit area of the disc be denoted by m_1, so that $\pi r^2 m_1 = m$.

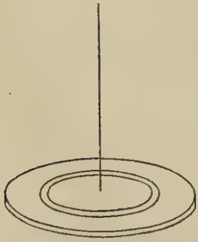

FIG. 12.—Method of calculating the moment of inertia of a disc.

Divide the disc into infinitely narrow circular strips (Fig. 12) by means of circles concentric with the disc and of radii $r_0, r_1, r_2 \ldots . r_{n-1}, r_n$. The mass of the circular strip lying between the circles of radii r_0 and r_1 is equal to $m_1(\pi r_1{}^2 - \pi r_0{}^2) = m_1\pi(r_1{}^2 - r_0{}^2)$, and the average value of the square of their distance from the centre may be taken as $(r_1{}^2 + r_0{}^2)/2$. Thus the part of the moment of inertia due to the strip is

$$m_1\pi\frac{(r_1{}^2 - r_0{}^2)(r_1{}^2 + r_0{}^2)}{2} = \frac{m_1\pi}{2}(r_1{}^4 - r_0{}^4).$$

Writing down similar expressions for the other elements, and adding these together, we find that the moment of inertia, I, of the whole disc is equal to

$$\frac{m_1\pi}{2}(r_n{}^4 - r_0{}^4) = \frac{m_1\pi r^4}{2},$$

since $r_n = r$, the radius of the disc, and $r_0 = 0$.

Hence $$I = \frac{(m_1 \,.\, \pi r^2)r^2}{2} = m\frac{r^2}{2}.$$

If we place a number of circular discs one on top of another, so as to form a circular cylinder, then the moment of inertia of the whole cylinder about its geometrical axis is equal to the sum of the moments of inertia of the discs about the same axis. Hence if m = mass of cylinder, and n = number of discs, the moment of inertia of the cylinder

$$= n \times \frac{m}{n} \,.\, \frac{r^2}{2} = \frac{mr^3}{2}.$$

The moment of inertia of a flat ring, that is, a circular disc from which the central portion has been removed by a concentric circular cut, can be found in a similar manner. Let r be the internal, and R the external radius of the ring, and let m be its mass, and m_1 its mass per unit area, so that $m = \pi(R^2 - r^2)m_1$.

By reasoning similar to that employed above—

$$\begin{aligned}\text{Moment of inertia of ring} &= \frac{m_1\pi}{2}(R^4 - r^4)\\ &= \frac{m_1\pi(R^2 - r^2)(R^2 + r^2)}{2}\\ &= m \,.\, \frac{R^2 + r^2}{2}.\end{aligned}$$

Moment of inertia of a thin circular ring about a diameter as axis.—Let ABCD, Fig. 13, represent the ring, of radius OA, and let AC be the diameter chosen as axis. Through the centre, O, draw DOB perpendicular to AC; then AC and DB divide the ring into four equal quadrants. Divide the quadrant, AB, into any very large even number of equal elements, and let E be one of these elements. Join E to O, and drop the perpendicular, EG, on to the axis, AC, and the perpendicular, FH, on to OB; then $(EG)^2$ is the quantity by which the mass of the element at E must be multiplied in order to obtain that part of the moment of inertia due to this element.

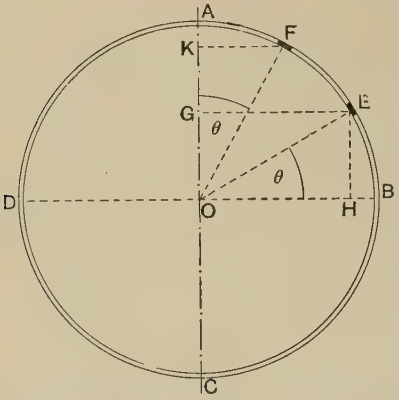

FIG. 13.—Method of calculating the moment of inertia of a circular ring about a diameter as axis.

From O draw OF, so that the angle FOA = angle EOB; then the point, F, will be the centre of an element. Drop the perpendicular, FK, on to the axis; then to obtain that part of the moment of inertia due to the element at F we must multiply the mass of that element by $(FK)^2$, and FK = EH. Hence, that part of the moment of inertia due to the elements at E and F = mass of either element × $\{(EG)^2+(EH)^2\}$ = mass of element × $(OE)^2$. In other words, the average value of the squares of the distances from E and F to the axis, is equal to half the square of the radius of the circle.

We may now group the elements comprised in the quadrant, AB, in pairs, one of each pair being as far from B as the other is from A, and the mean square distance of each pair from the axis is equal to half the radius squared. From this it follows that the average value of the squares of the distances of all elements of the ring from the axis AC is equal to half the square of the radius of the ring; hence, if m is the mass and r the radius of the ring, the moment of inertia, I, about the axis, AC, is given by

$$m\frac{r^2}{2},$$

that is, its value is equal to half the moment of inertia of the ring about an axis through the centre and perpendicular to its plane.

Let angle EOB $=\theta$; then EG = OH $= r\cos\theta$, and the average value of $(r\cos\theta)^2$ for values of θ between 0 and $\pi/2$ is equal to $r^2/2$. Thus, **the average value of $\cos^2\theta$, for values of θ between 0 and $\pi/2$, is equal to 1/2.** It can be proved, in a similar manner, that **the average value of $\sin^2\theta$, for values of θ between 0 and $\pi/2$ is equal to 1/2.** These very important results will be used frequently in subsequent investigations.

Moment of inertia of a circular disc about a diameter as axis.—Let the disc be divided into narrow circular strips by means of concentric circles; then the moment of inertia of each circular strip about its diameter is equal to half the moment of inertia of the same strip about an axis through the centre of the disc and perpendicular to its plane; hence the moment of inertia of all the circular strips (that is, of the whole disc) about a diameter is equal to $\frac{1}{2}\cdot m\frac{r^2}{2}$, where $m\frac{r^2}{2}$ is the moment of inertia of the disc about an axis through its centre and perpendicular to its plane. Hence

$$I = m\frac{r^2}{4}.$$

It is easily proved that the moment of inertia, I, of a flat ring, of which the mass is m and the internal and external radii are r and R respectively, about a diameter as axis, is given by the equation

$$I = m\frac{R^2 + r^2}{4}.$$

Moment of inertia of a thin spherical shell, about a diameter as axis.—Let ACBD (Fig. 14) represent the spherical shell, the centre of the sphere being O, and AB being the diameter chosen as axis. Draw any diameter, DC, perpendicular to AB, and draw a third diameter EF perpendicular to

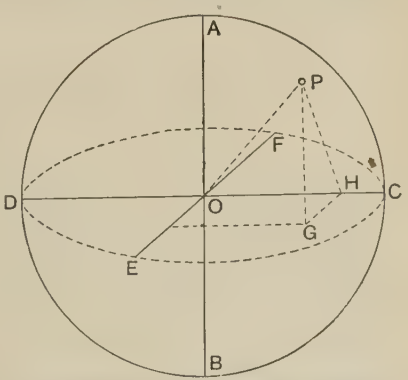

Fig. 14.—Method of calculating the moment of inertia of a thin spherical shell.

the plane containing AB and DC. Let P represent a point on the shell, and from this point drop the perpendicular PG on the plane containing the diameter DC and EF ; and from G draw GH perpendicular to DC. Let $OH=x$, while $HG=y$ and $GP=z$. Then the square of the distance from P to the diameter DC is equal to $(HG)^2+(GP)^2=y^2+z^2$, and similar reasoning shows that the square of the distance from P to the diameter EF is equal to x^2+z^2, and the square of the distance from P to the diameter AB is equal to x^2+y^2.

Now the shell is symmetrical about all three diameters ; hence the moment of inertia about one, has the same value as about any other diameter. If we suppose that a small element of surface about P has a mass m, then it contributes an amount equal to $m(x^2+y^2)$ towards the moment of inertia about AB, and the total moment of inertia about that axis will be found by dividing the whole of the surface into elements, multiplying the mass of each element by the sum of the squares of the corresponding values of x and y, and then adding together the results found for all the elements ; that is, the moment of inertia of the shell about AB is equal to $\Sigma m(x^2+y^2)$. Similarly, the moment of inertia about DC is equal to $\Sigma m(y^2+z^2)$, and that about EF is equal to $\Sigma m(x^2+z^2)$. Then if I is the moment of inertia about any diameter,

$$\begin{aligned} I &= \Sigma m(x^2+y^2)=\Sigma m(y^2+z^2)=\Sigma m(x^2+z^2) ; \\ \therefore\ 3I &= \Sigma m(x^2+y^2)+\Sigma m(y^2+z^2)+\Sigma m(x^2+z^2) \\ &= \Sigma\{m\ .\ 2(x^2+y^2+z^2)\} \\ &= 2\Sigma m\ .\ (OP)^2=2(OP)^2\ .\ \Sigma m. \end{aligned}$$

since $x^2+y^2+z^2=(PH)^2+(OH)^2=(OP)^2$, the triangle POH having a right angle at H. But OP is the radius of the sphere ; let this be denoted by r. Also let M be the mass of the shell. Then $\Sigma m=M$, and

$$3I=2r^2\ .\ \Sigma m=2Mr^2,$$

so that

$$I=\frac{2}{3}Mr^2.$$

Moment of inertia of a solid sphere about a diameter as axis.— Let r be the radius, and let m be the mass of the solid sphere, while m_1 is its mass per unit volume, so that $\frac{4}{3}\pi r^3 m_1=m$. Divide the sphere into thin spherical shells by concentric spherical surfaces. If r_a is the internal and r_b the external radius of one of these shells, its mass will be equal to $4\pi \mathrm{r}^2.(r_b - r_a)m_1$, where r^2 is less than r_b^2 and greater than r_a^2. The moment of inertia of this shell will be equal to

$$4\pi \mathrm{r}^2 m_1(r_b - r_a) \times \frac{2}{3}\mathrm{r}^2 = \frac{8}{3}\pi m_1 \mathrm{r}^4(r_b - r_a),$$

and we may write—

$$\mathrm{r}^4 = \frac{r_a^4 + r_a^3 r_b + r_a^2 r_b^2 + r_a r_b^3 + r_b^4}{5},$$

since this value lies between r_a^4 and r_b^4, and the thickness of each shell may be made as small as we please, and therefore r_a and r_b differ only infinitesimally in value. Substituting the value of r^4 in the value for the moment of inertia of the shell, we find that this is equal to

$$\frac{8}{15}m_1\pi(r_b^5 - r_a^5).$$

Let the radii of the concentric spherical surfaces, which divide the solid sphere into shells, be $r_0, r_1, r_2 \ldots . r_n$, where $r_0 = 0$ and $r_n = r =$ the radius of the sphere. Then, writing down values for the moments of inertia of the various shells, and adding these together, we obtain

$$\begin{aligned} \mathrm{I} &= \frac{8\pi m_1}{15}(r_n^5 - r_0^5) \\ &= \frac{8\pi m_1}{15}r^5 = \left(\frac{4}{3}\pi r^3 m_1\right) \times \frac{2}{5}r^2 \\ &= \frac{2m}{5}r^2. \end{aligned}$$

Moment of inertia of a solid, about an axis which does not pass through the centre of gravity of the solid. In the preceding investigations, moments of inertia of various bodies, about axes passing through their centres of gravity, have been determined; now we must see what modification in our procedure is required when the axis has any position whatsoever.

Let AB (Fig. 15) be a rod of which the centre of gravity is at G; let this rod rotate with uniform angular velocity ω about an axis, perpendicular to the plane of the paper, and passing through any point C. In a given interval of time let the rod change its position from AB to A′B′, the centre of gravity sweeping out the arc GG′.

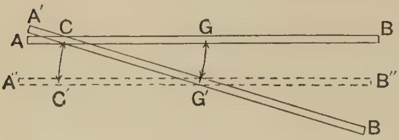

FIG. 15.—Rod rotating about an axis through the point C.

The same results would have been obtained if the rod had first moved to the position $A''B''$ whilst remaining parallel to its original direction, and had then rotated about G' till it acquired the position $A'B'$. Let $CG = h$, and let the angle $BCB' = \theta$. As the rod moves from the position AB to $A''B''$, every particle of it moves through a distance equal to that traversed by the centre of gravity, that is, a distance $GG' = h\theta$. As the rod rotates to the position $A'B'$ it turns through an angle $A''G'A = \theta$. To obtain the actual motion of the rod, we may suppose the two motions which have been just described to occur simultaneously. Thus the kinetic energy possessed by the rod, as it rotates with an angular velocity ω about the axis through C, can be resolved into two parts. The first part is due to the motion of every particle of the rod with a velocity equal to that of the centre of gravity G; and as G moves with a velocity $h\omega$, this part of the kinetic energy is equal to $\frac{1}{2}m(h\omega)^2$, where m is the mass of the rod. The second part of the kinetic energy is due to the rotation of the rod with an angular velocity ω about its centre of gravity G; if I_1 is the moment of inertia of the rod about an axis through G and parallel to the axis through C, then the second part of the kinetic energy is equal to $\frac{1}{2}I_1\omega^2$. Now let I be the moment of inertia of the rod about the axis through C; then the total kinetic energy of the rod is equal to $\frac{1}{2}I\omega^2$. Hence—

$$\tfrac{1}{2}I\omega^2 = \tfrac{1}{2}mh^2\omega^2 + \tfrac{1}{2}I_1\omega^2;$$

$$\therefore\ I = mh^2 + I_1.$$

Hence, if we know the moment of inertia I_1 of a body about an axis through its centre of gravity, we can obtain the moment of inertia I about any other parallel axis, by adding to I_1 the mass of the body multiplied by the square of the distance between the two axes.

Moment of inertia of a rectangular bar, about an axis through its centre of gravity and perpendicular to one of its faces. Let the mass of the bar be m, while its length and breadth, perpendicular to the axis AA′, are l and b respectively (Fig. 16). Divide the bar into an indefinitely large number n of equal slices by means of planes perpendicular to its length; then the mass of each slice will be m/n, and the moment of inertia of a slice about the axis AA′ will be equal to—

$$\frac{m}{n}x^2 + \frac{m}{n}\cdot\frac{b^2}{12},$$

where x is the distance from the centre of gravity of the slice to the axis AA′, and $\frac{m}{n}\cdot\frac{b^2}{12}$ is the moment of inertia of the slice, about an axis through its own centre of gravity and parallel to AA′. In adding together the moments of inertia of the various slices, the sum of the terms similar to $\frac{m}{n}x^2$ will be equal to the moment of inertia of a *thin* rod of mass m and length l, that is, to $m\frac{l^2}{12}$. The second term does not contain x, and the sum of the terms of this type will be equal to—

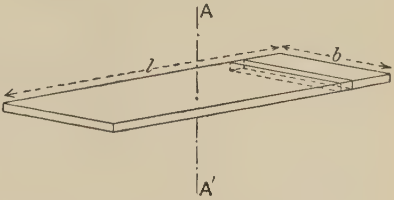

FIG. 16.—Method of calculating the moment of inertia of a bar.

$$n\cdot\frac{m}{n}\cdot\frac{b^2}{12} = m\frac{b^2}{12}.$$

Hence, the total moment of inertia I of the bar is given by the equation—

$$I = m\frac{l^2+b^2}{12}.$$

Notice that (l^2+b^2) is equal to the square of the diagonal length of the bar.

The student should now find no difficulty in proving that the moment of inertia I of a cylinder of circular section, about an axis through its middle point and perpendicular to its length, is given by—

$$I = m\left(\frac{l^2}{12}+\frac{r^2}{4}\right); \text{ (compare p. 53).}$$

where m is the mass, and l the length of the cyclinder, whilst r is the radius of its tranverse section.

Radius of gyration.—The moment of inertia I of a body about any specified axis may always be expressed in the form

$$I = mk^2,$$

where m is the mass of the body and k is proportional to its linear dimensions. For instance, in the case of a thin rod $k^2=(l^2/12)$, and in the case of a sphere $k^2=(2r^2/5)$. The quantity k is called the **radius of gyration** of the body about the given axis; if the total mass of the body were concentrated at a uniform distance k from the axis, then the moment of inertia of this arrangement of matter would have the same value as that of the body.

Problem. *A body rolls, without slipping, down a smooth plane inclined at an angle θ to the horizon: it is required to find the acceleration of the body, and the distance through which it moves in a given time.*

When the body is rolling along the plane with a velocity v (*i.e.* when its centre of gravity is moving parallel to the plane, with a velocity v), it must be rotating with an angular velocity equal to v/r, when r is the radius of the body; for in each complete rotation, every point of the circumference must come, in turn, in contact with the plane; that is, the body must advance through a distance $2\pi r$ units of length; and therefore if it is rotating at a rate of n turns per second, the centre of gravity must advance through $2\pi rn=v$ units of length per second, so that the angular velocity $\omega=2\pi n=v/r$.

After t seconds let the body be moving with a linear velocity v parallel to the plane; then, if its mass is m, its kinetic energy due to its linear velocity is equal to $\frac{1}{2}mv^2$, and that due to its angular velocity is equal to—

$$\tfrac{1}{2}mk^2\omega^2 = \tfrac{1}{2}mk^2\left(\frac{v}{r}\right)^2 = \tfrac{1}{2}mv^2 \,.\, \left(\frac{k}{r}\right)^2,$$

where k is the radius of gyration of the body about its axis of rotation. Hence its total kinetic energy—

$$= \tfrac{1}{2}mv^2\left\{1 + \left(\frac{k}{r}\right)^2\right\}.$$

If the body started from rest, and has travelled over a distance s parallel to the plane, the work done by gravity is equal to the component of the earth's attraction resolved parallel to the plane (that is, $mg \sin \theta$), multiplied by s; hence—

$$\tfrac{1}{2}mv^2\left\{1 + \left(\frac{k}{r}\right)^2\right\} = mgs \sin \theta.$$

At t_1 seconds from the start, let the linear velocity of the body be v_1, and let the distance traversed be s_1; and at t_2 seconds from the start, let the linear velocity of the body be v_2 and let the distance traversed be s_2. Then—

$$\tfrac{1}{2}v_1^2\left\{1+\left(\frac{k}{r}\right)^2\right\}=s_1\,.\,g\sin\theta\quad.\quad.\quad.\quad.\quad.\quad(1)$$

$$\tfrac{1}{2}v_2^2\left\{1+\left(\frac{k}{r}\right)^2\right\}=s_2\,.\,g\sin\theta\quad.\quad.\quad.\quad.\quad.\quad(2)$$

Subtract (1) from (2); then—

$$\tfrac{1}{2}\left(v_2^2-v_1^2\right)\left\{1+\left(\frac{k}{r}\right)^2\right\}=\left(s_2-s_1\right)g\sin\theta\quad.\quad.\quad(3)$$

If t_2 and t_1 are nearly equal, the value of (s_2-s_1) is equal to the distance traversed in the time (t_2-t_1) with the average velocity $(v_2+v_1)/2$. Therefore (3) may be re-written in the form—

$$\left(v_2-v_1\right)\left(\frac{v_2+v_1}{2}\right)\left\{1+\left(\frac{k}{r}\right)^2\right\}=\left(\frac{v_2+v_1}{2}\right)\left(t_2-t_1\right)g\sin\theta,$$

$$\therefore\quad\frac{v_2-v_1}{t_2-t_1}=g\sin\theta\,.\Big/\left\{1+\left(\frac{k}{r}\right)^2\right\},$$

and this gives the linear acceleration of the body, *i.e.*, the increase of velocity per unit time. Thus the acceleration is constant.

If the body were to slide down the plane without rotation, its acceleration would be $g\sin\theta$, so that the acceleration is diminished by the rolling of the body. For a solid cylinder $(k/r)^2=1/2$, (p. 51); in this case the acceleration would be only two-thirds that of a body sliding down the same plane.

Finally
$$s=\tfrac{1}{2}\,\frac{g\sin\theta}{1+\left(\frac{k}{r}\right)^2}\,t^2.$$

Problem. *A nut descends without friction along a vertical screw with n threads per unit length; find its acceleration.*

For every unit of length through which the nut descends it makes n complete rotations; hence, when it is moving vertically downwards with a linear velocity v, its angular velocity is equal to $2\pi nv$, and its total kinetic energy is equal to

$$\tfrac{1}{2}mv^2+\tfrac{1}{2}mk^2.(2\pi nv)^2=\tfrac{1}{2}mv^2\{1+(2\pi nk)^2\},$$

where k is the radius of gyration of the nut about its axis of rotation.

Hence, by reasoning similar to that used in the preceding problem, the acceleration is equal to—

$$\frac{g}{1 + (2\pi nk)^2}.$$

Moment of momentum.—When the angular velocity of a body, about an axis through its centre of gravity, is changing, the linear velocity of any particle which does not lie on the axis must be changing, and therefore all such particles must be acted upon by forces. If a particle of mass m_1 is at a distance r_1 from the axis of rotation, and the angular velocity of the body increases by ω in a time t, then the linear velocity of the particle increases by ωr_1 in that time, and f_1, the force acting on the particle, is given by the equation—

$$\omega r_1 = \frac{f_1 t}{m_1}, \text{ (see p. 18)},$$

and therefore $f_1 = (mr_1\omega)/t$.

This force is, of course, additional to the centripetal force (p. 27) necessary to keep the particle in a circular orbit.

Now, the direction of motion of the particle changes from instant to instant, and therefore the direction of the force which is increasing its velocity must be continually changing; further at any instant the forces acting on the various particles of the body will have different directions, and the resultant of all these forces will be equal to zero, if the centre of gravity of the body remains stationary. This is due to the fact that f_1 is proportional to r_1, and therefore changing the sign of r_1 changes the sign of f_1; thus oppositely directed forces must act on particles equidistant from, and on opposite side of the centre of gravity. From the investigation on p. 39, it follows that the tendency of a force to produce rotation about a given axis is measured by the moment of that force about the axis of rotation; and it is now obvious that if we multiply f_1 by r_1, and so obtain the moment of this force about the axis of rotation, we obtain—

$$f_1 r_1 = m_1 r_1^2 \cdot \frac{\omega}{t};$$

and since r_1 occurs to the second power on the right hand side of this equation, the value obtained is unaffected if we change $(+r_1)$ into $(-r_1)$. As a general rule, the applied forces do not act directly on the individual particles of the body, but each particle is acted upon by forces called into play by strains produced by the applied forces. If the strains produce forces $f_1, f_2, f_3 \ldots$ acting on particles of masses $m_1, m_2, m_3, \ldots$ at distances $r_1, r_2, r_3, \ldots$ from the axis of rotation, then the turning moment or torque which must be applied to the body is given by—

$$f_1r_1 + f_2r_2 + f_3r_3 + \ldots = \frac{\omega}{t}\{m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\}$$

$$= \frac{\omega}{t} . \mathrm{I},$$

where I is the moment of inertia of the body about the axis of rotation. Thus, **to find the torque which must be applied to a body in order to produce a given angular acceleration (ω/t), we must multiply (ω/t) by the moment of inertia of the body about the axis of rotation.**

Now $m_1r_1\omega$ is the momentum (p. 19) of a particle, and $m_1r_1\omega \times r_1 = m_1r_1^2\omega$ may be called the **moment of momentum** of the particle about the axis of rotation, so that $\omega\Sigma mr^2 = \omega\mathrm{I}$ may be called the moment of momentum of the body about the axis of rotation. Thus **the torque acting on the body is equal to the rate of change of the moment of momentum of the body.**

Moment of momentum is a vector quantity, since it cannot be defined completely without specifying the axis, and the direction of rotation about that axis. If we look at a rotating body along the axis of rotation, the moment of momentum of the body can be represented by a vector of which the length is equal to $\mathrm{I}\omega$, drawn parallel to the axis, and extending away from us if the rotation is clockwise, or toward us if the rotation is anti-clockwise. The torque acting on a body determines the rate at which its moment of momentum changes, just as the force acting on a body determines the rate at which its momentum changes (p. 19). Further, **a torque is needed to change either the direction or the magnitude of the moment of momentum of a body,** just as a force is needed to change either the magnitude or the direction of the momentum of a body.

Problem. *Determine the linear acceleration of a body which rolls down an inclined plane.*

It is supposed that that the body does not slip on the plane; therefore the point of the body which is in contact with the plane is stationary for an instant, and the body is turning about that point. The body must be either cylindrical or spherical in shape; the radius of the cylinder or sphere may be denoted by r. The centre of gravity of the body is at a distance r from the point of contact with the plane, and if the centre of gravity is moving with a velocity v parallel to the plane, the radius joining the centre of gravity to the point of contact must be rotating with an angular velocity ω, given by the equation—

$$r\omega = v; \quad \therefore \ \omega = v/r.$$

During a short interval of time t, let the linear velocity of the body change from v_1 to v_2, and let the angular velocity of rotation change from ω_1 to ω_2. Then—

$$\omega_1 = v_1/r,$$

$$\omega_2 = v_2/r,$$

$$\text{and} \quad \omega_2 - \omega_1 = (v_2 - v_1)/r.$$

$$\therefore \quad \frac{\omega_2 - \omega_1}{t} = \frac{v_2 - v_1}{t} \cdot \frac{1}{r} = \frac{a}{r},$$

where a denotes the linear acceleration of the body. The expression $(\omega_2 - \omega_1)/t$ denotes the angular acceleration of the body, and this is equal to a/r; the torque necessary to produce this angular acceleration is equal to $mk^2 \, a/r$, where mk^2 denotes the moment of inertia of the body about the axis of rotation.

The body is prevented from slipping on the plane by a force F, called into play at the point of contact of the body with the plane, the direction of the force being parallel to the plane and opposite to the direction of motion of the centre of gravity of the body (Fig. 17). Let it be imagined that two oppositely directed forces, each numerically equal and parallel to F, are applied to the centre of gravity of the body; these forces are in equilibrium, and therefore produce no change in the motion of the body.

The force F at the point of contact, together with the force $(-F)$ applied to the centre of gravity of the body, are equivalent to a torque equal to Fr; this is the torque which produces the angular acceleration of the body, and thus—

$$Fr = mk^2 \cdot \frac{a}{r},$$

$$\therefore \quad F = ma\frac{k^2}{r^2}.$$

The component of the force exerted on the body by gravity, resolved parallel to the plane, is equal to $mg \sin \theta$ (Fig. 17), and the resultant of this force and the oppositely directed force F applied to the centre of gravity, is equal to $(mg \sin \theta - \mathrm{F})$. This resultant force is equal to the product of the mass m and the linear acceleration α of the body; therefore—

$$m\alpha = mg \sin \theta - \mathrm{F} = m\left\{g \sin \theta - \alpha\left(\frac{k}{r}\right)^2\right\},$$

$$\therefore \quad \alpha\left\{1 + \left(\frac{k}{r}\right)^2\right\} = g \sin \theta,$$

$$\text{and} \quad \alpha = g \sin \theta \Big/ \left\{1 + \left(\frac{k}{r}\right)^2\right\}.$$

This result was obtained, by a different method, on p. 59.

The gyrostat.—The gyrostat is essentially a fly-wheel of large moment of inertia, mounted so that it can rotate freely about its axle, while both wheel and axle can also turn about any axis perpendicular to the axle. If the direction of the axle remains fixed, then the moment of momentum of the fly-wheel can only change in magnitude, and such change can only be produced by a torque acting about the axle as axis; this torque increases or decreases the kinetic energy of the fly-wheel, and therefore its application involves the performance of work: if it ceased to act the angular velocity of the fly-wheel would remain constant, in the absence of friction. Let us suppose that the fly-wheel is rotating with constant angular velocity about its axle; if the direction of the axle is altered, the direction of the moment of momentum must also be altered, and this involves the action of a torque. The magnitude and direction of the

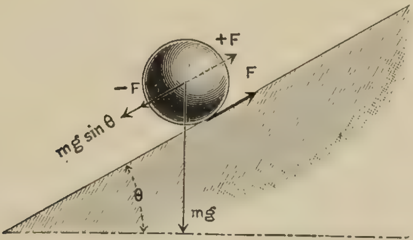

FIG. 17.—Body rolling down an inclined plane.

torque which must be applied, in order to change the direction in which the axle points, may be determined directly from the conditions explained above; but it is interesting to solve the problem, in the first place, from first principles,[1] as will now be done.

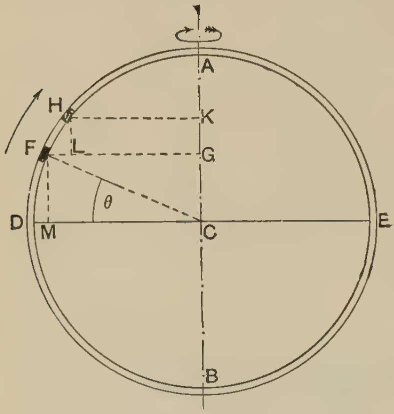

FIG. 18.—Action of the gyrostat.

Let Fig. 18 represent a fly-wheel rotating, with constant angular velocity ω, in a clockwise direction about an axis through C and perpendicular to the plane of the paper. Let the mass of the fly-wheel be concentrated in the rim at a uniform distance r from the axis of rotation. It is required to find the condition that the fly-wheel shall also turn with a uniform angular velocity ω_1 about the diameter AB, the direction of ω_1 being clockwise when we look along AB in the direction from A to B.

Let us suppose that the rim is divided into a great number of equal portions, each of mass m_1. Consider the forces which must act on one of these particles, say that at F.

Centripetal force.—Each particle must be acted upon by a force directed toward the centre C, in order that it may move in a circular path; the magnitude of this force has already been determined (p. 27), and it need only be mentioned that this force is called into play by strains in the material of the rim, and the spokes by which it is connected to the axle.

Torque due to change in the magnitude of the velocities of particles.—At the instant when the particle is at F, it is revolving about the diameter AB with a linear velocity equal to $\omega_1 \times$ FG.

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[1] The ensuing investigation is a mathematical development of a train of reasoning used by Dr. S. Tolver Preston in an article on *The Mechanics of the Gyroscope*, Technics, Vol. ii. p. 47.

where FG is the perpendicular let fall from F on the diameter AB. After a short time , let the particle arrive at the position H ; it is now revolving about AB with a linear velocity $\omega_1 \times \mathrm{HK}$, where HK is the perpendicular let fall from H on to AB. Hence its velocity, from front to back through the plane of the paper, is diminishing ; and a force must be acting on it from back to front through the plane of the paper, the magnitude of this force being equal to the rate at which the momentum of the particle is changing. The change in the velocity of the particle is equal to

$$\omega_1(\mathrm{FG}-\mathrm{HK})=\omega_1\mathrm{FL},$$

where L is the foot of the perpendicular let fall from H on to FG. Further, if t is sufficiently small, the arc FH will be small, and it may be treated as a straight line of length $r\omega t$, where ω is the angular velocity of the fly-wheel about its axle ; and since FH is perpendicular to the radius FC, the angle FHL = angle FCD = θ (say), so that FL = FH sin FHL = $\omega r t \sin\theta$. Therefore the *rate* at which the momentum of the particle is decreasing is equal to—

$$m_1 \cdot \frac{\omega_1\mathrm{FL}}{t} = \frac{m_1\omega_1\omega r t\sin\theta}{t} = m_1\omega_1\omega r\sin\theta \quad . \quad . \quad . \quad (1)$$

This gives the force, directed from back to front through the plane of the paper, that must act on the particle as it moves through the position F. Notice that $r\sin\theta$ = FM, the perpendicular distance of the particle from the diameter DE drawn at right-angles to AB. It follows that no force acts on the particle passing through D, but all particles between D and A are acted on by forces directed from back to front through the plane of the paper, the magnitudes of these forces being proportional to the respective distances of the particles from the diameter DE ; thus, the maximum force acts on the particle passing through A.

Similar reasoning applied to any particle in the quadrant AE shows that the velocity of each of these particles, directed from back to front through the plane of the paper, is increasing : therefore forces, directed from back to front through the plane of the paper, must act on these particles, the force acting on any particle being equal to $m\omega_1\omega$ multiplied by the perpendicular distance of the particle from the diameter DE.

The velocity of a particle in the quadrant EB is directed from back to front through the plane of the paper, and this velocity is diminishing, so that a force acting from front to back through the plane of the paper must act on it. The velocity of a particle in the quadrant BD is directed from front to back through the plane of the paper, and this velocity is increasing, so that a force directed from front to back through the plane of the paper must act on it. In each case the magnitude of the force is equal to $m_1\omega_1\omega$ multiplied by the perpendicular distance of the particle from the diameter DE. In other words, the expression (1) above gives the force that acts on any particle at the end of a radius which makes an angle θ with the radius CD, the angle θ varying from 0 to 2π for all particles of the wheel.

We thus see that all particles in the two quadrants DA and AE are acted on by forces directed from back to front through the plane of the paper, and all particles in the two quadrants EB and BD are acted on by forces directed from front to back through the plane of the paper. A particle in DA or AE at a distance d above DE may be paired with a particle in DB or BE at an equal distance below DE, and equal but oppositely directed forces must act on these particles; these forces, however, produce equal torques of the same sign, about the diameter DE; **hence the condition that the rotating wheel shall turn with a uniform angular velocity ω_1 about the diameter AB is, that a torque must be applied which would turn the wheel, if it were stationary, about the diameter DE in a clockwise direction looking from D to E.** The magnitude of the torque is easily found; for if a particle is at a distance d_1 from DE, the force acting on it is equal to $m_1\omega_1\omega d_1$, and the moment of this force about DE is found by multiplying it by d_1, thus giving $m_1\omega_1\omega d_1^2$. Hence the applied torque, which must be equal to the sum of the torques due to the forces acting on the individual particles, is equal to

$$\omega_1\omega\Sigma m_1 d_1^2 = \omega_1\omega I_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where I_1 is the moment of inertia of the wheel about the diameter DE as axis.

It must be noticed that this torque does no work; the work done by a torque acting about a given axis is equal to the torque multiplied by the twist produced *about the same axis;* and in the case considered the torque, and the twist due to it, are about mutually rectangular axes. Similarly, when a central force constrains a particle to move in a circular orbit, no work is done by the force, since the force and the motion are at right-angles to each other. It will also be noticed that the gain of energy due to the increasing velocities of some particles of the fly-wheel is just counterbalanced by the loss of energy due to the decreasing velocities of other particles.

Torque due to change in the plane of motion.—It is obvious that, as the wheel turns about the diameter AB, the plane in which a particle is moving changes continually. Let Fig. 19 represent the fly-wheel, seen in perspective from a point towards the left-hand side; then at a given instant the particle at F is moving in the direction FM in the plane containing AFD; shortly afterwards the particle will be at F′, moving in the direction F′M in the plane containing AF′D. In this time the velocity of the particle changes from ωr along FM, to ωr along F′M; and if we draw FM and F′M as in Fig. 20, and from F draw a line FP parallel to F′M and equal in length to $\mathrm{FN} = \omega r$, then the change of velocity is equal to NP, and the force acting on the particle is equal to $m_1 \frac{NP}{t}$, where t is the time in which the wheel turns from the position AFB to AF′B (Fig. 19). To find the value of NP/t (Fig. 20), notice that $\mathrm{NP} = \omega r \times$ angle $\mathrm{PFN} = \omega r \times$ angle FMF′.

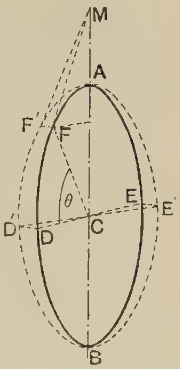

FIG. 19.—Action of the gyrostat.

Now from Fig. 19, angle FMF′—

$$= \frac{\mathrm{FF'}}{\mathrm{MF}} = \frac{r \cos \mathrm{FCD} \,.\, \omega_1 t}{r \tan \mathrm{MCF}} = \frac{\omega_1 \cos \theta}{\cot \theta} \,.\, t,$$

$$= \omega_1 t \sin \theta,$$

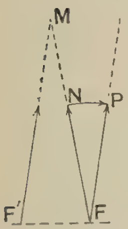

FIG. 20.—Action of the gyrostat.

and therefore the force necessary to change the plane of motion of the particle

$$= m_1 \frac{\mathrm{NP}}{t} = m_1 \omega r \,.\, \omega_1 \sin \theta = m_1 \omega_1 \omega r \sin \theta.$$

This force is just equal to that given by (1) which suffices to change the magnitude of the velocity of the particle about the axis AB. Hence the total force acting on each particle is twice that given by (1), and the total torque, which must act on the rotating fly-wheel in order to constrain it to turn uniformly about the diameter AB, is twice that given by (2) above, that is the total torque—

$$= \omega_1\omega \,.\, \times 2I_1 = \omega_1\omega I,$$

where I is the moment of inertia of the fly-wheel about an axis through C, and perpendicular to the plane of the paper; I is equal to twice the moment of inertia I_1 about the diameter DE (p. 52).

Direct determination.—The torque required to constrain a fly-wheel, which rotates about its axle with a constant angular velocity ω, to turn with uniform angular velocity ω_1 about a diameter, can be determined directly in terms of the rate of change of moment of momentum. Let DAE (Fig. 21) represent the fly-wheel as viewed from a point vertically above A (compare Fig. 18), the direction of rotation being clockwise when viewed from the side O of the fly-wheel. Then the moment of momentum can be represented by a vector drawn along the axis of rotation and equal to AV=Iω, where I is the moment of inertia about the axle (p. 61). If the wheel turns about a vertical diameter to the position D′AE′, the angular velocity ω remaining numerically constant, the moment of momentum changes to AV′ which is numerically equal to AV. Hence the change in the moment of momentum is represented by VV′, and if the angle VAV′ is small, VV′ will coincide with a circular arc with A as centre and AV as radius. If the wheel is turning with constant angular velocity ω_1 about the diameter through A perpendicular to the plane of the paper, then the angle VAV′=$\omega_1 t$, where t is the small interval of time required for the wheel to turn from the position DAE to D′AE′. Then VV′=AV×$\omega_1 t$=I$\omega\omega_1 t$, and this gives the change of momentum in the time t.

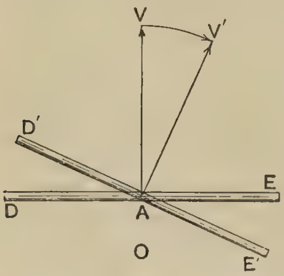

FIG. 21.—Action of the gyrostat.

To find the rate of change of momentum, which measures the torque required to constrain the fly-wheel to turn about the vertical diameter, we divide the change of momentum by t, and thus obtain $I\omega\omega_1$, which is the result previously obtained. Further, VV'/t represents a torque which, if acting on a stationary body, would tend to turn it in a clockwise direction, looking along an axis parallel to VV′ in the direction from V to V′ (p. 40), which is also in accordance with the results previously obtained.

There is, however, a case in which the direct method explained above cannot be used to determine the torque necessary to constrain a rotating body to turn about an axis perpendicular to the axis of rotation. Let FCF′ (Fig. 22) represent two heavy spheres F and F′, each of mass m_1, fastened to the ends of a massless rod which is rotating in a clockwise direction, with uniform angular velocity ω, about an axis through C and perpendicular to the plane of the paper. Let this arrangement turn with uniform angular velocity ω_1 about the line AB, in a clockwise sense when viewed from A to B. Then the reasoning explained on pp. 65 to 67 shows that the sphere F must be acted upon by a force $2m_1\omega_1\omega FM$, directed from back to front through the plane of the paper; while an equal but oppositely directed force must act on F′. The momentary torque is therefore equal to $4m_1\omega_1\omega(FM)^2$; this value is not constant, but varies from the value zero when the arrangement passes through the position DE, to $4m_1\omega_1\omega.(CA)^2$ when the arrangement passes through the position BA. If we had used the direct method explained above, we should have obtained $2m_1(CA)^2\omega_1\omega$ for the torque, which gives only the average value of the variable torque really required.

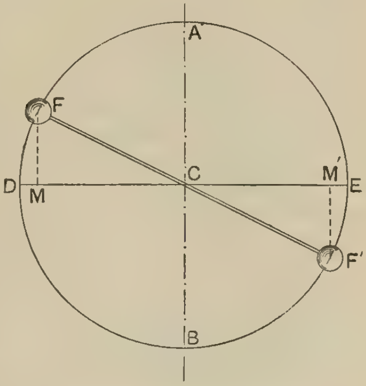

FIG. 22.—An anomalous type of gyrostat.

Lanchester's rule.—Mr. F. W. Lanchester has given a very useful rule for finding the direction of the torque that is required to constrain a rotating fly-wheel to turn about a specified diameter. **Let the fly-wheel be viewed from a point in its plane, the line of sight being perpendicular to the diameter about which the wheel is required to turn. Any point on the rim of the rotating wheel will appear to move in a straight line. Now let the wheel be turned through a small angle in the required direction, about the specified diameter; a point on the rim of the wheel will appear to move in an ellipse. The direction in which the ellipse is traversed gives the direction in which the required torque would turn a stationary body. The line of sight gives the axis of the torque.**

Let this rule be applied to Fig. 18, p. 64. Let it be supposed that the wheel is viewed in the direction from D to E, perpendicular to the diameter AB about which the wheel is required to turn. When the wheel has turned through a small angle about AB (in a clockwise direction viewed from A to B), a point on the rim will appear to traverse an elliptic path in a clockwise sense. Therefore the required torque must act in a clock-wise sense about the diameter DE, viewed in the direction from D to E.

Experimental study of gyrostatic action.—Let DAEB (Fig. 23) represent a fly-wheel rotating, without friction, with angular velocity ω about the axle CC′, the direction of rotation being clockwise when viewed in the direction from C to C′. In order that the fly-wheel may turn with uniform angular velocity ω_1 about the vertical axis AB, in a clockwise direction when viewed in the direction from A to B, a force f' must act vertically upwards on the end C′ of the axle, and an equal but oppositely directed force f must act on the end C of the axle, the torque τ due to these two forces being equal to $I\omega\omega_1$, where I is the moment of inertia of the fly-wheel about the axis CC′. The agent producing the torque τ will be acted upon by a numerically equal torque in an opposite direction; that is, the end C′ of the axle will exert on the agent a force numerically equal to f', in a direction vertically downwards;

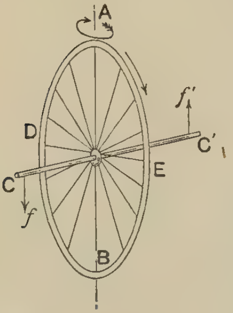

Fig. 23.—Action of the gyrostat.

and the end C will exert on the agent a force numerically equal to f, in a direction vertically upwards.

Conversely, if a torque τ is exerted on the axle of the fly-wheel the latter will turn about an axis perpendicular to the axis of the torque, with an angular velocity ω_1 given by the equation :

$$\omega_1 = \frac{\tau}{I\omega}.$$

This result is expressed by saying that the fly-wheel *precesses* with an angular velocity ω_1 about a vertical axis, when the torque τ is applied about a horizontal axis perpendicular to the axis of rotation. If no torque is applied to the fly-wheel, then no precession will be produced ; that is, the axis of rotation will maintain a fixed direction in space. Hence, if a fly-wheel is mounted so that when it is not rotating it is in equilibrium in any position (Fig. 24), then the direction of the axle will remain fixed in space when the fly-wheel is spinning ; in particular, as the earth rotates it will not impart its rotation to the fly-wheel. This gives one method by which the rotation of the earth can be observed experimentally. The approximate constancy in the direction of the earth's axis is explained in a similar manner ; its small precessional motion will be dealt with presently.

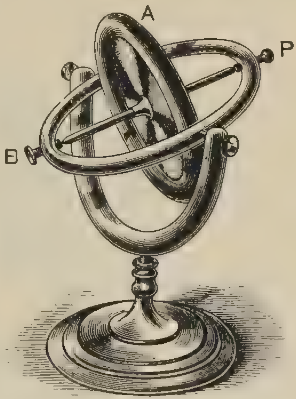

FIG. 24.—Gyrostat mounted so that it is in equilibrium in any position.

Fig. 25 represents a form of gyrostat in which the fly-wheel is mounted so that it can rotate freely within a metal ring ; the ring is supported only at one point P near an end of the axle, where it is pivoted on a vertical rod, being free to move in any direction about the pivot. If the fly-wheel were not rotating, it could not remain in the position shown in Fig. 25 unless further support were afforded to the ring. The action of the gyrostat when the fly-wheel is rotating can be easily deduced from the results of the preceding investigation.

Let us suppose that the ring, within which the fly-wheel is pivoted, is clamped while the wheel is set spinning with an angular velocity ω. On releasing the ring, gravity tends to rotate the whole arrangement downwards about the supporting pivot P. But directly the rotation commences, the direction of the axle commences to change; on applying the rule explained on p. 70, it is easily seen that if the wheel rotates in the direction of the arrow A, then the reaction due to the descent of the gyrostat will produce a torque which will tend to turn the ring about a vertical axis in the direction of the arrow B. If this reaction were counterbalanced by an equal but opposite torque, the wheel would continue to descend freely; this would be the case if the ring were hinged at P so that it could only swing up or down, without being free to turn about a vertical axis. But in the case represented in Fig. 25, no external force, except that of gravity, acts on the wheel and its supporting ring; thus, the reaction sets the supporting circle rotating about a vertical axis through P, in the direction of the arrow B. This rotation produces another reaction which opposes the torque due to gravity, and therefore the wheel descends more slowly; but so long as it continues to descend, the reaction due to its descent causes it to rotate more quickly in the direction of the arrow B, and this rotation, in its turn, produces a greater torque opposing that due to gravity. After a very short time the fly-wheel ceases to descend, its angular velocity ω_1 about the pivot P being sufficient to generate a torque equal and opposite to that exerted by gravity; henceforth, in the absence of friction, the axle of the gyrostat would remain inclined at a constant angle to the

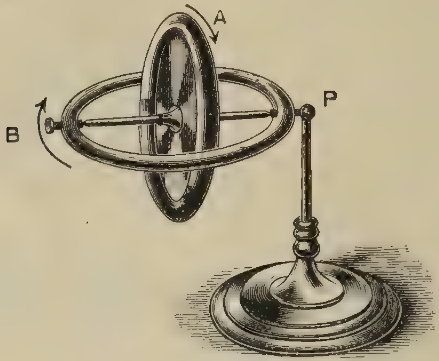

FIG. 25—Action of a gyrostat.

horizontal, but it would precess at a constant angular velocity ω_1 given by the equation

$$\omega_1 = \frac{\tau}{I\omega},$$

where τ is the torque about the point P due to gravity, ω is the angular velocity, and I is the moment of inertia of the fly-wheel about its axle. The fly-wheel has descended toward the earth only far enough for the work done by gravity to be sufficient to produce the extra kinetic energy due to the rotation of the wheel and its supporting ring about the vertical axis through P.

In practice, friction gradually diminishes the value of ω, with the result that the rate of precession ω_1 gradually becomes greater, while the fly-wheel gradually descends towards the earth.

The following instances of gyrostatic action may now be explained by the student.

When the axis of a spinning-top is not vertical, its inclination to the vertical remains constant (in the absence of friction), but the top precesses about a vertical axis in the same sense as that in which it spins about its own axis. In practice, the rate of precession increases, as the speed of the top diminishes owing to friction.

The axis of the earth is inclined to the ecliptic (that is, the plane of the earth's orbital motion about the sun). The earth is not exactly a sphere, its polar diameter being smaller than its equatorial diameter; the attraction exerted by the sun on the earth tends to pull the equatorial protuberance of the earth into the plane of the ecliptic, and hence the axis of the earth slowly precesses, that is, it points in different directions as time goes on. The rate of precession is about fifty seconds of arc per annum.

When a metal disc (such as a coin) is rolled along a smooth horizontal surface, its path is straight so long as the plane of the disc remains vertical; but if the disc inclines towards one side, its path curves away towards that side. It is partly owing to this fact that an expert cyclist can ride a bicycle without holding the handle-bars, although the customary "rake" of the forks exercises an influence which assists the gyrostatic action. The possibility of riding a bicycle at all depends, of course, on the gyrostatic properties of the rotating wheels.

Motion in circular and spiral orbits.—A body of mass m can move in a circular orbit with a linear velocity v, if it is acted upon by a force, directed towards the centre of the circle, of magnitude mv^2/r, where r is the radius of the circular orbit

(p. 26). If the angular velocity (p. 46) of revolution is equal to ω, then $v = \omega r$, and the central force must have the value $mr\omega^2$.

Let us suppose that a body is attracted towards a point with a force proportional to the distance of the body from the point; then if f_1 is the force that would be exerted on the body if it were at unit distance from the point, the force acting on it when it is at a distance r from the point is equal to $f_1 r$; and in order that the body may move in a circle of radius r around the point, we must have—

$$mr\omega^2 = f_1 r,$$

$$\therefore \ \omega = \sqrt{\frac{f_1}{m}}.$$

If a revolution is completed in a time T, then $\omega = 2\pi/\mathrm{T}$ and—

$$\mathrm{T} = 2\pi\sqrt{\frac{m}{f_1}}.$$

In this case the time occupied by a complete revolution is independent of the radius of the orbit.

It should be noticed that the kinetic and potential energies of the body are equal. If the body were to move from the circumference to the centre of the circle, the force acting on it would diminish uniformly from $f_1 r$ to $f_1 \times 0 = 0$, and therefore the average force $= f_1 r/2$. Hence, potential energy lost as the body moves from a distance r up to the point $= (f_1 r/2) \times r = f_1 r^2/2$. Further $f_1 = m\omega^2$. Thus potential energy $= m\omega^2 r^2/2 = mv^2/2$.

If a body revolves around a point, subject only to the action of a force directed towards that point, it will continue for all time to revolve with a uniform angular velocity. A very close approximation to this condition is supplied by the moon, which revolves in a nearly circular orbit, under the action of the gravitational attraction exerted upon it by the earth. There is no loss of energy in this case, since the motion of the body is always perpendicular to the only force acting on it.

When a body moves through air or any other material fluid, its energy continually decreases. This is due to the fact that the body sets the fluid in its neighbourhood in motion, and the friction between different parts of the fluid causes the kinetic energy communicated to it to be dissipated in the form of heat. In effect, the motion of the body is opposed by a force which, when the motion is slow (not more than a few feet per second) is proportional to the velocity of the body; for high speeds the force is more nearly proportional to the square of the velocity.

Let us now study the motion of a body which revolves around a point O (Fig. 26), under the action of a force directed toward that point, and proportional to the distance of the body from it; together with a force directly opposing the motion of the body, and proportional to its linear velocity. To fix our ideas, let us suppose that the body is originally moving in the circular orbit JAH subject only to the action of the central force $f_1 r$ directed towards O; and that when the body arrives at A an opposing force equal to κv (where κ is a constant, and v is the linear velocity of the body) commences to act. The opposing force diminishes the velocity of the body, with the result that the central force necessary to constrain the body to move in the circle JAH is less than the actual force which pulls the body toward O; hence the body gradually approaches O, and its path takes the form of the spiral APBCDE. . . .

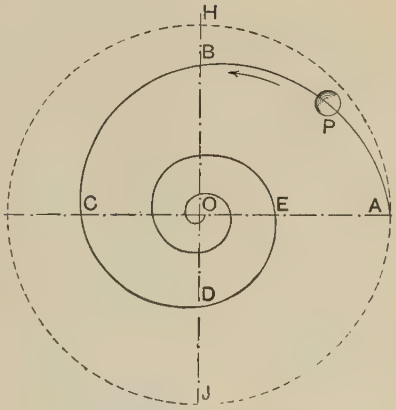

FIG. 26.—Motion in a spiral path.

In the first place it can be shown that, when the motion of the body has become steady, **its spiral path will cut all radii drawn from O at one uniform angle.** This proves that the path of the body is an equiangular spiral.

An **equiangular spiral** can be described by means of the apparatus represented diagrammatically in Fig. 27. The rod AB can slide freely through a cylindrical hole in the pillar O, and this pillar is pivoted so that it can rotate freely about an axis perpendicular to a sheet of paper strained on a drawing board. To the end B of the rod AB is attached a framework, on which is pivoted a small wheel. The axle of this wheel must remain parallel to the plane of the paper, but it can be set at any required inclination to the rod AB. The wheel is in contact with the paper, and supports the weight of the end B of the rod. The edge of the wheel is sharp, so that it cannot slip sideways over the paper. Thus, the end B of the rod can move only when the wheel rolls along the paper, and the direction in which the wheel can roll makes a constant angle with the rod AB. The curved path which the wheel traverses is of such a form, that an element of the curve is inclined at a constant angle to the line joining the element to O.

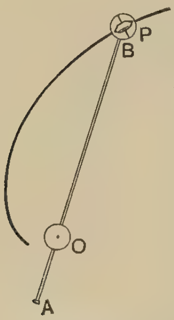

FIG. 27.—Apparatus for describing an equiangular spiral.

Let ACB (Fig. 28) be a portion of the curve ABCD (Fig. 26), described in a very small time t. As the body passes through A, its velocity is in the direction AD, inclined at an angle DAE $= \alpha$ to the line EA, which represents part of the radius vector drawn from the central point O to the body. The arc ACB (Fig. 28) is supposed to be magnified greatly, so that AE and BG, parts of the radii which join its ends to the central point, are sensibly parallel. Let the body, when at A, be at a distance r from the central point, the force acting along AE having the value $f_1 r$. This force has a component $f_1 r \cos \alpha$ acting along AD, and a component $f_1 r \sin \alpha$ perpendicular to AD. From B draw BD perpendicular to AD.

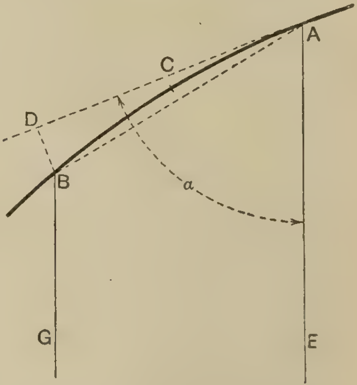

FIG. 28.—A small portion of the curve ABCD (Fig. 26) very much magnified.

In the time t the body actually moves from A to B, and its displacement is equivalent to the components AD and DB. When the body is at A, its linear velocity is parallel to AD; let the value of this velocity be v. Then the displacement AD, completed in the time t, is due to the velocity v, modified by the accelerating force $fr \cos \alpha$ and the opposing force κv. Thus—

$$AD = vt + \tfrac{1}{2}\frac{f_1 r \cos \alpha}{m} \,.\, t^2 - \tfrac{1}{2}\frac{\kappa v}{m} \,.\, t^2.$$

The second and third terms on the right of this equation, both of which are proportional to t^2, can be made as small as we please, in comparison with the first term which is proportional to t, by making the time t sufficiently small. Hence when t is infinitely small—

$$AD = vt.$$

Let the angular velocity of the body about O (Fig. 26) (that is, the rate at which the radius vector turns about that point) be equal to ω. To find the value of ω, notice that the velocity v of the body at A (Fig. 28) is inclined at an angle α to the radius AE, and therefore the linear velocity perpendicular to AE is equal to $v \sin \alpha$; that is, the end A of the radius vector of length r is moving perpendicular to AE with a velocity $v \sin \alpha$, and therefore—

$$\omega = \frac{v \sin \alpha}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \; v = \frac{\omega r}{\sin \alpha},$$

$$\therefore \; AD = \frac{\omega r}{\sin \alpha} t.$$

At the end of the time t the body is actually at B; hence the force $f_1 r \sin \alpha$, acting perpendicular to AD, has produced the displacement DB in the time t, so that—

$$DB = \tfrac{1}{2}\frac{f_1 r \sin \alpha}{m} \,.\, t^2.$$

Let C be a point midway between A and B on the arc ACB; then since ACB is very small, it may be taken as a very small part of a circle. The body will pass through C at a time $t/2$ later than the instant when it passes through A, and the tangent at C will be parallel to the straight line joining A and B; that is, when the body moves through C its velocity is parallel to AB. Thus in the time $t/2$ the direction of motion of the body turns through the angle BAD; and

$$\text{the small angle BAD} = \frac{DB}{AD} = \frac{\tfrac{1}{2} \,.\, \dfrac{f_1 r \sin \alpha}{m} \,.\, t^2}{\dfrac{\omega r}{\sin \alpha} t}$$

$$= \frac{\tfrac{1}{2} f_1 \sin^2 \alpha \,.\, t}{m\omega}.$$

In the time $t/2$ the radius vector joining the body to the central point O (Fig. 26) turns through an angle $\omega t/2$. Hence, while the radius vector turns through an angle $\omega t/2$, the direction of motion of the body turns through an angle equal to BAD or—

$$\tfrac{1}{2}\frac{f_1}{m\omega} . \sin^2\alpha . t.$$

If the direction of motion, and the radius vector, turn through equal angles in the time $t/2$, the path of the body will cut all radii vectors at one uniform angle ; in this case—

$$\tfrac{1}{2}\omega t = \tfrac{1}{2}\frac{f_1}{m\omega}\sin^2\alpha . t,$$

$$\therefore\ \omega^2 = \frac{f_1}{m} . \sin^2\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the direction of motion turns through a greater angle than the radius vector, the angle α at which the path cuts the radii must be diminishing, and therefore $\sin\alpha$ is diminishing, and the rate at which the direction of motion turns is diminishing ; this diminution continues until the direction of motion and the radius vector turn through equal angles, in which case (2) will be satisfied. If the direction of motion turns through a smaller angle than the radius vector, the angle α must be increasing, and therefore $\sin\alpha$ must be increasing, with the result that the rate at which the direction of motion turns is increasing ; this increase can only cease when (2) is satisfied. Hence, finally, when the motion has become steady, the path cuts all radii vectors at a constant angle α.

Since the angle α is constant, it follows that the value of ω deduced from (1) is constant ; hence **the body revolves with uniform angular velocity about the central point.**

In the absence of the frictional opposing force $\alpha = \pi/2$, and (2) then gives the value of ω deduced on p. 74.

It now becomes obvious that the kinetic and potential energies of the body remain equal during the steady motion of the body ; for—

$$\text{kinetic energy} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\frac{r^2\omega^2}{\sin^2\alpha} \text{ from (1) ; and}$$

$$\text{potential energy} = \frac{f_1 r}{2} \times r = \tfrac{1}{2}\frac{mr^2\omega^2}{\sin^2\alpha} \text{ from (2).}$$

Hence, **the energy lost in overcoming the force that opposes the motion of the body is equally distributed between the kinetic and potential energies of the body.**

Since the opposing frictional force is equal to κv, the total *rate* at which the body is losing energy is equal to κv^2. The rate at which the body is losing potential energy is equal to the product of the central force $f_1 r$ and the component velocity $v \cos \alpha$ parallel to this force; and since the loss of potential energy is equal to half the total loss of energy, we have—

$$f_1 r \,.\, v \cos \alpha = \tfrac{1}{2}\kappa v^2,$$

$$\therefore\ f_1 r \cos \alpha = \tfrac{1}{2}\kappa v = \frac{\kappa}{2}\frac{\omega r}{\sin \alpha}, \text{ from (1).}$$

$$\therefore\ f_1 \cos \alpha = \frac{\kappa \omega}{2 \sin \alpha}.$$

Substituting the value of ω obtained from (2) we have—

$$f_1 \cos \alpha = \frac{\kappa}{2 \sin \alpha} \,.\, \sin \alpha \,.\, \sqrt{\frac{f_1}{m}}$$

$$= \frac{\kappa}{2}\sqrt{\frac{f_1}{m}}$$

$$\therefore\ \cos \alpha = \frac{\kappa}{2}\sqrt{\frac{1}{f_1 m}},$$

which determines the angle α of the spiral.

When κ is small, $\cos \alpha$ will be nearly equal to zero, and therefore α will be nearly equal to $\pi/2$, and the spiral will approximate to a circle. Unless α is considerably less than $\pi/2$, $\sin \alpha$ will be nearly equal to unity, and the value of ω obtained from (2) will be nearly equal to $\sqrt{f_1/m}$, which is the value of the angular velocity that the body would have in the absence of friction (p. 74). Hence, **when the frictional force is small, it will produce no appreciable effect on the angular velocity of the body, and therefore the time occupied in a complete revolution will be sensibly the same as if friction were absent.**

Properties of the spiral.—At a given instant let the body be at a point on the spiral at a distance r_0 from the origin O (Fig. 26), and after successive intervals of time, each equal to $1/p$th of a second, let the body be in positions at distances $r_1, r_2, r_3, \ldots r_n$ from O. Now, during the first interval of time, the average velocity v of the body is equal to—

$$\frac{r_0 + r_1}{2} \,.\, \frac{\omega}{\sin \alpha} \text{ (compare equation (1), p. 77),}$$

and since the *rate* at which the body is losing energy, owing to the frictional opposition to its progress, is equal to κv^2, the total amount of energy lost in $1/p$th part of a second is equal to—

$$\kappa\left(\frac{\omega}{\sin\alpha}\right)^2\left(\frac{r_0+r_1}{2}\right)^2\cdot\frac{1}{p}.$$

At the commencement of the interval the kinetic energy of the body is equal to—

$$\tfrac{1}{2}m\left(\frac{r_0\omega}{\sin\alpha}\right)^2,$$

and at the end of the interval it is equal to—

$$\tfrac{1}{2}m\left(\frac{r_1\omega}{\sin\alpha}\right)^2.$$

The loss of kinetic energy in the interval is equal to the difference between these two quantities, and this loss is half of the total loss of energy in the interval; hence—

$$\tfrac{1}{2}m\left(\frac{v}{\sin\alpha}\right)^2\left(r_0^2-r_1^2\right)=\frac{\kappa}{2}\left(\frac{\omega}{\sin\alpha}\right)^2\left(\frac{r_0+r_1}{2}\right)^2\cdot\frac{1}{p},$$

$$\therefore\ m(r_0-r_1)=\frac{\kappa}{4p}(r_0+r_1),$$

$$\therefore\ r_1\left(m+\frac{\kappa}{4p}\right)=r_0\left(m-\frac{\kappa}{4p}\right),$$

$$\therefore\ r_1=\frac{m-\dfrac{\kappa}{4p}}{m+\dfrac{\kappa}{4p}}r_0=\mathrm{K}r_0,$$

where

$$\mathrm{K}=\frac{m-\dfrac{\kappa}{4p}}{m+\dfrac{\kappa}{4p}}.$$

The value of K depends only on the mass m of the body, the frictional coefficient κ, and the interval of time $1/p$.

Similar reasoning will show that at the end of the second interval of time the distance r_2 of the body from the origin is given by—

$$r_2=\mathrm{K}r_1=\mathrm{K}^2r_0,$$

and similarly

$$r_3=\mathrm{K}^3r_0,$$

and

$$r_n=\mathrm{K}^nr_0 \qquad \ldots\ldots\ldots \quad (3)$$

Hence $r_0, r_1, r_2, r_3, \ldots \ldots r_n$ form a geometrical progression. Let us suppose that n intervals of time, each equal to $1/p$th of a second, are occupied by the body in travelling from A to B (Fig. 26). Then if $OA = a$, and $OB = b$, we have—

$$b = K^n \, . \, a.$$

At the end of the next (n/p) sec., the body will arrive at C, since the angular velocity of the body about O is constant; and if $OC = c$—

$$c = K^n b = K^{2n} a.$$

Similarly if $OD = d$, and $OE = e$,

$$d = K^n c = K^{3n} a,$$

$$e = K^n d = K^{4n} a, \text{ \&c.}$$

$$\therefore \frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} \text{ \&c.} \quad \ldots \ldots \quad (4)$$

Equation to the Spiral.—Let r_0 be the distance of the body from the origin O at any given instant; then its distance r_t from the origin, t seconds afterwards, can be found by substituting $t = n/p$ in (3). Thus—

$$r_t = K^{pt} \, . \, r_0 = r_0 \left(\frac{m - \frac{\kappa}{4p}}{m + \frac{\kappa}{4p}} \right)^{pt} \quad \ldots \ldots \quad (5)$$

Now—

$$\left(\frac{m - \frac{\kappa}{4p}}{m + \frac{\kappa}{4p}} \right) = \left(\frac{1 - \frac{\kappa}{4pm}}{1 + \frac{\kappa}{4pm}} \right).$$

Let $1/p$ be infinitely small, so that p is infinitely great; then, dividing the denominator into the numerator of the quantity within the brackets, and neglecting terms containing the square and higher powers of $1/p$, we obtain for the quotient the value—

$$1 - \frac{\kappa}{2pm}.$$

We may now re-write (5) in the form—

$$r_t = r_0 \left(1 - \frac{\kappa}{2pm} \right)^{\frac{2pm}{\kappa} \cdot \frac{\kappa t}{2m}} = r_0 \left(1 + \frac{1}{\beta} \right)^{\beta \left(- \frac{\kappa t}{2m} \right)}, \quad \ldots \ldots \quad (6)$$

where $\beta = -\frac{2pm}{\kappa}$. When $p = \infty$, the quantity $\beta = -\infty$.

By the binomial theorem—

$$\left(1+\frac{1}{\beta}\right)^{\beta} = 1 + \frac{\beta}{1}\cdot\frac{1}{\beta} + \frac{\beta(\beta-1)}{1\,.\,2}\cdot\frac{1}{\beta^2} + \frac{\beta(\beta-1)(\beta-2)}{1\,.\,2\,.\,3}\beta^3 + \ldots$$

$$= 1 + 1 + \frac{1\left(1-\frac{1}{\beta}\right)}{1\,.\,2} + \frac{1\left(1-\frac{1}{\beta}\right)\left(1-\frac{2}{\beta}\right)}{1\,.\,2\,.\,3} + \ldots.$$

$$= 1 + 1 + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \frac{1}{1\,.\,2\,.\,3\,.\,4}\ldots$$

when β is infinitely great. The series just obtained is the base of the Napierian system of logarithms, and is denoted by ϵ. Substituting in (6), we have—

$$r_t = r_0 \cdot \epsilon^{-(\kappa t/2m)}, \quad \ldots \ldots \quad (7)$$

$$\therefore \log_\epsilon r_t = \log_\epsilon r_0 - \frac{kt}{2m},$$

and therefore the logarithm of the distance from the origin to the body decreases proportionately with the time t. For this reason the spiral is called a **logarithmic spiral.**

QUESTIONS ON CHAPTER II

1. Determine the moment of inertia of a solid sphere about an axis passing tangentially through a point on the surface of the sphere.

2. Two similar conical shells are joined together, apex to apex, in such a manner that the axes of the cones are collinear. Any generating line of either cone makes an angle $\tan^{-1}\sqrt{2}$ with the axis. Prove that the moment of inertia of the combination, about the axis of symmetry common to both cones, is equal to the moment of inertia about any perpendicular axis passing through their apexes.

3. A closed cubical box is made from thin sheet metal; determine its moment of inertia about an axis passing through the middle points of two opposite faces, if the mass of the box is equal to m, and the length of an edge of the cube is equal to l.

4. A heavy uniform rod, of mass m and length l, is hinged at one end so that it can swing freely in a vertical plane. The rod is held in a horizontal position, and then released; determine its angular velocity as it swings through the position in which it hangs vertically downwards.

5. A body is allowed to roll down a plane inclined at an angle θ to the horizon. Prove that the body will be just on the point of sliding if $\tan\theta = \mu\left(1 + \frac{r^2}{k^2}\right)$, where r is the radius of curvature, and k is the

radius of gyration of the rolling body; and μ is the coefficient of friction between the body and the plane. (If a body is pressed against a plane by a normal force F, the body will just slide along the plane under the action of a force equal to μF, parallel to the plane. Thus μF is the maximum tangential force that can be called into play at the point of contact of the body with the plane.)

6. A hoop 3 ft. in diameter weighs 2 lb. Find the kinetic energy of the hoop when it is rolling along a horizontal road at a linear speed of 7 miles per hour.

7. The moment of inertia of the pulley of an Attwood's machine is equal to mk^2, and each of the equal masses attached to the ends of the cord is equal to M. A rider, of mass **m**, is placed on one of the equal masses; prove that the linear acceleration α of the moving system is given by the equation

$$\alpha\{2\mathrm{M}+\mathbf{m}+m(k/r)^2\} = \mathbf{m}g,$$

where r is the distance from the axis of rotation of the wheel to the bottom of the groove in which the cord rests.

8. A large uniform sphere of lead, of mass M and radius r, rests on a plane horizontal surface. A small bullet, of mass m, is fired horizontally into the sphere with a velocity V, the line of motion of the bullet being directed towards the centre of the sphere. Prove that the sphere will be set rolling along the surface with a linear velocity v, given by the equation

$$\mathrm{V} = \frac{\mathrm{M}}{m}\left(1 + \frac{k^2}{r^2}\right)v.$$

(Assume that the sphere does not slide on the plane.)

9. A small heavy body is hung from a fixed point by a long thin cord. The body is held in such a position that the cord is horizontal, and the body is then released; prove that when the body is passing through the lowest point in its path, the tension of the cord is equal to three times the weight of the body.

10. A long thin rod is hinged at one end, so that it can swing freely in a vertical plane. The rod is held in a horizontal position, and is then released; prove that, when the rod is passing through the position in which it is vertical, the downward pull on the hinge is equal to two-and-a-half times the weight of the rod.

11. Prove that, if the earth were not rotating about its axis, the value of g at the equator would be greater than its present value by 3·36 dynes per gram.

(Equatorial semi-diameter of the earth = $6{\cdot}38 \times 10^8$ cm.)

CHAPTER III

SIMPLE HARMONIC MOTION

Periodic motion.—When a body moves in such a manner that it periodically retraces its path, its motion is said to be **periodic.** The time which elapses between successive passages *in the same direction* through any point is termed the **periodic time**, or the **period**, of the motion. The number of periods comprised in one second is called the **frequency** ; thus, if T is the period, and n is the frequency of a periodic motion, we have $n = 1/T$.

The motion of the hands of a clock is periodic, the period of the motion of the minute hand being one hour, or 3600 seconds. The bob of a pendulum moves periodically, the period being equal to the time of one complete (to and fro) oscillation.

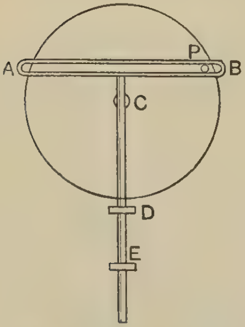

FIG. 29.—Nature of a simple harmonic motion.

Simple harmonic motion (s.h.m.).—A particular kind of periodic motion demands especial attention. In Fig. 29, AB represents a slotted bar rigidly connected to a rod CE, which works in guides at E and D ; thus AB is capable of moving only at right angles to its length, or parallel to the direction CE. A pin, P, is carried by a circular disc which can be rotated about C as centre ; this pin works in the slot of the bar AB. When the disc rotates, the pin P describes a circle ; thus the direction of its motion changes continually.

The component of its motion perpendicular to AB will be communicated to the slotted bar; on the other hand, the component parallel to AB will produce no effect on the motion of the slotted bar. The slotted bar will thus move up and down as the pin P describes its circular path: the bar is said to perform a **simple harmonic motion** (s.h.m.).

The characteristic properties of a s.h.m. can be studied by the aid of Fig. 30. Let a tracing point, P, revolve uniformly at a constant distance OP from O, the direction of motion being opposite to that of the hands of a clock. Through O draw the rectangular axes X′OX and Y′OY. From P draw PQ perpendicular to OY. Treating OP as a vector, we see that it is equivalent to the components OQ and QP. Thus, OQ is the component of OP resolved parallel to OY. As P revolves about O, the point Q moves up and down along Y′OY, its motion being of the simple harmonic type. When P passes across the axis X′OX, Q will pass through O. When P passes across the axis Y′OY, Q will be at its position of maximum displacement from its mean position, O. The maximum displacement of Q will thus be equal to the radius of the circular path of P; it is termed the **amplitude** of the s.h.m. The **phase** of the s.h.m. at any instant is equal to the angle which has been swept out by the line OP. If we measure the phase from the particular position of OP when Q is moving in a certain direction (say upwards) through O, the angle XOP $= \theta$ is equal to the phase of the s.h.m.

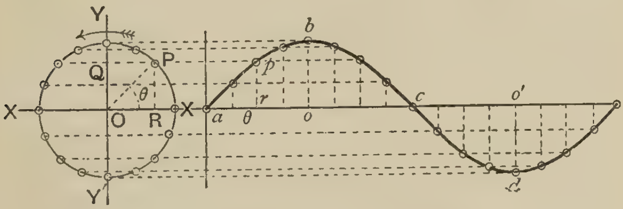

FIG. 30.—Characteristics of a simple harmonic motion.

The displacements of Q for various values of the phase angle θ, are shown by the curve to the right of Fig. 30. The distance *ar*, measured along the horizontal axis, is made equal to the circular measure of the angle XOP, and the distance *rp* plotted vertically above *r* is equal to the corresponding value of OQ. Other points on the curve are obtained in a similar manner.

Let the amplitude OP$=a$, while OQ$=y$. Then $y/a=$RP/OP$=\sin\theta$. Therefore $y=a\sin\theta$. This is the equation to the curve to the right of Fig. 30.

Further, let OP complete a rotation (*i.e.* rotate through an angle 2π) in the period T; then if the tracing point reaches P at a time t after passing through X, we have $\theta=2\pi t/\mathrm{T}$, and therefore $y=a\sin(2\pi t/\mathrm{T})$.

When the tracing point P crosses the axis OX, it is moving, for the instant, parallel to the axis OY. Consequently, at this instant the point Q is moving along OY with a velocity equal to that of the tracing point P in its circular path. The tracing point completes a revolution about O in a time T; since the length of its circular path is equal to $2\pi a$, the velocity of P is equal to $2\pi a/\mathrm{T}$. Hence, **the velocity of Q as it passes through its mean position O is equal to $2\pi a/\mathrm{T}$.**

When the tracing point P crosses the axis OY, it is moving, for the instant, in a direction perpendicular to OY. At this instant the point Q will be stationary. Consequently, **the point Q is stationary for an instant at the extremity of an excursion, on either side of its mean position O.**

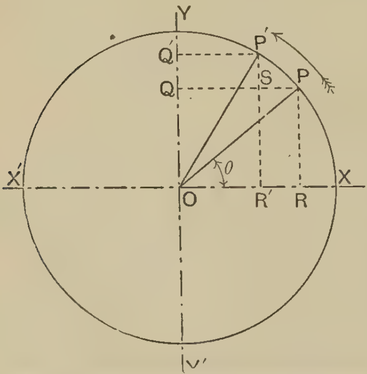

FIG. 31.—Method of calculating the velocity corresponding to any phase of a s.h.m.

The velocity corresponding to any phase of a s.h.m. can be determined readily. Let the vector OP (Fig. 31), of length a, revolve uniformly about O in an anticlockwise direction, its perod being equal to T; and let the projection Q of the point P on the axis YOY′ determine the simple harmonic motion. Thus OQ$=y=a\sin(2\pi t/\mathrm{T})$. If the tracing point moves from P to P′ in the very short interval of time t', its projection simultaneously moves from Q to Q′, and if t' is sufficiently small the velocity will be sensibly uniform over the short distance QQ′, its value being equal to QQ′$/t'$. From P and P′ drop perpendiculars PR and PR′ on OX, and let PR′ cut PQ in the point S. Treating the short circular arc PP′ as a straight line, PP′S forms a right angled triangle; and since PP′ is perpendicular to OP, while P′S is perpendicular to OX, it follows that $\angle$SP′P$=\angle$XOP$=\theta$, and QQ′$=$SP′$=$PP′ cos SP′P$=$PP′ $\cos\theta=$PP′ $\cos(2\pi t/\mathrm{T})$. Now the tracing point moves round the circle with a velocity $2\pi a/\mathrm{T}$, therefore in the time t' it moves through the distance $\mathrm{PP}'=2\pi at'/\mathrm{T}$, and—

$$\frac{\mathrm{QQ}'}{t'}=\left(\mathrm{PP}'\cos\frac{2\pi t}{\mathrm{T}}\right)\div t'=\left(\frac{2\pi at'}{\mathrm{T}}\cos\frac{2\pi t}{\mathrm{T}}\right)\div t'=\frac{2\pi a}{\mathrm{T}}\cos\frac{2\pi t}{\mathrm{T}}.$$

This gives the **instantaneous velocity of the simple harmonic motion** at the time t; in terms of the phase angle θ, the velocity is equal to $\frac{2\pi a}{\mathrm{T}}\cos\theta$. When $\theta=0$, $\cos\theta=1$, and the velocity is equal to $2\pi a/\mathrm{T}$, as already determined; when $\theta=\pi/2$, $\cos\theta=0$, and the velocity is equal to zero; and so on.

Thus, a point executing a s.h.m. moves with a maximum velocity equal to $2\pi a/\mathrm{T}$ on passing through its mean position. Its velocity diminishes as it recedes from its mean position, being equal to $(2\pi a/\mathrm{T})\cos\theta$ when the phase angle is θ, and attaining the value zero at the extremity of an excursion. Subsequently, as the point returns towards its mean position, its velocity increases, and once more attains the value $2\pi a/\mathrm{T}$ on moving through the mean position.

Resolution of a circular motion into its harmonic constituents.—The vector OP (Fig. 31) is equivalent to the two rectangular components $\mathrm{OR}=x$, and $\mathrm{RP}=y$. As proved already, $y=a\sin\theta$. Also, $\mathrm{OR}/\mathrm{OP}=x/a=\cos\theta$, and, therefore, $x=a\cos\theta$.

As P moves uniformly round its circular path, the point R moves backwards and forwards along the axis X′OX, and obviously executes a s. h. m. R will be at its position of maximum displacement at the instant when Q is passing through O; $\cos\theta$ acquires its maximum value (unity) when $\theta=0$, while $\sin\theta$ acquires its maximum value when $\theta=\frac{\pi}{2}$, *i.e.* a quarter period later. Therefore the phase of y lags behind that of x by $\pi/2$.

Thus, a uniform circular motion can be resolved into two s.h.m.'s at right angles to each other, their amplitudes being equal, while their phases differ by $\pi/2$. Conversely two s.h.m.'s at right angles to each other, of equal amplitudes but with phases differing by $\pi/2$, can be replaced by a uniform circular motion, the radius of the circle being equal to the amplitude of either s.h.m.

Resolution of a simple harmonic motion into two equal circular motions in opposite senses.—Let two tracing points traverse the circle

ACDB (Fig. 32) in equal times but in opposite senses; and let them pass simultaneously through the points A and B, at opposite ends of the diameter BOA of the circle. Each circular motion can be resolved into two simple harmonic constituents, one parallel to the axis XOX′, which passes through the points A and B, and the other parallel to the axis YOY′, which is perpendicular to the diameter BOA. When the tracing points are at A and B, the displacements parallel to YOY′ are both equal to zero, and the displacements parallel to XOX′ are equal and opposite; thus, at the given instant, the resultant displacement due to both circular motions is equal to zero. When the tracing point that moves in an anticlockwise sense passes through C, the other tracing point simultaneously passes through D, where ∠DOB = ∠COA. The component displacements due to the anticlockwise motion are then equal to OE parallel to YOY′, and EC parallel to XOX′, where E is the point in which the axis YOY¹ is cut by a straight line joining C to D. The component displacements due to the clockwise motion are equal to OE parallel to YOY′, and ED parallel to XOX′; and ED is numerically equal to EC. Thus the resultant displacement due to the two circular motions is equal to 2 × OE, since EC and ED are measured in opposite directions.

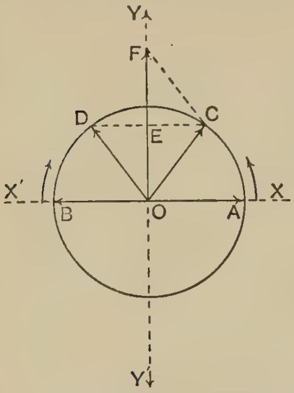

Fig. 32.—Equivalence between a simple harmonic motion and two equal and opposite circular motions.

The same result might have been obtained by drawing CF equal and parallel to OD; then OF = 2 × OE is equal to the resultant of OC and CF.

Let either tracing point traverse the circle in the time T, and let each of the arcs AC and BD be traversed in a time t; then ∠COA = $(2\pi t/\mathrm{T})$, and OE = OC sin $(2\pi t/\mathrm{T})$. Therefore the displacements due to the two circular motions are together equal to 2 × OC sin $(2\pi t/\mathrm{T})$; and if 2 × OC = a, the simple harmonic motion represented by the equation

$$y = a \sin (2\pi t/\mathrm{T})$$

is equivalent to two motions, in equal periods and in opposite senses, around a circle of radius OC = $(a/2)$, provided that the tracing points pass through the axis of y simultaneously.

The conical pendulum.—Let a small solid sphere (which will be called the bob) be attached to the end of a flexible filament AB (Fig. 33), which is so thin that it may be considered to be massless. Let the end B of the filament be fixed, and let the bob be displaced from its position of equilibrium and set in motion in the circular orbit ADC. Then the filament sweeps out a conical surface, and the bob and filament constitute a **conical pendulum.**

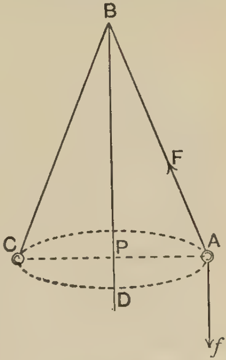

FIG. 33.—The conical pendulum.

From A drop the perpendicular AP on to the vertical axis through B. In order that the bob may revolve about P with uniform velocity, it must be acted upon by a force directed toward P, and equal to mv^2/PA; where m is the mass of the bob and v is its linear velocity (p. 27). Let $\mathrm{BA}=l$, and let $\angle \mathrm{ABP}=\alpha$; then if $\mathrm{PA}=a$, we have $a=l\sin\alpha$. If the bob completes each revolution in T seconds, we have—

$$v=2\pi a/\mathrm{T},$$

and the central force necessary to constrain the bob to move in the circle ADC is equal to

$$\frac{mv^2}{\mathrm{PA}}=\frac{m\left(\dfrac{2\pi a}{\mathrm{T}}\right)^2}{a}=m\left(\frac{2\pi}{\mathrm{T}}\right)^2 a=ml\left(\frac{2\pi}{\mathrm{T}}\right)^2\sin\alpha.$$

The forces actually exerted on the bob are—

(a) a vertical force f due to the pull of gravity, and

(b) the tension F of the filament. Since the filament is flexible, it can exert no force on the bob except a pull in the direction of the filament.

Now the vertical force f has no component along the horizontal line AP, and therefore the central force must be equal to the component of the tension F resolved along AP, *i.e.* to $\mathrm{F}\sin\alpha$.

The vertical component of F must be equal and opposite to the force f exerted by gravity on the bob; hence—

$$\mathrm{F}\cos\alpha=f,\ \text{and}\ \mathrm{F}=\frac{f}{\cos\alpha},$$

$$\therefore \text{ central force} = ml \sin\alpha \left(\frac{2\pi}{T}\right)^2 = F\sin\alpha = f\frac{\sin\alpha}{\cos\alpha}.$$

Cancelling sin α, we have

$$ml\left(\frac{2\pi}{T}\right)^2 = \frac{f}{\cos\alpha}.$$

$$\therefore\ T = 2\pi\sqrt{\left(\frac{m}{f}\cdot l\cos\alpha\right)}.$$

When α is small, the value of cos α will be sensibly equal to unity, and in this case the period of a revolution will be independent of the angle α; that is, so long as the circle around which the bob moves has a radius small in comparison with the length of the suspending filament, the period of revolution is independent of the exact value of the radius of the circle.

Experiment shows that the period of a revolution is the same whatever may be the mass of the bob; that is, a small bob of platinum will revolve in the same time as a bob of cork, provided that the suspending filament has the same length in each case and that the radii of the circular orbits are small. Let m_1 be the mass of the platinum bob, and f_1 the vertical force exerted on it by gravity; while m_2 and f_2 are the corresponding magnitudes for the cork bob; then since the period T is the same in both cases—

$$\frac{m_1}{f_1} = \frac{m_2}{f_2},$$

$$\frac{f_1}{m_1} = \frac{f_2}{m_2}.$$

This proves that **the gravitational attraction f exerted on a body is directly proportional to the mass m of the body**; we have already deduced this law from another experiment (p. 21), but the method just explained is much more accurate, since the period T can be determined accurately from an observation of the time which elapses during a large number of revolutions of the bob.

If g is the gravitational attraction exerted on unit mass of matter, $f = mg$; hence for small circular orbits—

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

The simple pendulum.—A circular motion is equivalent to two simple harmonic motions at right angles to each other

(p. 87); hence the circular motion of the bob of a conical pendulum is equivalent to two simple harmonic constituents, one along, and the other at right angles to CPA (Fig. 34). Both of these simple harmonic motions have the same period T as the circular motion.

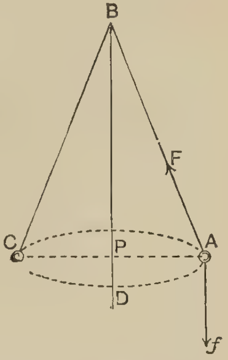

FIG. 34.—Relation between the conical and the simple pendulum.

If the eye of the observer were placed in the horizontal plane containing the circle ADC (Fig. 34) and at a considerable distance in front of the line CA, the simple harmonic motion in the straight line APC perpendicular to the line of vision would be observed, but the motion parallel to the line of vision would be unseen. If the radius PA of the circle is small, the straight line APC will differ only imperceptibly from the circular arc drawn through A and C in the plane ABC, with the point B as centre; and if the pendulum bob, instead of being set in motion in the circle ADC, were simply displaced to A and then released (thus forming a **simple pendulum**), it would oscillate to and fro along this circular arc, its motion being practically indistinguishable from the simple harmonic motion in the line APC. Hence we conclude that the period T of a complete (to and fro) oscillation of a simple pendulum is given by the equation—

$$T = 2\pi\sqrt{\frac{l}{g}};$$

provided that the arc of swing is small and that l is the length of the suspending filament, while g is the acceleration due to gravity.

EXPT. 5—Suspend balls of cork and lead, by means of silk threads of exactly equal lengths. Displace both from their positions of equilibrium and release them simultaneously. Observe that they swing to and fro together, although the amplitude of vibration of the cork ball dies down more rapidly than that of the lead ball.

EXPT. 6—Suspend a lead bullet by means of a silk thread, and determine the times of swing for various lengths of the suspension (measured to the centre of the bullet). From the results obtained, prove that T^2/l is constant. Also plot the value of T against l, and observe that the curve obtained is a parabola, of which the axis coincides with the axis along which l is measured.

EXPT. 7.—Using the mean value of T^2/l, obtained in the last experiment, calculate the value of g.

Mechanical condition for the execution of a s.h.m.—In order that a mass m may revolve uniformly in a circle of radius r, it must be acted on by a force directed toward the centre of the circle and equal to $mr\,(2\pi/T)^2$, where T is the period of a revolution (p. 74). This circular motion is equivalent to two s.h.m.'s at right angles to each other, given by the equations—

$$y=r\sin\theta \text{ and } x=r\cos\theta \text{ (see p. 87).}$$

Resolving the central force in the direction of the s.h.m.'s, we obtain—

$$m\left(\frac{2\pi}{T}\right)^2 r\sin\theta = m\left(\frac{2\pi}{T}\right)^2 y,$$

parallel to the axis of y, and—

$$m\left(\frac{2\pi}{T}\right)^2 r\cos\theta = m\left(\frac{2\pi}{T}\right)^2 x,$$

parallel to the axis of x.

Now, the motion of a body in a straight line can be effected only by forces acting in that line. Thus, the force necessary for the execution of the s.h.m. along the axis of x, is equal to $m\left(\frac{2\pi}{T}\right)^2 x$. This force is directed toward the origin, and is proportional to x, the displacement from that point.

Similarly, the s.h.m. executed along the axis of y is controlled by the force $m\left(\frac{2\pi^2}{T}\right)y$, acting along the axis of y, and directed toward the origin.

Thus, **a body moving in a straight line will execute a s.h.m. if it is acted on by a force, directed towards a point in the line of motion and proportional to the displacement of the body from that point.**

Since the force acting on a body is equal to the product of the mass and the acceleration of the body, it follows that when the displacement along the axis of x is equal to x, the **acceleration** is equal to $(2\pi/T)^2x$. Hence, **when a body is executing a s.h.m., its instantaneous**

acceleration is equal to its displacement from its mean position, multiplied by $(2\pi/T)^2$.

The potential energy possessed by the body at any point in its path is equal to the work done in moving it from the origin to that point, against the controlling force. Let a be the amplitude of the s.h.m., and let f_1 be the force which pulls the body towards the origin when it is at unit distance from that point. Then, since the controlling force is proportional to the distance from the origin, it increases uniformly from zero to f_1a as the displacement increases from zero to a, and therefore during this displacement the average force which opposes the motion of the body is equal to $(f_1a/2)$: this average force is overcome through the distance a, and therefore **the potential energy possessed by the body when it is at the extremity of an oscillation of amplitude** a, **is equal to** $(f_1a^2/2)$. Hence, the potential energy at the extremity of an oscillation is proportional to the square of the amplitude. This gives a useful criterion to determine whether the motion produced in any given circumstances, will, or will not, be of the simple harmonic type.

Oscillations of body suspended by an elastic filament.—Let A (Fig. 35) represent the equilibrium position of a heavy body suspended by an elastic filament OA. Let m be the mass of the body; then the downward pull of gravity on the body is equal to mg. At A this force is just counterbalanced by the tension of the stretched elastic filament. If the body is displaced downwards to B, experiment shows that within certain limits (called the limits of elasticity) the increase in the tension of the filament is proportional to the displacement AB; consequently, the resultant force on the body, when at B, is equal to $f_1 \times \text{AB}$, where f_1 is the force of restitution called into play by unit displacement from A. This resultant force acts toward A, the position of equilibrium of the body.

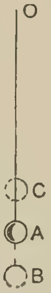

FIG. 35.—Body suspended by an elastic filament.

If the body is displaced upwards to C, the tension of the filament will be diminished by $f_1 \times \text{AC}$, so that the upward pull of the filament is now less than the downward pull of gravity by that amount. The resultant force acting on the body at C will thus be equal to $f_1 \times \text{AC}$, directed toward A.

Consequently, when the body is displaced in the line OA through a distance equal to a, and then given its freedom, it will be acted on at each instant by a force directed toward A, and proportional to the displacement from that point. It will therefore execute a s.h.m. about the position of equilibrium A. The amplitude of the s.h.m. will be equal to a, and the force acting on the body when its displacement is equal to x will be equal to f_1x. But (p. 92) when a body of mass m executes a s.h.m. of period T, the force corresponding to a displacement x from its position of equilibrium, is equal to $m\left(\frac{2\pi}{\mathrm{T}}\right)^2 x$. Thus, we have—

$$f_1 x = m\left(\frac{2\pi}{\mathrm{T}}\right)^2 x; \therefore \mathrm{T} = 2\pi\sqrt{\frac{m}{f_1}}.$$

Thus, **the time of vibration is found by multiplying 2π into the square root of the ratio of the mass to the restoring force per unit displacement.**

We may obtain the same result in another manner. As the body is displaced from its position of equilibrium, work is performed, and the potential energy of the body is increased. At the limit of an excursion the body is for an instant stationary; it then possesses only potential energy, the value of this being equal to $f_1 a^2/2$.

When the body passes through its position of equilibrium the restoring force vanishes, and then the body possesses only kinetic energy. Its velocity at that instant is equal to $2\pi a/\mathrm{T}$ (p. 86), and the kinetic energy of the body is consequently equal to $\frac{1}{2}m(2\pi a/\mathrm{T})^2$. Now, if no energy is lost during the passage from B to A, the kinetic energy at A must be equal to the potential energy at B, by the law of conservation of energy. Thus—

$$f_1\frac{a^2}{2} = \frac{1}{2}m\left(\frac{2\pi a}{\mathrm{T}}\right)^2; \quad \therefore \mathrm{T} = 2\pi\sqrt{\frac{m}{f_1}},$$

as before.

It should be noticed that the performance of a s.h.m. is characterised by the continual transference of energy from the potential to the kinetic form, and back again to the potential form. At all points in an excursion, the total energy of the body remains constant.

If any frictional forces act on the body, the total energy of the body decreases continually; when the body passes through its position of equilibrium, its kinetic energy is less than its potential energy at the extremity of the preceding excursion: the amplitude continually diminishes, and the body finally comes to rest.

The simple pendulum.—We have already derived the period of vibration of a simple pendulum from the period of revolution of a conical pendulum; we can now derive it directly.

When the arc of oscillation is very small, no appreciable error is introduced by assuming that the bob moves to and fro along the chord of the arc, instead of along the arc itself. In this case the restoring force may be considered to act along the chord, its value being equal to

$mg \sin \alpha / \cos \alpha$ (p. 90); and α being very small, we may write $\cos \alpha = 1$, and $\sin \alpha = \alpha$. If a is the linear amplitude of the vibration, $a = l\alpha$; and the restoring force is equal to $mg\alpha = mga/l$, so that the restoring force per unit displacement $= mg/l$. Then, as on p. 94—

$$\frac{mga}{l} = m\left(\frac{2\pi}{T}\right)^2 a,$$

$$\therefore\ T = 2\pi\sqrt{\frac{l}{g}}.$$

Notice that l/g is equal to the ratio of the mass m of the bob to the restoring force per unit displacement, mg/l.

As, on many occasions, we shall find it convenient to determine times of vibration by means of the energy equations, we may use these equations in connection with the simple pendulum. Let the bob, at the extremity A or C (Fig. 36) of a swing, be raised a distance $EF = h$ above its position of equilibrium E; then the potential energy of the bob at A or $C = mgh$ (p. 34). With the point of support, B, as centre, and l as radius, describe the circle ADC; then—

$$(EF) \times (FD) = (FA)^2.$$

If the oscillations are so small that the arc AEC and its chord AFC are sensibly equal, $AF = a$; then—

$$a^2 = h(2l - h) = h \times 2l,$$

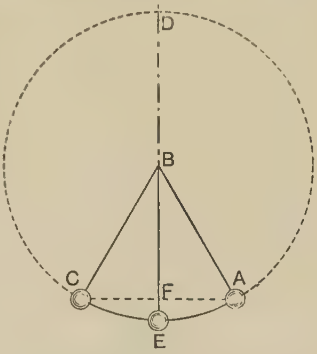

FIG. 36.—Simple pendulum at the extremities and the mid-point of an oscillation.

since h is so small that the value of $(2l - h)$ does not differ appreciably from $2l$. Thus—

$$h = a^2/(2l),$$

and the potential energy at A or $C = mgh = \dfrac{mga^2}{2l}$.

The kinetic energy at $E = \frac{1}{2}m\left(\dfrac{2\pi a}{T}\right)^2$,

$$\therefore\ \frac{1}{2}\frac{mg}{l}a^2 = \frac{1}{2}m\left(\frac{2\pi}{T}\right)^2 a^2,$$

$$\therefore\ T = 2\pi\sqrt{\frac{l}{g}},\ \text{as before.}$$

In the above investigation we have taken no account of the fact that, as the bob moves bodily to and fro, it also rotates about a horizontal axis through its centre ; in reality, therefore, the kinetic energy of the bob, as it passes through its position of equilibrium, is greater than that which we have assigned to it. The correction necessary on this account will be derived readily from the next stage in our investigation.

The compound pendulum.—Let a rigid body be supported so as to be free to rotate about a horizontal axis (Fig. 37) ; then if the centre of gravity of the body does not lie on this axis, the body will oscillate to and fro when it is displaced from its position of stable equilibrium and then released. A body supported in the manner described constitutes a **compound pendulum** : we must now investigate the period in which it oscillates.

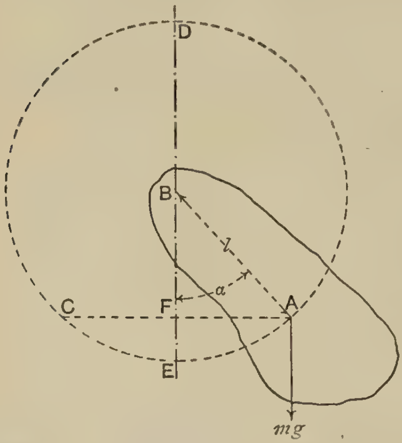

FIG. 37.—A compound pendulum at the extremity of a swing.

When the compound pendulum is in its position of stable equilibrium (p. 42) its centre of gravity is at the lowest point possible in the circumstances. Any displacement, therefore, raises the centre of gravity ; and since the total force exerted by gravity on the body may be considered to act at its centre of gravity, it follows that if this point is raised through a distance h, the work done (or the potential energy gained by the body) is equal to mgh, where m is the mass of the body. As the body oscillates to and fro, the centre of gravity describes a small arc of a circle of radius l, where l is the distance from the point of support to the centre of gravity. Let the angle swept out by the line joining the point of support to the centre of gravity, as the body moves from its position of equilibrium to the extremity of a swing, be equal to α ; then if the linear amplitude of oscillation of the centre of gravity is small and equal to a, we have $a = l\alpha$. Then—

$$h=\frac{a^2}{2l}=\frac{la^2}{2}.$$

(Compare with the result obtained on p. 95).

Therefore the potential energy of the body at the extremity of a swing $=mg(a^2/2l)$. This is proportional to the square of the amplitude a, and therefore we conclude that the oscillations are of the simple harmonic type (p. 93); and since the centre of gravity executes a simple harmonic motion, every other point in the body must execute a simple harmonic motion, and therefore the angular displacement of the body varies harmonically. Thus the angular displacement of the body at any instant can be found if we draw a circle XPX′ (Fig. 38) of radius α, and suppose that a tracing point P revolves uniformly around this circle in a period T. The distance from O to the projection Q of the tracing point P on the line YOY′ gives the angular displacement of the body at the given instant. Thus, at the instant when the phase angle XOP is equal to θ,

$$\text{the angular displacement} = \alpha\sin\theta=\alpha\sin\frac{2\pi}{\mathrm{T}}.$$

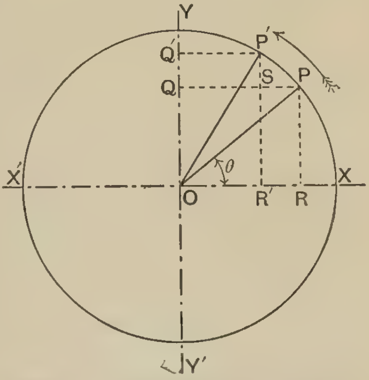

Fig. 38.—Angular oscillation of a compound pendulum.

Using reasoning similar to that explained on p. 87, it can be proved that, when the phase angle is equal to θ, the angular velocity of the body is equal to $(2\pi\alpha/\mathrm{T})\cos\theta$. Hence the angular velocity attains its maximum value $(2\pi\alpha/\mathrm{T})$ when the body is just swinging through its position of equilibrium.

The kinetic energy possessed by a body when it is rotating with an angular velocity ω, is equal to $(1/2)\mathrm{I}\omega^2$, where I is the moment of inertia of the body about the axis of rotation (p. 47). Let mk^2 be the moment of inertia of the compound pendulum about an axis through its centre of gravity and parallel to the axis of rotation; where k is the radius of gyration about the axis through the centre of gravity. Then the moment of inertia about the axis of rotation, which is at a distance

l from the centre of gravity, is equal to $m(k^2+l^2)$ (p. 56). Hence, as the pendulum swings through its position of equilibrium, its kinetic energy is equal to—

$$\tfrac{1}{2}m(k^2+l^2)\left(\frac{2\pi\alpha}{T}\right)^2,$$

and since this is equal to the potential energy at the extremity of a swing, we have—

$$\tfrac{1}{2}m(k^2+l^2)\left(\frac{2\pi\alpha}{T}\right)^2 = \frac{mga^2}{2l} = \tfrac{1}{2}mgl\alpha^2,$$

$$\therefore\ T = 2\pi\sqrt{\frac{k^2+l^2}{lg}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We can now apply this result to determine the period of vibration of a simple pendulum, on the assumption that the bob and filament move as if, together, they constituted a rigid body. Let the bob be spherical, its radius being r; then if the filament is so thin that its mass may be neglected, the centre of gravity of the pendulum will coincide with the centre of the sphere; and since the moment of inertia of a sphere, about a diameter as axis, is equal to $(2/5)\ mr^2$ (p. 55), it follows that $k^2=(2/5)\ r^2$. Let l be the distance from the fixed end of the filament to the centre of the sphere; then—

$$T = 2\pi\sqrt{\frac{l^2+\frac{2}{5}r^2}{lg}}.$$

If r is so small that r^2 may be neglected in comparison with l^2, the quantity under the radical sign becomes equal to l/g.

In applying equation (1) to a simple pendulum, we have assumed that the bob and filament move as if they constituted a rigid body. Such, however, is not the case; as the bob approaches the end of a swing, its inertia causes it to rotate through an angle somewhat greater than the angle α swept out by the filament.

Further, in order that the pendulum may continue to oscillate for a considerable time, so that the period of vibration may be determined with accuracy, the bob must be fairly heavy, and as a consequence the suspending filament must be fairly thick; in practice a metal wire is generally used. In this case the centre of gravity of the pendulum will not coincide with the centre of the spherical bob, and some difficulty will arise in determining its exact position. For these reasons, the most exact determinations of g have been made by the aid of a rigid body oscillating about a fixed axis, and not by the aid of a body suspended by a filament.

The reversible pendulum.—In order to understand the principle on which the compound pendulum acts, let the time of vibration T be plotted for various values of l; we then obtain curves similar to those reproduced in Fig. 39.

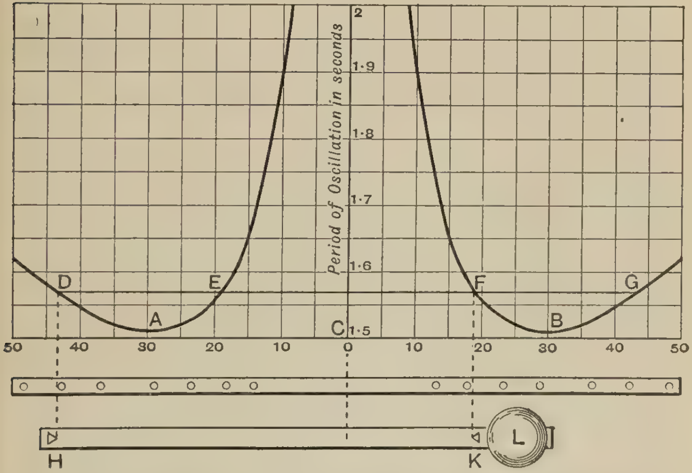

FIG. 39.—Graph exhibiting the relation between the period of oscillation and the position of the point of support of a compound pendulum.

The point C corresponds to the centre of gravity of the pendulum, and l is measured to the right or to the left of C, according as the axis about which rotation occurs is on one side or the other of the centre of gravity. It must be remembered, however, that negative values must not be assigned to l, for if they were the quantity under the radical sign in (1) would be negative. Hence the curves will always comprise two similar branches symmetrical with respect to the vertical ordinate drawn through C.

Substituting various values of l in the equation—

$$T = 2\pi \sqrt{\frac{k^2 + l^2}{lg}},$$

it is seen that when $l=0$, T is infinitely great. This is due to the fact that when $l=0$, the axis of rotation passes through the centre of gravity; in this case the body is in neutral equilibrium, and if it is displaced from any position and then released, gravity has no tendency to make it return to that position.

If we now suppose l to increase from zero, at first the effect will be to increase the value of the denominator of the quantity under the radical sign without materially increasing the value of the numerator; hence the value of T at first decreases as l increases. T does not, however, continue to decrease as l increases; with a certain value of l, a minimum value of T is attained, and a further increase in l increases the value of the numerator to a greater extent than it increases the denominator; hence the curve ultimately slopes upwards, and T increases with l.

Hence there will be two points A and B corresponding to a minimum time of vibration of the pendulum. Corresponding to a time of vibration T greater than this minimum value, there will be, in general, four possible positions of the axis of rotation; their positions can be found by drawing a horizontal line DEFG, at a distance equal to T above the axis of l. The four points may be grouped in two pairs E and F, D and G; the points grouped in each pair are equidistant from the centre of gravity.

Let the pendulum oscillate with a period T_1, when the distance of the axis of rotation from the centre of gravity is equal to l_1; then rearranging the terms in (1) we have—

$$\left(\frac{T_1}{2\pi}\right)^2 l_1 g = k^2 + l_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the pendulum oscillates with a period T_2 when the distance from the axis of rotation to the centre of gravity is equal l_2, we have—

$$\left(\frac{T_2}{2\pi}\right)^2 l_2 g = k^2 + l_2^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now let T_1 and T_2 be each equal to T; substituting in (2) and (3), and subtracting (3) from (2), we obtain—

$$\left(\frac{T}{2\pi}\right)^2 (l_1 - l_2)g = l_1^2 - l_2^2 = (l_1 - l_2)(l_1 + l_2) \quad . \quad . \quad . \quad (4)$$

If l_1 and l_2 are not equal (that is, **if the positions of the axes of rotation are not symmetrically situated with respect to the centre of**

gravity) we may divide both sides of (4) by $(l_1 - l_2)$; when this is done, we obtain—

$$\left(\frac{T}{2\pi}\right)^2 g = l_1 + l_2,$$

$$\therefore \ T = 2\pi \sqrt{\frac{l_1 + l_2}{g}}.$$

Let D and F, or E and G, be the two points, asymmetrically situated with regard to the vertical line through C, which correspond to the period of vibration T; then $(l_1 + l_2) = DF$ or EG. Denote this distance by λ. Then an ideal simple pendulum of length λ would oscillate in the period T, equal to the period of the compound pendulum when its axis of rotation has the position corresponding to any of the points D or F, E or G. Thus, if the pendulum were supported so as to be free to oscillate about the axis corresponding to D, its period of oscillation would be the same as if the whole of its mass were concentrated at the point corresponding to F. The point corresponding to D is called the **centre of suspension**, and the point corresponding to F is called the **centre of oscillation**: Further it is clear that if the point corresponding to F is made the centre of suspension, then D will be the centre of oscillation; that is, **the centres of suspension and oscillation are convertible.** The length $\lambda = DF$ or EG, is called the length of the **equivalent simple pendulum.**

Hence, **in a solid body of any shape whatsoever, we can find two points, asymmetrically situated with respect to its centre of gravity, about which the body oscillates in one and the same period** T. **The distance** λ **between these points is equal to the length of an ideal simple pendulum** (that is, of a simple pendulum without the defects described on p. 98) **that will oscillate in the same time** T. Hence, if we measure λ, we can determine g from the equation—

$$T = 2\pi \sqrt{\frac{\lambda}{g}}.$$

EXPT. 8.—To determine the value of g by the aid of a bar pendulum. Procure a brass rod about 100 cm. long, 2 cm. wide, and 0·5 cm. thick, pierced transversely with a number of holes. Suspend the bar by hanging it from any hole on a knife-edge. Hang it in turn from each hole, and determine its period of oscillation. From the results you obtain, plot a curve similar to Fig. 39, and thence determine the value of g. It is best to plot horizontally the distance from one end of the bar to that side of each hole on which the bar hangs, the times of vibration being plotted vertically.

Kater's reversible pendulum.—As already explained, it is only necessary, in determining g, to find the two points on the pendulum corresponding to D and F (Fig. 39) and to measure the distance $DF = \lambda$; this distance, with the period of oscillation T common to both points, serves to determine g. If we find the points corresponding to D and F, it is unnecessary to find that corresponding to G; hence the portion of the pendulum corresponding to the part of the curve F B G is not directly used in the experiment, and may be dispensed with. In order to attain this end, the pendulum may consist of a bar H K (Fig. 39) with a massive bob L near one end; the bob merely serves to keep the centre of gravity at a sufficient distance from the end H.

The centre of gravity of the pendulum must lie somewhere between the middle of the bob and H, and the point corresponding to F on the curve will generally lie between the middle of the bob and H. The point corresponding to D is necessarily further from the centre of gravity than the point corresponding to F. This serves to explain the nature of the pendulum used by Captain Kater in 1817 in determining g.

Two knife edges, turned inwards toward the centre of gravity, were provided for supporting the pendulum at the points corresponding to D and F. These knife edges were fixed, and the period of swing was adjusted by moving the bob and two small weights along the bar. Kater carried out this adjustment so accurately that the periods of swing about the two knife edges were exactly equal, within the limits of experimental error; but it was afterwards shown by Bessel that this refinement is unnecessary, since the value of g can be obtained with equal accuracy when T_1 and T_2, the periods of oscillation about the two knife edges, are only approximately equal. From equations (2) and (3), p. 100—

$$\frac{g}{(2\pi)^2}(T_1^2 l_1 - T_2^2 l_2) = l_1^2 - l_2^2,$$

$$\therefore \frac{4\pi^2}{g} = \frac{T_1^2 l_1 - T_2^2 l_2}{l_1^2 - l_2^2} = \frac{(T_1^2 + T_2^2)(l_1 - l_2) + (T_1^2 - T_2^2)(l_1 + l_2)}{2(l_1^2 - l_2^2)}$$

$$= \frac{T_1^2 + T_2^2}{2(l_1 + l_2)} + \frac{T_1^2 - T_2^2}{2(l_1 - l_2)}.$$

Now, (l_1+l_2) denotes the distance between the two knife edges, and this can be measured directly; the periods T_1 and T_2 can be observed accurately. The second term contains (l_1-l_2), and the magnitude cannot be measured with accuracy, since the exact position of the centre of gravity is unknown; but since the term containing (l_1-l_2) is multiplied by $(T_1^2-T_2^2)$, and T_1 and T_2 are very nearly equal, it suffices to find the approximate position of the centre of gravity by balancing the pendulum on a knife edge, and then to measure the distances l_1 and l_2 from this point.

A type of reversible pendulum frequently found in laboratories has the bob and knife-edges all adjustable. In using a pendulum of this kind, the following procedure may be followed. Fix the bob and knife-edges in positions, similar to those shown in Fig. 39. Swing the pendulum about the knife-edge near the bob, and adjust a simple pendulum so that it swings in the same period as the reversible pendulum. Then the second knife-edge must be placed at a distance from the first equal to the length of the simple pendulum. This adjustment will alter the period of the reversible pendulum, so again adjust the simple pendulum, and afterwards readjust the second knife-edge, if necessary. It will now be found that the reversible pendulum swings in approximately equal times about the two knife-edges, and a further adjustment may be made so as to bring the two periods more exactly to equality.

The adjustment of the pendulum so that it oscillates in exactly equal times about the two knife-edges would be very tedious, and is unnecessary. Let the pendulum be said to be "erect" when the bob is at its lower position, and "reversed" when the bob is at its higher position. Let us assume that the period of swing is slightly smaller in the reversed than in the erect position, that these periods have been determined accurately, and that the exact distance between the knife-edges has been measured. It will be obvious, from reference to Fig. 39, that the knife-edge remote from the bob is too far from the centre of gravity; move it nearer to the centre of gravity, so that the period is now slightly greater in the reversed than in the erect position, and let these periods be determined, and the distance between the knife edges measured. On a piece of squared paper plot the periods of the various swings above the respective distances between the knife-edges (Fig. 40); join the two points corresponding to the reversed positions, and also those corresponding to the erect positions. The co-ordinates of the intersection of these two lines gives the correct distance between the knife-edges when the pendulum swings with the same period about either, and also the value of that period.

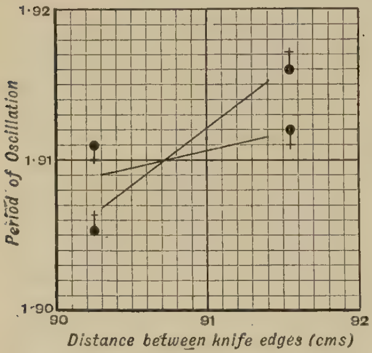

FIG. 40.—Graphical method of dealing with observations made in connection with a reversible pendulum.

If the knife-edges have been adjusted so that the periods of oscillation of the pendulum, in the erect and the reversed positions, are very nearly equal, the value of g can be determined still more simply by using Bessel's formula (p. 102).

Ball rolling on a concave surface.—If a ball A (Fig. 41) is placed on a concave surface EF, its position of equilibrium will be such that its centre is at the lowest possible point. If the ball is displaced and then released, it will oscillate about its position of equilibrium; in the absence of friction the amplitude would remain constant. Let it be required to determine the period of an oscillation.

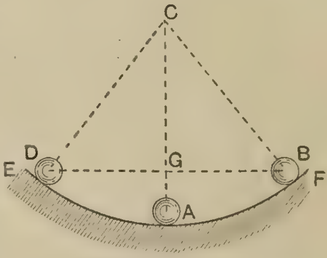

FIG. 41.—Ball rolling on a concave surface.

The potential energy at the end of an excursion can be found in the manner explained in connection with the simple pendulum (p. 95). Let m be the mass of the ball A, while R is the distance from its centre to the centre of curvature C of the surface EF. If a is the linear amplitude of an oscillation (*i.e.* the distance AB through which the centre of the ball moves on either side of its position of equilibrium) then potential energy of ball at B $= \frac{1}{2}mg\frac{a^2}{R}$.

Since the potential energy at the end of an excursion is proportional to the square of the amplitude, the oscillation of the ball will be of the simple harmonic type (p. 93). Let the ball roll on the surface without slipping; then, as it passes through its position of equilibrium A, its energy will be due, partly to the bodily motion of the ball, and partly to its angular velocity about its centre. The velocity of the centre of the ball as it passes through A is equal to $2\pi a/T$, where T is the period of an oscillation (p. 86). The angular velocity of the ball about its centre can be found from the following considerations. Since the ball does not slip, that point of it which is in contact with the surface will be stationary for the instant. If the ball is rolling with an angular velocity ω, the centre of the ball must be moving forward with a linear velocity ωr, where r is the radius of the ball; for ω gives the rate at which a radius of the ball rotates, and the end of the radius in contact with the surface is stationary. Thus the angular velocity of the ball as it passes through the point A is equal to $(2\pi a/T) \div r$. The moment of inertia of the ball about a diameter is equal to $(2/5)mr^2$ (p. 55); hence—

The kinetic energy of the ball at A

$$= \tfrac{1}{2}m\left(\frac{2\pi a}{T}\right)^2 + \tfrac{1}{2}\cdot\tfrac{2}{5}mr^2\cdot\left(\frac{2\pi a}{T}\right)^2\cdot\frac{1}{r^2} = \tfrac{1}{2}m(1+\tfrac{2}{5})\left(\frac{2\pi a}{T}\right)^2,$$

$$\therefore\ \tfrac{1}{2}mg\frac{a^2}{R} = \tfrac{1}{2}m\cdot\tfrac{7}{5}\cdot\left(\frac{2\pi a}{T}\right)^2,$$

$$\therefore\ T = 2\pi\sqrt{\frac{\frac{7}{5}R}{g}}.$$

If λ is the length of the equivalent simple pendulum (*i.e.* of the simple pendulum that oscillates in the same period as the ball), we have—

$$\lambda = \tfrac{7}{5}R;\quad \therefore\quad R = \tfrac{5}{7}\lambda.$$

The radius of curvature of the surface is obviously equal to $R + r$.

Expt 9.—Determine the radius of curvature of a concave mirror by rolling a ball on it.

A $\frac{3}{8}$th-inch steel ball (such as is used for ball bearings) answers well; the ball and surface must both be rubbed with clean tissue paper to remove all traces of grease. If the surface has a radius of about 7 to 10 cm., an accurate result can be obtained.

To determine the moment of inertia of a fly-wheel.—Let it be supposed that the axle of the fly-wheel is supported in bearings, in a horizontal position. If the wheel is properly balanced, it will remain at rest in any position; that is, its centre of gravity lies on the axis of rotation. Let a small body, of mass m, be fixed to the wheel at a distance d from the axis of rotation, and let I' be the moment of inertia of this mass about the axis of rotation, while I is the moment of inertia of the wheel itself about the same axis. Then the total moment of inertia of the wheel and the attached mass is equal to $I+I'$. The wheel will rest so that the centre of gravity of the attached mass occupies the lowest position possible; if it is displaced from this position and then released, it will oscillate to and fro as a compound pendulum.

If the angular amplitude of oscillation is equal to α, it follows that the potential energy of the whole arrangement at the end of an excursion is equal to the work done in raising the attached mass from its lowest position to that which it occupies at the end of an excursion, that is, it is equal to $mgd\alpha^2/2$ (compare p. 97). Hence, equating this expression to the kinetic energy possessed by the arrangement as it swings through its position of rest, we have—

$$\tfrac{1}{2}(I + I')\left(\frac{2\pi\alpha}{T}\right)^2 = \tfrac{1}{2}mgd\alpha^2,$$

where T is the period of an oscillation;

$$\therefore\ I = mgd\left(\frac{T}{2\pi}\right)^2 - I'.$$

EXPT. 10—Determine the moment of inertia of the wheel of an Attwood's machine.

A lead bullet weighing about 100 grms. may be attached to the rim of the wheel with soft red wax. If r is the radius of the bullet, and a is the distance of its centre from the axis of rotation, we have—

$$I' = \tfrac{2}{5}mr^2 + md^2.$$

Observe the value of T, and then calculate the value of I.

Notice that the moment of inertia can also be obtained by determining the inertia of the wheel in the manner explained on p. 17, and then multiplying this into the square of the distance from the axis of rotation to the bottom of the groove on the edge of the wheel.

The bifilar suspension.—Let AB (Fig. 42) represent a body suspended by two parallel flexible fibres ED and GF, attached to points E, G, equidistant from, and on opposite sides of, the centre of gravity C of the body. Then the body will be in stable equilibrium when it hangs so that the fibres are vertical and lie in a vertical plane ; in this case the centre of gravity of the body is at the lowest point consistent with the nature of the suspension. If the body is twisted about the vertical axis CH, each fibre assumes a position inclined to the vertical, and the centre of gravity of the body rises ; if the body is released, it executes angular oscillations about the vertical axis through C.

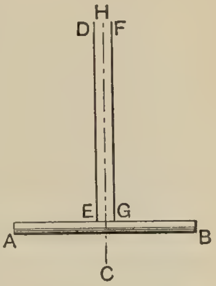

FIG. 42.—Body supported by a bifilar suspension.

The period T of small oscillations about the vertical axis CH can be determined readily. Let l be the length of each fibre, and let d be the distance EG separating them ; then when the body is twisted through a small angle α about CH, the points E and G must lie on an imaginary cylinder, of radius $d/2$, and with CH as axis ; the displacement of E or G perpendicular to the plane of the paper is obviously equal to $\alpha d/2$. Thus, when α is very small, the motion of E or G is practically identical with that of the bob of a simple pendulum suspended by a fibre of length l, when it oscillates with a linear amplitude equal to $\alpha d/2$; hence, employing the reasoning explained on p. 95, we find that each of the points E and G rises through a distance equal to—

$$\frac{\left(\frac{\alpha d}{2}\right)^2}{2l},$$

and this gives the distance through which the centre of gravity of the body is raised. Hence, the potential energy of the body at the extremity of an oscillation—

$$= \frac{1}{2}\frac{mg}{l}\left(\frac{\alpha d}{2}\right)^2;$$

and since this quantity is proportional to the square of the amplitude α, the oscillations will be of the simple harmonic type. If the moment of inertia of the body is equal to mk^2, where m is its mass and k is its radius of gyration about the axis CH, the kinetic energy of the body as it swings through its position of equilibrium is equal to—

$$\frac{1}{2}mk^2\left(\frac{2\pi\alpha}{T}\right)^2.$$

Hence—

$$\tfrac{1}{2}mk^2\left(\frac{2\pi\alpha}{\mathrm{T}}\right)^2 = \tfrac{1}{2}\frac{mg}{l}\left(\frac{\alpha d}{2}\right)^2,$$

$$\therefore \mathrm{T} = 2\pi \cdot \frac{2k}{d}\sqrt{\frac{l}{g}}.$$

Hence, by making d small enough, we can make T as large as we please.

Torsional oscillations.—A torque is needed to twist one end of a wire of which the other end is fixed; experiment shows that, within certain limits which depend on the material of which the wire is composed, the torque is proportional to the angle of twist. The twist in the wire may be maintained if the torque continues to act; in other words, the twisted wire remains in stable equilibrium under the action of the applied torque. Hence, the twist in the wire must produce a restoring torque, equal and opposite to the applied torque; and the restoring torque is also proportional to the twist. Hence, if we divide the torque by the circular measure of the twist which it produces, we obtain a constant value for the given wire; this value may be called the **torque per unit twist** and it will be denoted by τ_1.

In twisting a wire, work must be done in overcoming the restoring torque called into play; as the twist increases from zero to α, the restoring torque increases uniformly from zero to $\tau_1\alpha$; the average value of the torque overcome is therefore equal to $\tau_1\alpha/2$, and the work done (p. 42), or the energy gained, is equal to $(\tau_1\alpha/2) \times \alpha = \tau_1\alpha^2/2$. This energy is stored as potential energy in the wire. Let a body be suspended by a wire, and let the body be twisted through an angle α about a vertical axis coinciding with the axis of the wire. Then the potential energy gained is equal to $\tau_1\alpha^2/2$, and since this quantity is proportional to α^2, the body will execute simple harmonic oscillations about its position of equilibrium when it is released. Hence, if I is the moment of inertia of the body, and T is the period of oscillation—

$$\tfrac{1}{2}\mathrm{I}\left(\frac{2\pi\alpha}{\mathrm{T}}\right)^2 = \tfrac{1}{2}\tau_1\alpha^2,$$

$$\therefore \mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I}}{\tau_1}}.$$

This result will apply, not only to very small oscillations, but to all oscillations which do not strain the material of the wire beyond its limits of elasticity.

In general, **the period of a torsional oscillation is equal to 2π multiplied by the square root of the ratio**—

$$\frac{\textit{moment of inertia of oscillating body}}{\textit{torque per unit twist of the suspension}}.$$

This result, obtained from considering the oscillations of a body suspended by a wire, is generally applicable to all small angular oscillations.

In the case of a simple pendulum, the moment of inertia I of the bob, about the fixed point to which the suspending filament is attached, is equal to ml^2, if the bob is very small; or to $(2/5mr^2)+ml^2$, if the bob is a sphere of finite radius r.

When the bob is displaced so that the filament makes an angle α with the vertical, the downward force of gravity, mg, exerts a torque $mg\,.\,l\sin\alpha$ about the fixed point; and when α is small, $\sin\alpha=\alpha$, so that the torque divided by the twist (that is, τ_1) is equal to mgl.

Thus $$\frac{\mathrm{I}}{\tau_1}=\frac{ml^2}{mgl}=\frac{l}{g},\ \text{for an infinitely small bob,}$$

or $$\frac{m(\frac{2}{5}r^2+l^2)}{mgl}=\frac{l}{g}+\frac{2}{5}\frac{r^2}{gl},\ \text{for a spherical bob of radius } r.$$

Compare these results with those obtained on pp. 95 and 98. In the case of the bifilar suspension, each filament supports half of the weight mg of the suspended body; and when this body is rotated through an angle α, each filament assumes a position inclined at an angle ϕ (say) to the vertical, and the horizontal component of the tension of each filament is equal to—

$$(mg/2)\sin\phi=\frac{mg}{2}\left(\frac{\alpha d}{2}\div l\right)=mg\frac{\alpha a}{4l}.$$

The resultant torque about the axis of rotation, exerted by the horizontal components of the tensions of both filaments—

$$=2\,.\,mg\frac{\alpha d}{4l}\times\frac{d}{2}=\frac{mg\alpha}{l}\left(\frac{d}{2}\right)^2,$$

so that τ_1, the torque per unit twist, is equal to $\frac{mg}{l}\left(\frac{d}{2}\right)^2$.

Hence $$\frac{\mathrm{I}}{\tau_1}=\frac{mk^2}{\frac{mg}{l}\left(\frac{d}{2}\right)^2}=\left(\frac{2k}{d}\right)^2\frac{l}{g}.$$

Compare the result with that found on p. 108.

Problem.—*A body is suspended by means of two similar flexible filaments which are inclined at equal angles to the vertical. Determine the torque necessary to twist the body through a small angle* θ *about its vertical axis.*

Let the upper ends of the filaments be fixed at points B, D (Fig. 43), in a horizontal plane; and let the distance BD$=c$. Let the lower ends of the filaments be attached to the body at points E, F, where the distance EF$=d$. When the body is in its position of equilibrium, the filaments will both lie in the vertical plane drawn through B and D. Draw the vertical line AC, through the point A midway between B and D; then if C is the point in which the vertical line AC cuts a horizontal plane drawn through E and F, the point C will be midway between E and F. It is obvious that AC is the axis about which the body can rotate.

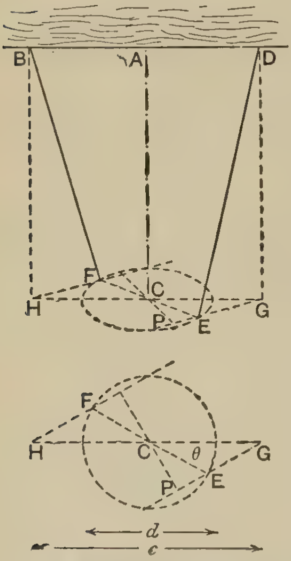

FIG. 43.—Suspension consisting of two similar filaments, equally inclined to the vertical. (Perspective view above, plan below.)

Let the body be rotated through an angle θ about AC as axis; the filaments will now be in the positions DE and BF (Fig. 43). Through the filament DE, draw a vertical plane cutting the vertical plane through BD in the vertical line DG; and from E draw EG perpendicular to DG. Let the filament DE be inclined at an angle EDG$=\phi$ to the vertical line DG; the other filament will be equally inclined to the vertical. Then if the suspended body has a mass m, gravity pulls it down with a force mg, and this force must be equal to the upward pull exerted on the body by the two filaments. Thus, if the tension of each filament is equal to f, we have

$$2f\cos\phi=mg.$$

When ϕ is small, $\cos\phi$ is practically equal to unity. Let it be assumed that, when the suspended body is in its position of equilibrium, the inclination of the filaments to the vertical is small; and also that θ is small. Then ϕ will be small, and $f=(mg/2)$.

The force f acting on the body in the direction ED has a component parallel to EG, equal to $(f \times \mathrm{EG}/l)$, where l is the length ED of the filament. From C drop a perpendicular CP on GE produced; then $\mathrm{CP} = \mathrm{CG} \sin \angle \mathrm{CGP} = (c/2) \sin \angle \mathrm{CGP}$. The torque exerted about the axis AC by the tension of the filament is equal to

$$(f \times \mathrm{EG}/l) \times (c/2) \sin \angle \mathrm{CGP}.$$

Now, from the triangle CGE—

$$\frac{\sin \angle \mathrm{CGE}}{\sin \angle \mathrm{GCE}} = \frac{\sin \angle \mathrm{CGP}}{\sin \theta} = \frac{\mathrm{CE}}{\mathrm{EG}} = \frac{(d/2)}{\mathrm{EG}},$$

and, since θ is small, $\sin \theta = \theta$.

$$\therefore \quad \sin \angle \mathrm{CGP} = \frac{\theta d}{2 \,.\, \mathrm{EG}},$$

and the torque exerted about the axis AC by the filament ED is equal to—

$$\frac{f \,.\, \mathrm{EG}}{l} \,.\, \frac{c}{2} \sin \mathrm{CGP} = \frac{f \,.\, \mathrm{EG}}{l} \,.\, \frac{c}{2} \,.\, \frac{\theta a}{2 \,.\, \mathrm{EG}}$$

$$= f \,.\, \frac{cd}{4l} \theta.$$

A similar and equal torque is exerted about AC by the filament FB. Thus the total torque τ exerted on the body by the two filaments is given by the equation—

$$\tau = 2f \frac{cd}{4l} \theta = mg \,.\, \frac{cd}{4l} \,.\, \theta,$$

and the torque per unit twist, τ_1, is given by the equation—

$$\tau_1 = mg \frac{cd}{4l}.$$

If the moment of inertia of the suspended body about the axis AC is equal to mk^2, the periodic time T of oscillation about that axis is given by the equation—

$$\mathrm{T} = 2\pi \sqrt{\frac{mk^2}{\frac{mgcd}{4l}}}$$

$$= 2\pi \,.\, \frac{2k}{\sqrt{(cd)}} \sqrt{\frac{l}{g}}.$$

When $c = d$, we obtain the value of T already found for an ordinary bifilar suspension.

Expt. 11—To determine, experimentally, the moment of inertia of a body about a given axis.

Suspend the body by a wire, in such a manner that the axis of the moment of inertia coincides with the axis of the wire. Set the body oscillating about the given axis, and determine its period of oscillation T_1. Then if I is the moment of inertia of the body about the given axis—

$$T_1 = 2\pi\sqrt{\frac{I}{\tau_1}},$$

where τ_1 is the restoring torque called into play by unit twist of the wire.

Obtain a body, of which the moment of inertia I_1 about some axis can be calculated, and attach this body rigidly to the first one in such a manner that the axes of both bodies coincide. Then the moment of inertia of the compound body is equal to $(I + I_1)$. Observe the period of oscillation T_2 of the compound body; assuming that the restoring torque called into play by unit twist of the wire has not been altered—

$$T_2 = 2\pi\sqrt{\frac{I + I_1}{\tau_1}},$$

Then $$\frac{T_2^2}{T_1^2} = \frac{I + I_1}{I},$$

and $$I = I_1\frac{T_1^2}{T_2^2 - T_1^2}.$$

EXPT. 12—Determine the restoring torque called into play by unit twist of a wire.

Suspend a body of known moment of inertia I by the wire, and obtain its period of torsional oscillation T. Then—

$$T = 2\pi\sqrt{\frac{I}{\tau_1}},$$

$$\therefore\ \tau_1 = \left(\frac{2\pi}{T}\right)^2 I.$$

Maxwell's needle.—In determining the moment of inertia of a body by the experimental method explained above, it is assumed that the restoring torque called into play by unit twist is independent of the longitudinal tension of the wire. This assumption is only approximately correct, and where very great accuracy is required it is desirable to use a method in which no such assumption is made. Further, the moment of inertia of the body suspended from the wire is determined by comparing it with the moment of inertia of a body of regular shape, of which the moment of inertia has been calculated by one of the methods explained on pp. 47 to 57. In calculating the moment of inertia of a body, it is assumed that the body is homogeneous, that is, that it is of uniform density throughout; and even in the case of a cylinder of rolled brass, this condition is not complied with absolutely. Maxwell invented a device by means of which both of the above sources of error are eliminated; it involves the use of a body of special construction such that its moment of inertia can be determined without making any assumption as to its absolute homogeneity; this body is called **Maxwell's needle.**

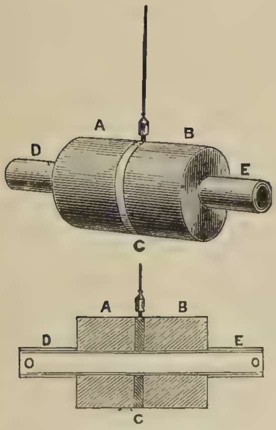

FIG. 44.—Maxwell's needle. (Perspective view above, longitudinal section below.)

Maxwell's needle may be constructed in several different forms, but of these the one which can be manipulated most easily will be described. A brass tube OO (Fig. 44) has a discoidal collar C attached to it, midway between its ends; and by this the tube can be suspended from a wire. Four hollow cylinders, A, B, D, and E, all exactly equal in length, slide on the tube OO. The cylinders A, B are made from rolled brass, and are fairly massive; besides being equal in length, A and B are also equal in mass. The cylinders D and E are thin-walled tubes, equal in length and in mass; they are used as distance pieces.

Let the mass of A or B be M, and let d_1 be the distance of the middle transverse section of either from the axis of the wire. Although A and B are equal in mass and in length, they may not be quite homogeneous, with the result that their centres of gravity will not lie necessarily in their middle transverse sections, and their moments of inertia about axes through their centres of gravity and perpendicular to their lengths may differ. Let the centre of gravity of A be between its middle section and the axis of the wire in the arrangement represented in Fig. 44; let the distance of its centre of gravity from its middle section be equal to α, and let its moment of inertia about an axis through its centre of gravity and perpendicular to its length be equal to I_a. Then its moment of inertia about the axis of the wire is equal to $\{I_a + M(d_1 - \alpha)^2\}$ (p. 56). Let I_b be the moment of inertia of B about an axis through its centre of gravity and perpendicular to its length, and let the centre of gravity of B be at a distance β from its middle section, its position being between the latter and the wire; then the moment of inertia of B about the axis of the wire is equal to $\{I_b + M(d_1 - \beta)^2\}$.

Let the middle transverse sections of D and E be at a common distance d_2 from the axis of the wire, and let their centres of gravity be between the wire and their middle transverse sections, at distances δ and ϵ from the latter. If the mass of D or E is m, and their moments of inertia about transverse axes through their centres of gravity are I_d and I_e, then their moments of inertia about the axis of the wire will be respectively equal to—

$$\{I_d + m(d_2 - \delta)^2\} \text{ and } \{I_e + m(d_2 - \epsilon)^2\}.$$

Now, if the moment of inertia of the tube OO and its collar is equal to I_o, the moment of inertia of OO, with the tubes A, B, C, and D in the positions represented in Fig. 44, will be equal to

$$I_o + I_a + I_b + I_d + I_e + M\{(d_1 - \alpha)^2 + (d_1 - \beta)^2\} + m\{(d_2 - \delta)^2 + (d_2 - \epsilon)^2\}.$$

Let T_1' be the period of vibration of the arrangement; then if the torque per unit twist of the wire is equal to τ_1, we have, as on p. 109,

$$\tau_1\left(\frac{T_1'}{2\pi}\right)^2 = I_o + I_a + I_b + I_d + I_e + M\{(d_1 - \alpha)^2 + (d_1 - \beta)^2\} \\ + m\{(d_2 - \delta)^2 + (d_2 - \epsilon)^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now turn the tubes A, B, C, and D end for end, and replace them without otherwise altering their arrangement. Their centres of gravity will now be at distances $(d_1 + \alpha)$, $(d_1 + \beta)$, $(d_2 + \delta)$, and $(d_2 + \epsilon)$ from the axis of the wire, and the period of vibration T_1'' will be given by the equation—

$$\tau_1\left(\frac{T_1''}{2\pi}\right)^2 = I_o + I_a + I_b + I_d + I_e + M\{(d_1 + \alpha)^2 + (d_1 + \beta)\} \\ + m\{(d_2 + \delta)^2 + (d_2 + \epsilon)^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Adding the equation (1) and (2) together, dividing by two, and writing T_1^2 for $\{(T_1')^2 + (T_1'')^2\}/2$, we have—

$$\tau_1\left(\frac{T_1}{2\pi}\right)^2 = I_o + I_a + I_b + I_d + I_e + M(2d_1^2 + \alpha^2 + \beta^2) \\ + m(2d_2^2 + \delta^2 + \epsilon^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Next, interchange the positions of A and D, and also those of B and E. Since A, B, D, and E are all equal in length, the middle transverse sections of A and B will now be at a distance d_2 from the axis of the wire, and the middle transverse sections of D and E will be at a distance d_1 from the same axis. If T_2^2 is the mean value of the squares of the periods of vibration obtained before and after the tubes have been turned end for end as before, we have—

$$\tau_1\left(\frac{T_2}{2\pi}\right)^2 = I_o + I_a + I_b + I_d + I_e + M(2d_2^2 + \alpha^2 + \beta^2) \\ + m(2d_1^2 + \delta^2 + \epsilon^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Subtracting (3) from (4), we obtain—

$$\tau_1\left\{\left(\frac{T_2}{2\pi}\right)^2 - \left(\frac{T_1}{2\pi}\right)^2\right\} = 2(M - m)(d_2^2 - d_1^2).$$

This equation serves to determine the value of τ_1 in terms of magnitudes which can be measured directly. Then, if the moment of inertia of any particular arrangement (say the first) is required, it can be obtained by substituting the value τ_1 in the left-hand side of (1), when the value of the right-hand side of that equation becomes known.

Theory of the balance.—The beam of a balance is supported on a central knife-edge, and two scale-pans are hung from knife-edges near to its ends and equidistant from the central knife-edge. If the beam of the balance by itself is in stable equilibrium in a horizontal position, and the masses of the scale-pans are equal, it follows that the beam will remain horizontal if the two pans are loaded equally (p. 39).

In order that the beam by itself may be in stable equilibrium in a horizontal position, its centre of gravity must be at a small distance vertically beneath the central knife-edge: if its centre of gravity were to coincide with the central knife-edge the beam would be in neutral equilibrium, and would remain in any position, horizontal or inclined, in which it was placed; and if its centre of gravity were above the central knife-edge, the beam would be in unstable equilibrium (p. 42). Hanging the scale-pans on their knife-edges makes no difference in the position of equilibrium of the beam, since gravity exerts equal vertical forces on the pans. If the pans are unequally loaded, the beam will attain equilibrium in a position inclined to the vertical; the difference between the opposite torques due to the force of gravity acting on the loaded pans, is then just equal to the restoring torque called into play by the displacement of the centre of gravity of the beam. It is clear that when the beam is tilted, its centre of gravity describes a circular arc, just as the centre of gravity of a pendulum does. The beam oscillates about its position of equilibrium in a period T, given by the equation—

$$T = 2\pi\sqrt{\frac{I}{\tau_1}},$$

where I is the moment of inertia of the beam, scale-pans, and load ; and τ_1 is the restoring torque called into play by unit angular displacement of the beam.

Loading the scale-pans equally makes no difference in the restoring torque, unless, indeed, it bends the beam and so alters the position of its centre of gravity. Since each scale-pan and its load moves with the same velocity as the knife-edge on which it is supported, the moment of inertia is increased by loading the pans. Hence the period of oscillation of the balance is increased by equally loading the scale-pans.

The **sensitiveness of the balance** is measured by the angular deflection of the beam produced by a given increase in the load added to one scale-pan ; it is clear that the sensitiveness is inversely proportional to τ_1, the restoring torque called into play by unit angular displacement of the beam.

In general, the beam is provided with a pointer, which moves over a small graduated scale. It is often necessary to determine the position which the pointer would occupy if the beam were at rest, from observations made while the beam is oscillating. If no frictional forces acted on the beam, it would execute simple harmonic oscillations about its position of equilibrium, and the equilibrium position of the pointer would be midway between the points on the scale at which the motion of the pointer is reversed. The effect of friction is to gradually diminish the amplitude of the oscillations ; this will be studied in detail in an ensuing chapter. For the moment, however, we may assume that, if the friction is small and the amplitude of the oscillations decreases slowly, the rate of decrease remains practically constant during the time occupied in performing one or two oscillations.

In this case, let a particular turning point of the pointer be at a distance a to the right of its equilibrium position ; and let the next turning point be at a distance $(a - \delta)$ to the left of the equilibrium position. Then in half an oscillation the amplitude has decreased by δ, and therefore in a complete oscillation the amplitude will decrease by 2δ, and the next turning point will be at a distance $(a - 2\delta)$ to the right of its equilibrium position. Hence, a point half-way between the two consecutive turning points *to the right* will be at distance $(a - \delta)$ from the equilibrium position, and the equilibrium position will be midway between this point and the intermediate turning point to the left.

EXPT. 13—Determine the sensitiveness of a chemical balance, that is, the change in the equilibrium position of the pointer produced by increasing the load on one side of the balance by a milligram.

EXPT. 14—Determine the value of g, the acceleration due to gravity, from observations made with a balance.

Observe the period of oscillation T_1 of the balance when it is unloaded, and the period T_2 when each scale-pan is loaded with M grams. Then, if each of the knife-edges supporting the scale-pans is at a distance d from the central knife-edge, loading the pans increases the moment of inertia of the moving system by $2Md^2 = i$ (say). Then—

$$T_1 = 2\pi\sqrt{\frac{I}{\tau_1}},$$

$$T_2 = 2\pi\sqrt{\frac{I + i}{\tau_1}}.$$

These equations suffice to determine I. Insert the value obtained for I in the first equation, and calculate the value of τ_1. Now observe the equilibrium position of the extremity of the pointer, first when the pans are unloaded, and then when one pan contains a small mass m. Then the distance (expressed in centimetres) between these positions, divided by the distance from the extremity of the pointer to the central knife-edge, gives the angular deflection θ produced by an applied torque equal to mgd; therefore τ_1, the torque per unit angular deflection, is equal to mgd/θ. Then, calculate the value of g with care. An accuracy of about two per cent. can be attained.

A balance provided with a nickel or steel pointer must not be used for this experiment, on account of the magnetic properties of these metals.

Oscillations of a magnet in a magnetic field.—Let H be the strength of a magnetic field, measured in dynes per unit pole; then a N. pole of strength m placed in this field will be acted upon by a force equal to $(+mH)$ dynes, the positive sign denoting that the direction of the force is the same as that of the field. A S. pole of equal strength will be acted upon by a force equal to $(-m\text{H})$ dynes, the negative sign denoting that the direction of the force is opposite to that of the field. A magnet suspended so that it is free to turn in a horizontal plane, will rest in stable equilibrium with its axis coinciding with the direction of the field. Let the magnet be turned through a small angle α; then the parallel and oppositely directed forces acting on its two poles are separated by a perpendicular distance $l \sin \alpha$, where l is the distance between the poles. Hence the restoring torque called into play—

$$= m\text{H}l \sin \alpha = ml.\text{H}.\alpha,$$

when α is small. The product ml is called the **magnetic moment** of the magnet, and may be denoted by M; then, the restoring torque τ_1 per unit twist = MH.

Hence, if the moment of inertia of the magnet about its axis of rotation is equal to I, we have—

$$\text{Period of one oscillation} = \text{T} = 2\pi\sqrt{\frac{\text{I}}{\tau_1}} = 2\pi\sqrt{\frac{\text{I}}{\text{MH}}}.$$

It may be left as an exercise for the student to obtain the same result, using the energy equations (compare p. 94).

Observation of oscillations.—A simple harmonic oscillation is defined in terms of its amplitude and its period. The amplitude is the distance from the mean position (or position of rest) to the end of an excursion; the body moves very slowly near the end of an excursion, therefore there is, in general, no difficulty in observing its amplitude. Torsional oscillations may be observed with ease if a small mirror is attached to the oscillating body, and a beam of light is reflected from this mirror on to a scale. In this case the beam of light rotates through twice the angle of twist of the body; however, the period of oscillation of the spot of light on the scale is equal to the period of oscillation of the body.

In determining the period of oscillation, it is advisable to note the instant at which the body swings through its position of rest, and the instant at which it again swings in the same direction through this position after n oscillations have been performed.

For a rough determination of the period, a stop-watch may be started as the body swings through its position of rest, and stopped after n oscillations have been performed. If the period of oscillation is to be determined with an error not exceeding one per cent., the time between starting and stopping the watch should not be less than 100 sec. For accurate observation of the period of a pendulum, the "ear and eye" method may be employed. For this purpose a chronometer or other time-keeper, which beats half seconds audibly, is required. First of all, make a rough determination of the period, using a stop-watch. Then, using the chronometer, write down the hour and minutes, and start counting in time with the ticks. The tick which occurs as the second hand passes through 60 is called naught, and we then proceed to count as follows :—

tick, *tick*,	*tick*, *tick*,	*tick*, *tick*,	*tick*, *tick*,	. . .
naught—and,	one—and,	two—and,	three—and,	. . .
tick, *tick*,	*tick*, *tick*,	*tick*, *tick*,	*tick*, *tick*,	*tick*, *tick*,
ten—and,	'leven—and,	twelve—and,	thir—teen,	four—teen.

Continuing to count in this manner, watch the pendulum, and observe the instant at which it swings through its position of rest ; if, for instance, this occurs on the first syllable of "nineteen," the time is nineteen seconds, and if on the second syllable, nineteen and a half seconds from the instant at which we commenced to count.

At the end of about two minutes, again observe and write down the time at which the pendulum swings through its position of rest, using the ear and eye method as before ; the number of oscillations which have been completed between the two observations can be determined, by dividing the interval of time which has elapsed, by the approximate value of the period previously determined. In this manner continue to make observations at intervals of about two minutes, and write each observation down ; then the number of oscillations completed in (say) an hour can be determined accurately, without performing the laborious task of actually counting the oscillations completed in that time. A similar procedure should be followed in accurately determining the moment of inertia of a body by the oscillation method.

QUESTIONS ON CHAPTER III.

1. Draw curves to show the value of the displacement, the velocity, and the acceleration at each instant during a simple harmonic oscillation.

2. Prove the truth of the following statement:—When a bar pendulum is supported so that its period of oscillation has a minimum value, the distance from the point of support to the centre of gravity of the bar is equal to the radius of gyration of the bar about its centre of gravity.

3. A uniform hoop, of radius r, is supported on a thin horizontal peg; determine the period of oscillation of the hoop when it is displaced through a small distance in its own plane, and then released. (Assume that the hoop does not slip on the peg.)

4. A uniform rectangular sheet of metal is supported by frictionless hinges attached to one edge, which is horizontal; determine the period of oscillation of the sheet, if l denotes the length of the side of the rectangle which hangs downwards.

5. A uniform rectangular sheet of metal is supported at one corner, so that it can oscillate, like a pendulum, about an axis through that corner and perpendicular to the plane of the sheet. Determine the period of oscillation of the sheet.

6. A simple pendulum is formed by supporting a small metal sphere by means of a very thin steel wire, and the number of oscillations which would be performed in twenty-four hours, when the temperature has a certain value, is determined. How many oscillations would be lost in twenty-four hours if the temperature were to rise through 10° C?

(Coefficient of linear expansion of steel $= 0{\cdot}000012$ per degree C.)

7. A solid hemisphere, of radius r, rests with its curved surface on a horizontal plane. Prove that the hemisphere will be in stable equilibrium when its plane surface is horizontal, and determine the period of small oscillations about this position, on the assumption that the hemisphere does not slip on the plane.

(The distance from the centre of a solid hemisphere to its centre of gravity is equal to three-eighths of the radius of the hemisphere.)

8. A uniform rectangular bar, of length l, breadth b, and depth (or thickness) d, is supported like a see-saw on a cylinder of radius r. Prove that the bar will be in stable equilibrium when it is horizontal and its centre of gravity is vertically above the axis of the cylinder, provided that $(d/2) < r$; and determine the period of small oscillations about this position.

9. A body is supported by three similar flexible filaments, equally inclined to the vertical, and arranged symmetrically about a vertical axis passing through the centre of gravity of the body. Determine the value of the restoring torque per unit twist about the vertical axis, and the period of small angular oscillations about that axis.

10. A (*simple*) pendulum is hanging at rest, when its point of support is suddenly shifted through a horizontal distance *d*. Prove that the first swing of the pendulum will carry the bob through a distance $2d$ in the direction of the displacement of the point of support, and the pendulum will come to rest ultimately with the bob at a distance *d* from its initial position.

11. A body is executing simple harmonic oscillations of amplitude *a* and period T. Prove that the body will travel from its position of rest to a distance $(a/2)$, in the time $(T/12)$; and will reach its extreme displacement in an additional time $(T/6)$. In returning towards its position of rest, the first half of the journey will be completed in $(T/6)$ sec., and the second half in $(T/12)$ sec.

12. A pendulum, which can oscillate in a period T, is hanging at rest when its point of support is shifted suddenly through a horizontal distance *d*. After the lapse of $(T/6)$ sec., the point of support is shifted suddenly to its original position; and after another $(T/6)$ sec., the first displacement *d* of the point of support is repeated. Prove that, after the third displacement of the point of support, the pendulum is left hanging at rest, at a distance *d* from its initial position.

13. A galvanometer, whose needle can oscillate freely, is included in an electric circuit which can be closed or broken at will by means of a suitable key. It is necessary to observe the steady deflection of the needle as soon as possible after the electric current has commenced to flow; prove that the needle can be brought to rest in its deflected position in a time $(T/3)$ after the current first commences to flow, where T is the period of oscillation of the needle.

14. Prove that the period of oscillation of a compound pendulum will not be affected by fixing a small mass to the pendulum at its centre of oscillation.

15. A body is supported in bearings, so that it can oscillate freely like a compound pendulum. The distance of its centre of gravity from the axis of the bearings is required: the mass of the body is not known, and the body cannot be removed from its bearings. Prove that the required distance can be obtained, by the following procedure: (1) Determine the period of oscillation of the body. (2) After fixing a small mass at a distance L_1 from the axis, determine the period again. (3) Determine its period when the same mass is fixed at a distance L_2 from the axis.

CHAPTER IV

THE GYROSTATIC PENDULUM AND THE SPINNING TOP

The gyrostatic pendulum.—Let a rod be supported at one end, A (Fig. 45), by means of a universal joint, so that its other end, B, can move freely in any direction. To the end B of the rod let a fly-wheel be attached in such a manner that it can rotate freely about the length of the rod as axis. When the fly-wheel is not rotating, the suspended system may oscillate in a plane, thus forming an ordinary compound pendulum; or the lower end of the rod may move in a circular path, thus forming a compound conical pendulum. When the fly-wheel is rotating, the suspended system forms a gyrostatic pendulum.

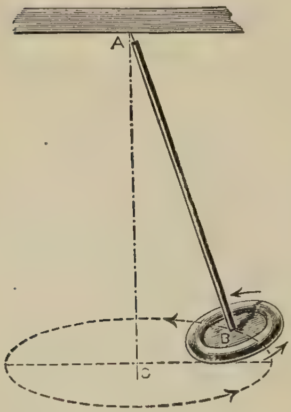

FIG. 45.—The gyrostatic pendulum.

Let it be supposed that the gyrostatic pendulum is set in motion in such a manner that the end B of the rod describes a circular path about the vertical line AC as axis, and let this circular path be described in an anti-clockwise sense when viewed from A ; also, let the rotation of the fly-wheel be in an anti-clockwise sense when viewed from A.

Gravity pulls each particle of the pendulum vertically downwards, and its action is equivalent to a single force, equal to the weight of the rod and fly-wheel, acting vertically downwards at the centre of gravity of the combination. This force, together with an equal force acting upwards at A, produces a torque, which will be called the **gravitational torque.** Since no other external forces act on the suspended system, it follows that the gravitational torque must be equal to the resultant of the various torques which are necessitated by the motion of the rod and fly-wheel.

In order that any particle of the pendulum may move in a circle about the vertical axis AC, it must be acted upon by a force directed towards the centre of its circular path ; this force, together with an equal force acting in an opposite direction at A, constitutes a torque. The resultant of the torques of this kind, due to the centripetal forces which must act on the various particles into which the pendulum may be supposed to be divided, will be called the **centripetal torque.** This torque is independent of the speed of rotation of the fly-wheel.

As the end B of the rod moves in its circular path, the direction of the rod continually alters ; and since the axis of the rod forms the axis of rotation of the fly-wheel, it follows that the direction of the axis of rotation of the fly-wheel continually alters. This necessitates a torque which will be called the **gyrostatic torque.**

No other torques are necessitated by the motion of the pendulum, and therefore it becomes evident that **the gravitational torque must be equal to the resultant of the centripetal and gyrostatic torques.**

The gyrostatic torque can be determined from the results obtained on p. 68. At a given instant let the axis of the pendulum be in the plane of the paper, in the position represented in Fig 46, *a*. To an eye looking from a distance, in a direction perpendicular to the plane of the paper, each particle in the rim of the wheel will appear to be moving in a straight line. Shortly afterwards the centre of the fly-wheel will be at a small distance behind the plane of the paper, and each particle of the rim of the wheel will appear to move in an anti-clockwise sense round an ellipse (Fig. 46, *b*). Hence, by Lanchester's rule (p. 70) the gyrostatic torque must act in an anti-clockwise sense about an axis perpendicular to the plane of the paper. It is clear that, as the centre of the fly-wheel passes through the plane of the paper, the wheel is turning about the diameter which lies in the plane of the paper.

Let I be the moment of inertia of the wheel about its axis of rotation, and let the wheel be rotating with an angular velocity ω; then when the wheel is turning about a diameter with an angular velocity ω', the value of the gyrostatic torque is equal to $I\omega\omega'$ (p. 68).

Let the centre of the fly-wheel describe its circular path in a time T; then if the length of the rod is equal to l, and the angle $BAC = \theta$, the radius of the circular path of the centre of the wheel is equal to $l\sin\theta$, and the linear velocity of the centre of the wheel is equal to $(2\pi l\sin\theta/T)$. In a very small time t the centre of the wheel travels from the plane of the paper through a distance $(2\pi l\sin\theta/T)t$, and its path approximates to a short straight line perpendicular to the plane of the paper. If we divide the length of this line by the length l of the rod, we obtain the angle through which the rod has rotated about A. Hence in the time t the rod has turned through an angle $(2\pi\sin\theta/T)t$, and therefore its angular velocity is equal to $(2\pi\sin\theta/T)$. The rod always remains perpendicular to the plane of the fly-wheel, so that the angular velocity of the rod, about A as centre, is

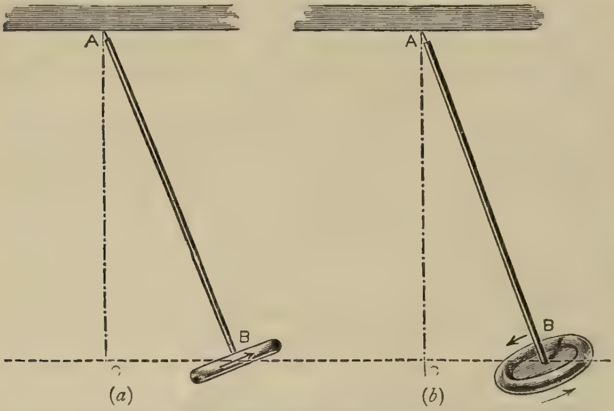

FIG. 46.—Method of determining the gyrostatic torque necessitated by the conical motion of a gyrostatic pendulum.

equal to the angular velocity ω' with which the fly-wheel turns about its diameter. Thus when θ is small—

$$\omega' = \frac{2\pi\theta}{T},$$

$$\therefore \quad \textbf{gyrostatic torque} = I\omega\frac{2\pi\theta}{T}.$$

Under the conditions indicated in Fig. 46, the gyrostatic torque acts in an anti-clockwise sense about an axis perpendicular to the plane of the paper. The sense in which it acts would be reversed by reversing the sense of the circular motion of B.

In determining the value of the **centripetal torque**, it will be assumed, in the first place, that the fly-wheel is small, and that the rod is practically without mass. The centre of the wheel revolves in a circular orbit of radius $l\sin\theta$, or $l\theta$ when θ is small, and a revolution is completed in the time T; thus the centripetal force is equal to $ml\theta(2\pi/T)^2$, (p. 74) where m is the mass of the fly-wheel. The perpendicular let fall from A on to this force is equal to $l\cos\theta$, or l when θ is small, so that—

$$\textbf{centripetal torque} = ml^2\theta\left(\frac{2\pi}{T}\right)^2.$$

Under the conditions indicated in Fig. 46, the centripetal torque must act in a clockwise sense about an axis perpendicular to the plane of the paper; its direction is unchanged by reversing the sense of the circular motion of B about the vertical axis AC.

When the weight of the rod is neglected, the vertical force of gravity, equal to mg, acts at the centre of the fly-wheel; and the perpendicular, let fall on to this force from A, is equal to $l\sin\theta$, or $l\theta$ when θ is small; hence—

$$\textbf{gravitational torque} = mgl\theta.$$

This torque acts in a clockwise sense about an axis perpendicular to the plane of the paper (Fig. 46); its direction would be unchanged by reversing the direction of the circular motion of B.

Let the moment of inertia I of the fly-wheel, about its axis of rotation, be denoted by mk_1^2 where k is the radius of gyration of the fly-wheel. Then, remembering that the gravitational torque is equal to the resultant of the centripetal and gyrostatic torques, both of which act about axes perpendicular to the plane of the paper, but in opposite senses, we have—

$$mgl\theta = ml^2\theta\left(\frac{2\pi}{T}\right)^2 - mk^2\omega \cdot \frac{2\pi\theta}{T};$$

$$\therefore \quad gl = l^2\left(\frac{2\pi}{T}\right)^2 - k^2\omega\left(\frac{2\pi}{T}\right).$$

If the motion of the pendulum about the vertical axis AC had been clockwise instead of anti-clockwise, the gyrostatic torque would have acted in the opposite sense, that is, its sign would have been changed; in this case we should have obtained the equation—

$$gl = l^2\left(\frac{2\pi}{T}\right)^2 + k^2\omega\left(\frac{2\pi}{T}\right).$$

Hence, the general form of the equation will be—

$$gl = l^2\left(\frac{2\pi}{T}\right)^2 \mp k^2\omega\left(\frac{2\pi}{T}\right),$$

where **the minus sign must be prefixed to the second term on the right hand side, when the circular motion of the pendulum and the rotation of the fly-wheel are both clockwise or both anti-clockwise; while the positive sign must be prefixed when one is clockwise and the other is anti-clockwise.**

Write p for $(2\pi/T)$; then—

$$p^2 \mp \frac{k^2}{l^2}\omega \cdot p = \frac{g}{l};$$

$$\therefore \quad p = \pm\frac{k^2\omega}{2l^2} \pm \sqrt{\frac{g}{l} + \left(\frac{k^2\omega}{2l^2}\right)^2}.$$

When the fly-wheel is not rotating, $\omega = 0$, and in this case the pendulum becomes a simple conical pendulum, for which $p = \sqrt{(g/l)}$. Thus the second term on the right-hand side must be preceded by the positive sign, and we obtain—

$$p = \frac{2\pi}{T} = \pm\frac{k^2\omega}{2l^2} + \sqrt{\frac{g}{l} + \left(\frac{k^2\omega}{2l^2}\right)^2} \quad . \quad . \quad . \quad (1)$$

When the circular motion of the pendulum and the rotation of the fly-wheel are both anti-clockwise, or both clockwise, the positive sign must be prefixed to the first term on the right-hand side ; when one is clockwise and the other is anti-clockwise, the negative sign must be prefixed.

When the positive sign is prefixed to the first term on the right-hand side of (1), the value of p will be greater than when the negative sign is prefixed ; and since T, the period of the conical motion of the pendulum, is inversely proportional to p, it follows that **when the fly-wheel moves round its circular path in the same sense as that in which it rotates, the period of its motion is less than when the sense of the circular motion is opposite to that of the rotation.**

Let it be supposed that the plane of the diagram (Fig. 47) is horizontal, and that the pendulum is hanging vertically with the centre of the fly-wheel over the point C ; and let it be supposed that an impulse is imparted to the pendulum which would cause the centre of the fly-wheel to oscillate between K and K′ if the fly-wheel were not rotating. A simple harmonic motion of amplitude CK, is equivalent to two circular motions of radius (CK/2), performed in opposite senses (p. 88). Thus the effect of the impulse imparted to the pendulum is equivalent to the resultant of two circular motions, one starting from L in an anti-clockwise sense, and the other starting from M in a clockwise sense ; where CL=(CK/2). If the fly-wheel is rotating in an anti-clockwise sense, viewed from above, the anti-clockwise circular motion will be performed in a shorter period than the clockwise circular motion ; let it be supposed that the period of the anti-clockwise circular motion, which starts from L, is equal to three-quarters of the period of the clockwise motion, which starts from M. Then, when the anti-clockwise displacement is equivalent to the circular arc LN, the clockwise displacement is equivalent to the circular arc MR, where MR =(3/4) × LN. From N draw NP parallel and equal to CR ; then the point P gives the instantaneous position of the centre of the fly-wheel. The position of the centre of the fly-wheel at other instants is deter-

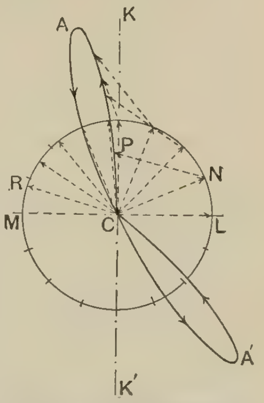

FIG. 47. — Graphical method of determining the resultant of two circular motions of unequal periods.

mined in a similar manner; the construction lines for five points are shown in Fig. 47, and it becomes evident that the centre of the fly-wheel describes a series of loops similar to CACA′C. The complete series of loops is represented in Fig. 48; the order in which the loops are described is AA′, BB′, CC′, DD′, . . . HH′, the direction of motion being that indicated by the arrow-heads.

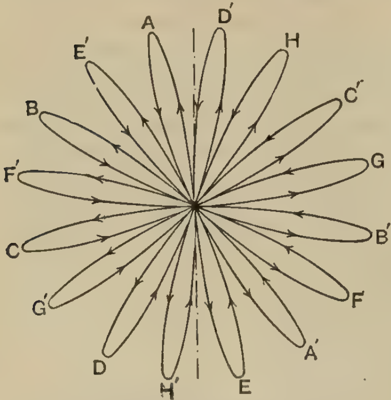

Fig. 48.—Graph of the path described by a gyrostatic pendulum.

The compound gyrostatic pendulum.—In the preceding investigation the mass of the rod was neglected, and it was supposed that the fly-wheel is small; in this case, when the fly-wheel is not rotating, the system forms a simple pendulum. In the following investigation these restrictions are discarded; the only condition imposed on the shape and dimensions of the pendulum is that of symmetry about the axis of rotation. Thus the pendulum may take the form of an ordinary peg-top suspended by means of a universal joint at the extremity of the peg; or it may take the form of a large fly-wheel which can rotate about the axis of a massive rod.

If I denotes the moment of inertia of the rotating part of the pendulum about its axis of rotation, the **gyrostatic torque** has the value already found, viz., $I\omega(2\pi\theta/T)$, where ω denotes the angular velocity of rotation, T denotes the period of the conical motion, and θ denotes the small angle of inclination of the axis of symmetry to the vertical. If the conical motion and the rotation are performed in the senses indicated in Fig. 46, the gyrostatic torque will act in an anti-clockwise sense about an axis perpendicular to the plane of the diagram.

Let M be the mass of the pendulum as a whole, and let the centre of gravity of the pendulum be at a distance L from the point of support A (Fig. 46). Then the **gravitational torque** about A is equal to $MgL \sin \theta$, or $MgL\theta$ when θ is small. In the position of the pendulum represented in Fig. 46, the torque acts in a clockwise sense about an axis perpendicular to the plane of the paper.

The centripetal forces, which must act on the particles into which the pendulum may be supposed to be divided, in order that these particles may revolve about the vertical axis AC, produce a resultant torque in a clockwise sense about an axis perpendicular to the paper. The direct determination of the torque, in terms of the centripetal forces acting on the particles, is a matter of some difficulty, owing to the fact that the particles are at various distances from the axis AC. Thus it is best to determine the centripetal torque by an indirect method.

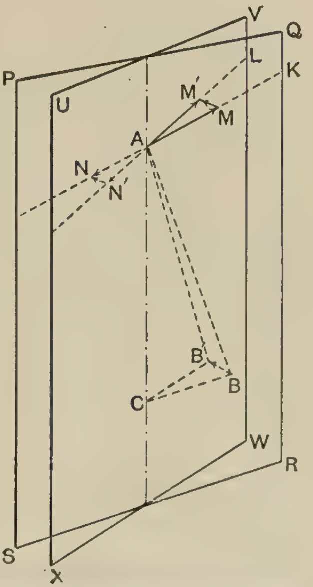

Fig. 49.—Method of determining the resultant centripetal torque necessitated by the motion of a compound gyrostatic pendulum.

Let PQRS and UVWX (Fig. 49) represent two vertical planes which intersect at a small angle in the vertical line AC; and let A be the point of support of the pendulum, while AB represents the axis of the pendulum at the instant when it passes through the plane PQRS. After a small time t, let the axis of the pendulum pass through the plane UVWX, the position of the axis at this instant being denoted by AB′. Since the lower end of the pendulum completes its circular path about the vertical axis AC in T seconds, the small arc BB′ is equal to 2π AB $\sin \theta \, (t/T)$; and since BB′ approximates to a small straight line when t is very small, the angle BAB′, through which the axis turns about A in the time t, is equal to $BB'/AB = 2\pi \sin \theta \, (t/T)$, or $(2\pi\theta t/T)$ when the angle $BAC = \theta$ is small. Hence, as the axis of the pendulum passes through the plane PQRS, it is rotating about A with an angular velocity equal to $2\pi\theta/T$. In the plane PQRS, draw the line AK from A, perpendicular to the line AB; then AK represents the axis about which the axis of symmetry of the pendulum is rotating as it passes through the plane PQRS. In the plane UVWX, draw the line AL from A, perpendicular to AB′: then AL represents the axis about which the axis of symmetry of the pendulum is rotating as it passes through the plane UVWX.

It must be remembered that we are not concerned, at present, with the rotation of the fly-wheel about the axis of symmetry of the pendulum; in determining the value of the centripetal torque we may suppose that the fly-wheel is not rotating at all.

Let the moment of inertia of the pendulum about an axis through A, perpendicular to the axis of symmetry, be denoted by I′; then, when the pendulum is passing through the plane PQRS, the moment of momentum of the pendulum about the axis AK is equal to $I'(2\pi\theta/T)$, (p. 61), and may be represented by the vector AM, equal in length to $I'(2\pi\theta/T)$, measured off along AK. After the small interval of time t, the moment of momentum of the pendulum may be represented by the vector AM′, equal in length to AM, measured off along AL. Thus, in the time t the moment of momentum of the pendulum changes from AM to AM′, and therefore the rate of change of the moment of momentum is represented by MM'/t, where MM′ is the vector drawn from M to M′ (p. 68). Since MM′ extends perpendicularly from front to back of the plane PQRS, and the rate of change of moment of momentum is equal to the torque that must be applied in order to produce that change (p. 61), it follows that the applied torque must be equal to (MM'/t), and must act in a clockwise direction about a normal to the plane PQRS; this applied torque is the centripetal torque which is required.

If the conical motion of the axis of symmetry were executed in a clockwise sense when viewed from A, the axis of symmetry would pass from AB′ to AB in the time t; in this case the moment of momentum would change from AN′ in the plane UVWX to AN in the plane PQRS, and the rate of change of the moment of momentum would be equal to $(N'N/t)$, which is equal to MM'/t both in magnitude and direction. Hence the centripetal torque is independent of the sense in which the conical motion is performed.

Since the angle BAC $=\theta$ is small, AM and AM′ will be very nearly perpendicular to AC, and therefore the angle MAM′ will not differ perceptibly from the angle BCB′, that is, from $(2\pi t/\mathrm{T})$. Thus—

$$\mathrm{MM}' = \mathrm{AM} \times \text{angle } \mathrm{MAM}',$$

and

$$\frac{\mathrm{MM}'}{t} = \mathrm{I}'\frac{2\pi\theta}{\mathrm{T}} \cdot \frac{2\pi t}{\mathrm{T}} \div t = \mathrm{I}'\left(\frac{2\pi}{\mathrm{T}}\right)^2\theta,$$

and therefore

$$\textbf{centripetal torque} = \mathrm{I}'\left(\frac{2\pi}{\mathrm{T}}\right)^2\theta.$$

From a cursory reading of the argument developed on p. 69, it might appear that the centripetal torque ought to have a value twice as great as that just found. The following reasoning proves that the result obtained above is correct.

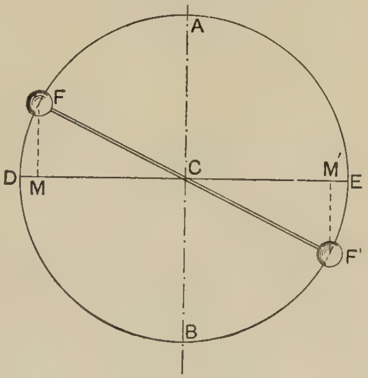

Fig. 50.—Two particles revolving about the point C in a plane which turns about the fixed line ACB.

Let F, (Fig. 50)[1] represent a particle revolving in a clock-wise sense, with an angular velocity ω, about an axis drawn through C, perpendicular to the plane of the paper; and let the plane in which the particle is revolving be turning, with an angular velocity ω', about the fixed line ACB, in a clockwise sense when viewed in the direction from A to B.

Draw CD perpendicular to AB. Then at the instant when the particle is at F, the forces which are necessary to constrain it to move in the manner described are as follows:—

(1) The centripetal force, acting toward the centre C about which the particle is revolving; this force does not concern us here.

(2) As the particle approaches A, its distance from the axis AC diminishes; therefore the linear velocity with which the particle revolves around AC diminishes, and this necessitates the application of

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[1] This diagram is a duplicate of Fig. 22, reproduced on p. 69.

a force, equal to $m.\mathrm{FM}\omega\omega'$, acting perpendicularly from back to front through the plane of the paper (p. 65); where m denotes the mass of the particle, and FM is its distance from the line CD. The torque about the axis CD due to this force is equal to $m\,(\mathrm{FM})^2.\ \omega\omega'$. When the particle is passing through A the value of this torque is equal to $m\ (\mathrm{CA})^2\ \omega\omega'$, and $m\ (\mathrm{CA})^2$ is equal to the moment of inertia of the particle about an axis through C perpendicular to the plane of the paper.

(3) The plane in which the particle revolves is rotating about AC as axis, and this necessitates a change in the direction of motion of the particle (p. 67). To produce this change, a force equal to $m.\mathrm{FM}\omega\omega'$ must act perpendicularly from back to front through the plane of the paper, and the corresponding torque is equal to $m.(\mathrm{FM})^2\omega\omega'$. When the particle is passing through A the value of this torque is equal to $m\,(\mathrm{CA})^2.\ \omega\omega'$.

The results obtained in (2) and (3) can be summarised as follows. The total torque that must act on the particle, when it is in the immediate neighbourhood of the point A, comprises two equal components, one necessitated by the bodily motion of the particle toward the axis AC (this axis may be called the *axis of turning*) and the other by the change in the direction of the plane in which the particle revolves, that is, by the change in the direction of the axis about which the particle revolves (this axis may be called the *axis of rotation*). Each component is equal to $i\ \omega\omega'$, where i is the moment of inertia of the particle about the axis of rotation.

With reference to the pendulum, the lines AK and AL (Fig. 49) represent the positions of the axis of rotation at the beginning and end of the small interval of time t. As the pendulum passes through the plane PQRS, it is rotating about AK, and it is therefore moving in a plane drawn through AB, perpendicular to the plane PQRS. At the instant when the pendulum passes through the plane UVWX it is rotating about AL, and it is therefore moving in a plane drawn through AB′ perpendicular to the plane UVWX. Hence AB represents the axis of turning when the pendulum passes through the plane PQRS, and AB′ represents the axis of turning when the pendulum passes through the plane UVWX; that is, the axis of turning is not fixed, as the line AC (Fig. 50) was supposed to be, but it moves with the pendulum. Now the axis of rotation of the pendulum is continually changing in direction, and this change necessitates the application of the component torque discussed in (3) above. But *the particles, into which the pendulum may be supposed to be divided, do not move towards, or away from, the axis of turning*, for this axis moves with the pendulum; therefore the component torque, discussed in (2) above, is not necessitated. Thus the torque required is equal to I′, the moment of inertia of the pendulum about the axis AK (Fig. 49), multiplied by the product of the angular velocity of rotation about the axis AK (that is, $2\pi\theta/T$), and the angular velocity with which the axis of rotation is turning about A (that is, $2\pi/T$).

The gravitational torque is equal to the resultant of the gyrostatic and centripetal torques; thus, when the conditions are those represented in Fig. 46—

$$MgL\theta = I'\left(\frac{2\pi}{T}\right)^2\theta - I\omega\frac{2\pi\theta}{T}.$$

If the sense of the conical motion is opposite to that of the rotation of the fly-wheel, the sign of the second term on the right-hand side must be changed; hence, in general—

$$MgL = I'\left(\frac{2\pi}{T}\right)^2 \mp I\omega \,.\, \frac{2\pi}{T}.$$

Let $(2\pi/T)=p$; then—

$$p^2 \mp \frac{I\omega}{I'}p = \frac{MgL}{I'};$$

$$\therefore \quad p = \pm\frac{I\omega}{2I'} + \sqrt{\frac{MgL}{I'} + \left(\frac{I\omega}{2I'}\right)^2}.$$

The interpretation of this equation is similar to that given on p. 126. The sign of the first term on the right-hand side must be positive when the conical motion and the rotation of the fly-wheel are identical in sense, and the negative sign is used when the conical motion and the rotation of the fly-wheel are opposite in sense.

When $\omega = 0$, the pendulum becomes a compound conical pendulum. Let MK^2 be the moment of inertia of the pendulum about an axis through its centre of gravity, and perpendicular to the axis of symmetry. Then $I' = M(K^2 + L^2)$ (p. 56) and

$$p = \frac{2\pi}{T} = \sqrt{\frac{gL}{K^2+L^2}},$$

which agrees with the result obtained for the ordinary compound pendulum on p. 98.

The spinning top.—Let it be supposed that an ordinary peg-top is spinning with the tip of its peg in a conical hole

(Fig. 51); in these circumstances the top becomes an inverted gyrostatic pendulum. If the axis of symmetry of the top is inclined to the vertical, its inclination will remain constant so long as the angular velocity of rotation of the top is constant; the axis of symmetry will not remain stationary, but will describe a conical surface.

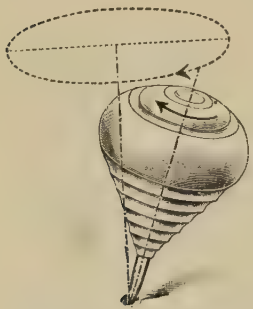

FIG. 51.—Top spinning with the tip of its peg in a conical hole.

In this case, as in the previous investigation, the gravitational torque must be equal to the resultant of the centripetal and gyrostatic torques necessitated by the motion of the top.

If M denotes the mass of the top, and L denotes the distance of its centre of gravity from the tip of the peg, the **gravitational torque** is equal to $MgL\theta$. This torque tends to deflect the axis of the top away from the vertical; let it be agreed that torques tending to deflect the axis from the vertical shall be positive, while those tending to deflect it towards the vertical shall be negative.

The centripetal torque is due to the forces which must pull the various particles of the top towards the vertical axis about which the conical motion is executed; this torque tends to deflect the axis towards the vertical, and must therefore be negative. The value of the centripetal torque is found by a method precisely similar to that used on p. 131; it may be left as an exercise to the student to make the necessary modifications in Fig. 49. Hence, it follows that the **centripetal torque** is equal to—

$$-I'\left(\frac{2\pi}{T}\right)^2\theta,$$

where I′ denotes the moment of inertia of the top about an axis through the tip of the peg, and perpendicular to the axis of symmetry, and T is the period of the conical motion of the axis.

The gravitational torque, which is positive, is equal to the resultant of the centripetal and gyrostatic torques; therefore since the centripetal torque is negative, it follows that *when the motion of the top is steady, the gyrostatic torque must be positive.*

Applying Lanchester's rule (p. 70), it is found that the upper end of the axis of symmetry must travel round a circle in the sense in which the top rotates about its axis (Fig. 51).

Let I denote the moment of inertia of the top about its axis of symmetry, while ω denotes the angular velocity with which the top is rotating about that axis ; then, by a reasoning similar to that used on p. 125, it follows that the **gyrostatic torque** is equal to—

$$\mathrm{I}\omega\frac{2\pi}{\mathrm{T}}\theta.$$

Then

$$\mathrm{M}g\mathrm{L}\theta = -\ \mathrm{I}'\left(\frac{2\pi}{\mathrm{T}}\right)^2\theta + \mathrm{I}\omega\ .\ \frac{2\pi}{\mathrm{T}}\ .\ \theta.$$

Let MK^2 be equal to the moment of inertia of the top about an axis, perpendicular to the axis of symmetry, and passing through the centre of gravity of the top. Then, (p. 56)—

$$\mathrm{I}' = \mathrm{M}(\mathrm{K}^2+\mathrm{L}^2).$$

Also, let $\mathrm{M}k^2=\mathrm{I}$, the moment of inertia of the top about its axis of symmetry. Then–

$$\mathrm{M}g\mathrm{L}\theta = -\ \mathrm{M}(\mathrm{K}^2+\mathrm{L}^2)\left(\frac{2\pi}{\mathrm{T}}\right)^2\theta + \mathrm{M}k^2\ .\ \omega\ .\ \frac{2\pi}{\mathrm{T}}\ .\ \theta.$$

Dividing through by M and θ, re-arranging the terms, and writing p for $2\pi/\mathrm{T}$, we obtain the equation—

$$p^2 - \frac{k^2\omega}{\mathrm{K}^2+\mathrm{L}^2}p = -\frac{g\mathrm{L}}{\mathrm{K}^2+\mathrm{L}^2};$$

$$\therefore\ \ p = \frac{k^2\omega}{2(\mathrm{K}^2+\mathrm{L}^2)} \pm \sqrt{\left\{\frac{k^2\omega}{2(\mathrm{K}^2+\mathrm{L}^2)}\right\}^2 - \frac{g\mathrm{L}}{\mathrm{K}^2+\mathrm{L}^2}}\ \ldots\ (1)$$

Now, the expression under the radical sign must be positive, for the steady motion of the top to be possible ; hence—

$$\left\{\frac{k^2\omega}{2(\mathrm{K}^2+\mathrm{L}^2)}\right\}^2 > \frac{g\mathrm{L}}{\mathrm{K}^2+\mathrm{L}^2},$$

and therefore

$$\omega > \sqrt{\frac{4(\mathrm{K}^2+\mathrm{L}^2)g\mathrm{L}}{k^4}}$$

If this condition is not complied with, the axis of symmetry will fall away from the vertical until the side of the top touches the ground. It follows that the taller and more slender a top is, the greater will be the angular velocity necessary to make it spin steadily; for when the top is small and slender, $(K^2+L^2)L$ will be large in comparison with k^4, and therefore ω will be large.

When the angular velocity of rotation is greater than the critical value given above, there are two possible values of p, and therefore two possible values for the period T of the conical motion of the axis. If we imagine that a top is spun at a high speed, and released with its axis slightly inclined to the vertical, at first the axis will begin to fall away from the vertical, and the angular velocity p of the precessional motion will increase (p. 72) until it attains the smaller of the two values given by equation (1) above; that is, the value of p obtained by prefixing the negative sign to the second term on the right-hand side of (1). If the velocity of rotation of the top remained constant, no further change would occur in the value of p; hence it appears that for the larger value of p (equation (1)) to be possible, an enhanced precessional motion would have to be imposed on the top by some external agency.

The smaller of the two values of p may be expressed in the form—

$$p = \frac{k^2\omega}{2(K^2+L^2)}\left\{1 - \sqrt{\left(1 - \frac{4(K^2+L^2)gL}{k^4\omega^2}\right)}\right\}.$$

When the second term under the radical sign is small in comparison with unity, (that is, when ω is very large) we may expand the quantity under the radical sign by the aid of the binomial theorem, and neglect powers of the small quantity that are higher than unity. We then find that—

$$p = \frac{k^2\omega}{2(K^2+L^2)} \cdot \frac{2(K^2+L^2)gL}{k^4\omega^2}$$

$$= \frac{gL}{k^2\omega}.$$

This result gives the angular velocity of the precessional motion when the top is spinning at a speed much in excess of the critical value necessary for steady motion.

Top spinning on a flat horizontal surface.—In carrying out the above investigation, it was assumed that the top is spinning with the tip of its peg in a conical hole; in these circumstances the top can spin freely about its axis, and the axis can execute a conical motion about the vertical, but the tip of the peg cannot move horizontally. When a top is spun on a flat horizontal surface it is generally found that the tip of the peg moves around a circle on the surface (Fig. 52); this is due partly to the fact that the peg does not end in a geometrical point, but in a small curved surface which approximates to a part of a sphere (Fig. 53). When the axis of the top is inclined to the vertical, the point of contact of the peg and the flat surface on which it rests is at some distance from the axis of rotation of the top; consequently the end of the peg rolls along the surface, as indicated in Figs. 52 and 53.

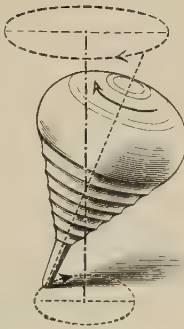

FIG. 52.—Top spinning on a horizontal plane.

If the top is spinning at a high speed, the rolling motion at the tip of the peg produces an acceleration in the precessional motion of the axis: and it is easily seen that this acceleration of the precessional motion makes the axis approach the vertical until the top "goes to sleep." When a top spins with its axis at a constant inclination to the vertical, the angular velocity of the precessional motion must have the value obtained above. If the angular velocity of the precessional motion is retarded, the axis of the top falls away from the vertical: if the angular velocity of the precessional motion is stopped, the top falls freely, just as it would if it were not spinning (p. 72). On the other hand, if the angular velocity of the precessional motion is greater than that required for the steady motion of the top, the axis of the top must approach the vertical.

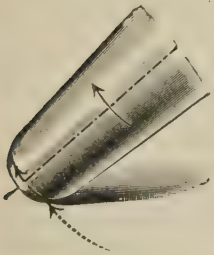

FIG. 53.—Peg of a top rolling on a plane.

If there were no friction between the peg and the horizontal plane on which it rests, the rolling motion would not occur, but the tip of the peg would still move in a circle. In this case the plane could exert no horizontal force on the tip of the peg, and therefore the centre of gravity of the top could not describe a circular orbit, for there would be no agent to pull the centre of gravity towards the centre of the orbit. Thus the top would spin and precess with its centre of gravity in a fixed position; in this case the axis would remain at a constant inclination to the vertical, and the top would not "go to sleep."

Disc rolling on a horizontal plane.—When a thin disc rolls, without slipping, on a horizontal plane, a point on the rim of the disc can suffer no horizontal displacement so long as it is in contact with the plane. At any instant, two neighbouring points on the rim of the disc will be in contact with the plane; as the disc rolls through a small angle, one point will remain in contact with the plane, while the other will rise vertically from it; simultaneously another point on the rim will descend vertically and come into contact with the plane. As a consequence, a disc can roll in a straight line only when its plane is vertical; when its plane is inclined to the vertical, the disc must roll in a curved path which will now be determined.

Let a number of straight lines be drawn from the apex of a right circular cone to points on the circumference of its base (Fig. 54); these lines are called generating lines. When a cone lies with its side on a horizontal plane, two neighbouring generating lines are in contact with the plane; when the cone rolls through a small angle, without slipping, its apex must remain in a fixed position; one generating line must remain in contact with the plane, while the other rises from it, the motion of each point on the moving generating line being in a vertical direction; simultaneously another generating line must descend and come into contact with the plane. The cone may be divided, in imagination, into an indefinitely large number of circular discs; and when the cone rolls without slipping, each of its constituent discs must also roll without slipping; each generating line of the cone cuts the rim of a disc in a point, and it is obvious that the conditions already deduced for the rolling of a disc are complied with. Thus, **when a disc, whose plane is inclined to the vertical, rolls without slipping, on a plane, the motion of the disc is the same as if it were the base of a cone which rolls with its curved surface in contact with the plane**; the angle of the cone must be such that the inclination of its base to the plane is equal to that of the disc.

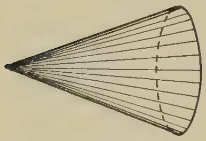

FIG. 54.—Perspective view of right circular cone, showing generating lines.

Let a circular disc, of radius r, roll on a horizontal plane; at a given instant let the centre A of the disc be moving in a perpendicular direction through the vertical plane PQRS (Fig. 55). The axis AC of the disc must lie in the plane PQRS; let it cut the horizontal plane, on which the disc rolls, in the point C. Then the motion of the disc is the same as if it were the base of a right circular cone with apex at C. After a short interval of time t, let the centre of the disc be passing through B, a point in the vertical plane UVWX which cuts PQRS in the vertical line KL; then KL must pass through C, and BC will represent the new position of the axis of the disc.

Let $AC = BC = l$; then if v denotes the linear velocity of the centre of the disc, the arc AB is equal to vt, and the angle ACB, through which the axis of the disc has turned in the time t, is equal to (vt/l); thus, the angular velocity with which the axis of the disc is turning about the point C is equal to (v/l). Let the disc be inclined to the vertical at a small angle θ; then $\tan\theta = \theta = (r/l)$. The arc DE, described on the horizontal plane by the rim of the disc, is sensibly equal to AB; and the angle DCE, at which the planes PQRS and UVWX intersect, is approximately equal to (vt/l). Further, the angular velocity of rotation of the disc about the axis AC is equal to (v/r).

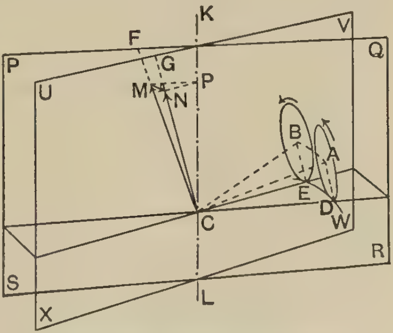

FIG. 55.—The rolling motion of a disc on a horizontal plane.

Let mk^2 be the moment of inertia of the disc about the axis AC; then the disc is rotating about AC with an angular velocity equal to (v/r), and it is turning about the diameter, in which it is cut by the plane PQRS, with an angular velocity equal to (v/l); thus—

$$\textbf{gyrostatic torque} = mk^2 \cdot \frac{v}{r} \cdot \frac{v}{l}.$$

This torque must act in an anti-clockwise sense about an axis perpendicular to the plane PQRS.

In the plane PQRS, draw CF perpendicular to AC; and in the plane UVWX, draw CG perpendicular to BC. As the centre of the disc passes through the plane PQRS, the moment of momentum of the disc about the axis CF, is equal to—

$$m\left(\frac{k^2}{2} + l^2\right)\frac{v}{l};$$

for the moment of inertia of the disc about a diamenter parallel to CF is equal to $(mk^2/2)$, and the moment of inertia about CF is equal to $m\{(k^2/2) + l\}$, (p. 56). Now k must be less than r, the radius of the disc; and therefore when $\theta = (r/l)$ is small, $(k^2/2)$ must be negligibly small in comparison with l^2. Thus, when θ is small, the moment of momentum of the disc about the axis CF, is equal to $ml^2(v/l)$.

Along the lines CF and CG, measure off distances CM and CN numerically equal to $ml^2(v/l)$; then in the time t, the moment of momentum changes from CM to CN; therefore the change in the value of the moment of momentum is equal to MN (p. 61). The torque required to produce this change in the moment of momentum is equal to MN/t; and since MN is perpendicular to the plane PQRS, and is directed from back to front of that plane, it follows that the torque must act in an anti-clockwise sense about an axis perpendicular to PQRS.

To evaluate MN/t, draw MP and NP perpendicular to the vertical line KL. Then $PN = CN \sin NCP = CN . \theta$, when θ is small. The angle MPN is equal to the angle DCE, that is, to vt/l. Thus, $MN = PN \times \angle MPN = (CN . \theta . vt/l)$, and $MN/t = CN . \theta . (v/l)$. Then, remembering that $\theta = r/l$, it follows that—

$$\textbf{centripetal torque} = ml^2\frac{v}{l} . \frac{r}{l} . \frac{v}{l} = mv^2 . \frac{r}{l}.$$

The only applied forces which exert a torque about the point C, are (1) the downward pull of gravity mg, exerted at the centre of the disc; and (2) the equal force, acting vertically upwards at the point D on the rim of the disc. The resultant torque about C may be called the gravitational torque; then—

$$\textbf{gravitational torque} = mgr\theta = mgr . (r/l).$$

This torque acts in an anti-clockwise sense, about an axis perpendicular to the plane PQRS.

Since the applied torque must be equal to the resultant of the torques necessitated by the motion of the disc, it follows that—

$$mgr\frac{r}{l} = mk^2\frac{v}{r}\frac{v}{l} + mv^2.\frac{r}{l}.$$

$$\therefore\quad gr = \frac{k^2}{r^2}v^2 + v^2 = v^2\left(\frac{k^2}{r^2} + 1\right).$$

In this equation, v is the only unknown quantity; and—

$$v = \sqrt{\left\{gr\Big/\left(\frac{k^2}{r^2} + 1\right)\right\}}.$$

Thus, **for a disc, inclined at a small angle to the vertical, to roll steadily along a horizontal plane, the velocity of the centre of the disc must have a definite value.** When the disc is uniform, $(k^2/r^2) = (1/2)$, and—

$$v = \sqrt{(2gr/3)}.$$

For a uniform circular hoop, $k^2 = r^2$, and—

$$v = \sqrt{(gr/2)}.$$

When the velocity of the disc has the critical value just determined, the disc will roll along the plane without any alteration in its inclination to the vertical; in this case, the gravitational torque is just equal to the resultant of the centripetal and gyrostatic torques necessitated by the motion of the disc. When v is greater than the critical value, the disc can only roll with its plane vertical; in this case the disc travels in a straight line, and each of the torques discussed above becomes equal to zero. When v is less than the critical value, the gravitational torque is greater than the resultant of the centripetal and gyrostatic torques, and the inclination of the disc to the vertical increases continually, while the radius of the path described by the disc decreases, until the disc finally falls on the plane. The phenomena discussed can be observed by rolling a coin along a flat table; when the speed of the coin is high, the plane of the coin remains vertical, and its path is a straight line; when the speed falls below a certain value, the coin travels along a spiral of constantly increasing curvature, and its plane continually falls away from the vertical; finally the coin falls flat on the table.

QUESTIONS ON CHAPTER IV

1. A gyrostatic pendulum comprises a thin rod, 100 cm. in length, weighing 500 gm.; a wheel is attached to one end of the rod, so that it can rotate about the rod as axis, and the mass of the wheel is equal to 500 gm., which may be considered to be at a uniform distance of 10 cm. from the axis of rotation. The end of the rod remote from the wheel is attached, by a universal joint, to a firm support. Calculate the period of a small conical oscillation of the pendulum, when the wheel is rotating about its axis at the rate of 600 rotations per minute.

2. A uniform circular disc, of 10 cm. radius, is spun as a top about a thin rod which projects axially through the centre of the disc to a distance of 10 cm. on either side. Calculate the smallest angular velocity which must be imparted to the disc in order that it may spin steadily with its axis inclined at a small angle to the vertical. At what rate will the axis precess when the disc spins at the rate of 10 rotations per second? (Neglect the mass of the rod.)

3. A wheel-barrow is to be wheeled round a curve; why is it advisable to tilt the barrow towards the concave side of the curve?

4. A bicycle of the old fashioned type, with a front wheel 6 ft. in diameter and a very small hind wheel, is to be ridden on a smooth horizontal road. What is the minimum speed at which this can be done, if the moment of inertia of the front wheel is equal to 200 lb. ft.^{2}, the mass of the rider and bicycle being equal to 200 lb., and the centre of gravity of the rider and bicycle being at a height of 6 ft. from the ground?

5. A paddle steamer is steered round a curve; prove that the gyrostatic properties of the paddle wheels will cause the steamer to tilt towards the convex side of the curve.

6. The speed of an engine can be prevented from fluctuating rapidly, by providing the engine with a massive fly-wheel. Why are such fly-wheels seldom provided for marine engines, although constant speed is as desirable in a marine as in a land engine?

7. Would a ship be prevented from pitching or rolling, if a fly-wheel were fitted, in the ordinary manner, to an engine carried by the ship? Discuss the possibility of preventing the rolling of a ship by means of a fly-wheel supported in a suitable manner.

CHAPTER V

GENERAL THEORY OF OSCILLATIONS

Reduction of observations.—Simple harmonic motion is the type of oscillatory motion which can be investigated mathematically with the greatest ease. In Chapter III, a number of problems, relating to oscillations, were solved by making certain assumptions in order to reduce the mechanical conditions to those appertaining to simple harmonic motion; it now becomes necessary to determine the magnitude of the errors thereby introduced into the results obtained. In general, we shall find that observations must be multiplied by certain factors, in order to allow for the discrepancy between the actual conditions under which an experiment can be performed, and the ideal conditions assumed in the simplified mathematical investigations; this process enables us to reduce observations made under experimental conditions to those corresponding to ideal conditions, and consequently is called the **reduction of observations** to ideal conditions.

Reduction for finite arc of swing.—In investigating the motion of the pendulum, it was assumed that the arc of swing is small (p. 95). In this case the restoring force called into play is approximately proportional to the displacement, and the smaller the arc the closer is the approximation; therefore the results obtained are absolutely true for infinitely small arcs, but only approximately true for finite arcs of swing. We cannot observe infinitely small arcs of swing, and therefore we must reduce observations made on finite arcs, to the conditions corresponding to infinitely small arcs.

Let A, Fig. 56, be the position of equilibrium of the bob, and let C be the point to which the suspending filament is attached. Let B be the position of the bob at the extremity of an oscillation, the angle of oscillation ACB being equal to α. In the absence of friction the bob will swing symmetrically about A, and it is obvious that the period of a complete oscillation is four times as great as the time which elapses while the bob travels from B, where it is for an instant stationary, to A, where it acquires its maximum velocity. Hence, we must calculate the time which elapses while the bob travels from B to A.

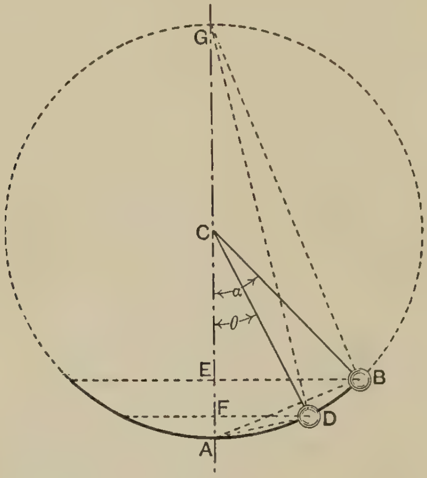

Fig. 56.—The motion of the bob of a simple pendulum.

Let us first determine the velocity acquired by the bob when it arrives at D, a point intermediate between B and A.

From B and D draw perpendiculars to CA, cutting that line in the points E and F. Then if $EF = h$, the kinetic energy of the bob when at D is equal to the potential energy lost in travelling from B to D, that is, to mgh; therefore, if the linear velocity of the bob has the value v, we have—

$$\tfrac{1}{2}mv^2 = mgh,$$

$$\therefore\ v = \sqrt{2gh}.$$

Let CB$=l$. Then, if $\angle$ACD$=\theta$,

$$h = l(\cos\theta - \cos\alpha)\,;$$

and writing $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, and $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$, we have—

$$h = 2l\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right).$$

Produce AC to G, where it cuts the circle drawn with C as centre and CA as radius; and join B and D to G. Then $\angle$AGB$=\alpha/2$, and $\angle$AGD$=\theta/2$. Join B and D to A; then each of the angles GBA and GDA is a right angle, and—

$$\mathrm{AB} = \mathrm{AG}\sin(\mathrm{AGB}) = 2l\sin\frac{\alpha}{2} \text{ while } \mathrm{AD} = 2l\sin\frac{\theta}{2}.$$

Thus—

$$h = \frac{1}{2l}\left\{\left(2l\sin\frac{\alpha}{2}\right)^2 - \left(2l\sin\frac{\theta}{2}\right)^2\right\} = \frac{1}{2l}\{(\mathrm{AB})^2 - (\mathrm{AD})^2\}.$$

With A as centre, and AD as radius, describe the arc DH cutting AB in H (to avoid confusion, this and the remaining parts of the construction are drawn separately in Fig. 57). Then—

$$h = \frac{1}{2l}\{(\mathrm{AB})^2 - (\mathrm{AH})^2\} = \frac{1}{2l}(\mathrm{AB}+\mathrm{AH})(\mathrm{AB}-\mathrm{AH})$$

$$= \frac{1}{2l}(\mathrm{AB}+\mathrm{AH})(\mathrm{HB}).$$

With A as centre, and AB as radius, describe the semicircle BKL, cutting BA (produced) in L. From H draw HK perpendicular to BA. Then

$$(\mathrm{HK})^2 = (\mathrm{LH}) \times (\mathrm{HB})$$
$$= (\mathrm{LA}+\mathrm{AH}) \times \mathrm{HB}$$
$$= (\mathrm{AB}+\mathrm{AH})(\mathrm{HB})—$$

$$\therefore\ h = \frac{1}{2l}(\mathrm{HK})^2,$$

and

$$v = \sqrt{2gh} = \mathrm{HK}\sqrt{\frac{g}{l}}.$$

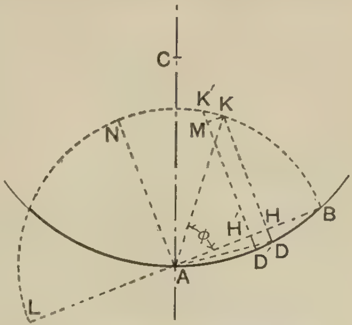

FIG. 57.—Details to be added to Fig. 56.

We have now obtained a geometrical construction, by means of which the exact velocity of the bob, at any point in its path, can be determined. Let D′ be the position of the bob a very short time after it passes through D;

then if DD′ is sufficiently small, the velocity of the bob will be uniform over the path DD′, and the time occupied in travelling from D to D′ can be found by dividing DD′ by the value of v obtained above.

Find the points H′ and K′ corresponding to D′, by the construction previously employed in obtaining H and K. *To a first approximation*, DD′ = HH′. Draw KM perpendicular to H′K′; then HH′ = DD′ = KM. Join K to A, and let ∠ BAK = ϕ. Then the approximate time which elapses while the bob travels from D to D′—

$$= \sqrt{\frac{l}{g}} \cdot \frac{\text{KM}}{\text{HK}} = \sqrt{\frac{l}{g}} \cdot \frac{\text{KM}}{\text{AK} \sin \phi},$$

and since ∠ KK′M = ∠ KAB = ϕ, and KM/sin ϕ = KK′, we find that, to a first approximation, the bob travels from D to D′ in the time—

$$\sqrt{\frac{l}{g}} \cdot \frac{\text{KK}'}{\text{AK}}.$$

From A draw AN perpendicular to AB, cutting the semicircle BKL in N. Divide the quadrant NKB into n small elements, each equal in length to KK′; from the points of division drop perpendiculars on to AB, and (with A as centre) draw circular arcs, passing through the points of intersection of AB (Fig. 58) similar to the arcs HD and H′D′. We have divided the arc BDA into n elements, each of which corresponds to an element of the chord BA, just as DD′ (Fig. 57) corresponds to HH; and if we assume that each element of the arc BDA is equal to the corresponding element of the chord BHA, the time occupied by the bob in travelling over each of the n elements of the arc is equal to—

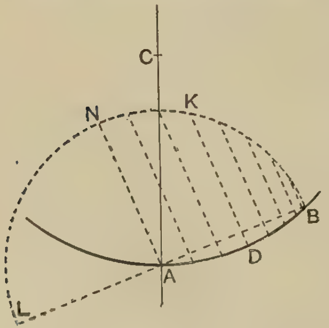

FIG. 58.—Details to be added to Fig. 56.

$$\sqrt{\frac{l}{g}} \cdot \frac{\text{KK}'}{\text{AK}},$$

and therefore the time which elapses while the bob travels from B to A—

$$= n\sqrt{\frac{l}{g}} \cdot \frac{\text{KK}'}{\text{AK}} = \sqrt{\frac{l}{g}} \cdot \frac{n \times \text{KK}'}{\text{AK}} = \sqrt{\frac{l}{g}} \cdot \frac{\pi}{2},$$

since $n \times \mathrm{KK}' =$ the quadrant BKN. Consequently, the time occupied by a complete oscillation of the bob $= 2\pi\sqrt{(l/g)}$, the result obtained by assuming that the oscillations of the bob are of the simple harmonic type (p. 95).

To determine the exact time required for the bob to travel from B to A, we must take account of the fact that the elements into which the arc BDA has been divided are not exactly equal in length to the corresponding elements of the chord BHA. Let an element of the arc BDA be p times as long as the corresponding element of the chord BHA; then for the various elements of BDA, the value of p will vary from unity at A to a value somewhat greater than unity at B; the time which elapses while the bob travels over an element—

$$= \sqrt{\frac{l}{g}} \cdot \frac{p \,.\, \mathrm{KK}'}{\mathrm{AK}},$$

and the time which elapses while the bob travels from B to A—

$$= n\sqrt{\frac{l}{g}} \cdot \frac{\mathrm{KK}' \times \textit{average value of } p \textit{ between } \mathrm{A} \textit{ and } \mathrm{B}}{\mathrm{AK}}$$

$$= \frac{\pi}{2}\sqrt{\frac{l}{g}} \,.\, \times \textit{average value of } p \textit{ between } \mathrm{A} \textit{ and } \mathrm{B}.$$

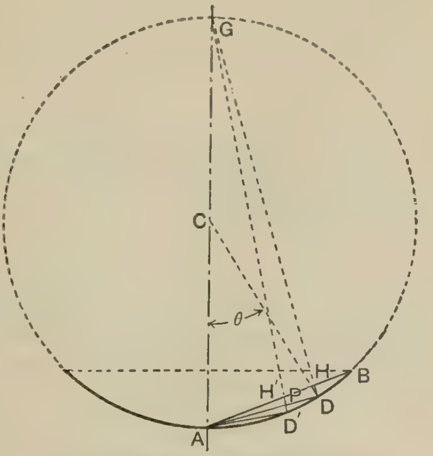

Fig. 59.—Details to be added to Fig. 56.

We must now determine the average value of p for all elements between A and B.

Join D to C, and join D and D′ to G (Fig. 59; the lettering of the diagram agrees with that of Figs. 56, 57 and 58). Let DA cut D′G in the point P; then by making DD′ as small as we please, we can make DP as nearly equal to HH′ as we please. Also, GD is perpendicular to AD, and CD is perpendicular to DD′; hence $\angle D'DP = \angle GDC = \theta/2$. Thus $HH' = DP = DD' \cos\theta/2$, and the ratio of the element DD′ of the arc BDA, to the corresponding element HH′ of the chord BA, is equal to $1/\cos(\theta/2)$.

The average value of $1/\cos(\theta/2)$ for all the elements of the arc BA can be found with various degrees of approximation. To the first degree of approximation the value is unity; we have already used this approximation. We may proceed as follows when the second degree of approximation is required; that is, when α is so small that powers of $\sin(\alpha/2)$ above the second may be neglected—

$$\frac{1}{\cos\frac{\theta}{2}} = \frac{1}{\left(1 - \sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}} = 1 + \tfrac{1}{2}\sin^2\frac{\theta}{2} + \ldots$$

When $\theta = 0$, $\sin\theta = 0$, and $\dfrac{1}{\cos\frac{\theta}{2}} = 1$.

When $\theta = \alpha$, $\dfrac{1}{\cos\frac{\theta}{2}} = 1 + \tfrac{1}{2}\sin^2\frac{\alpha}{2}$.

Taking the arithmetical mean of these values, we obtain a second approximation to the average value of $1/\cos\frac{\theta}{2}$ between the limits $\theta = 0$ and $\theta = \alpha$; namely, $\left(1 + \tfrac{1}{4}\sin^2\frac{\alpha}{2}\right)$. Writing $\sin(\alpha/2) = \alpha/2$, this becomes equal to $\{1 + (\alpha^2/16)\}$.

Hence, if T_α is the period of oscillation of a simple pendulum when the angular amplitude is equal to α, while T_0 is the period of oscillation of the same pendulum when the arc of oscillation is infinitely small,

$$T_\alpha = T_0\left(1 + \frac{\alpha^2}{16}\right),$$

the result originally obtained by Bernouilli.

If, as is generally the case, we wish to determine T_0 from the observed value of T_α, we have—

$$T_0 = \frac{T_\alpha}{1 + \frac{\alpha^2}{16}} = T_\alpha\left(1 - \frac{\alpha^2}{16}\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If, during the experimental determination of T, the amplitude varies from a_1 to a_2, it is best to use the reduction factor—

$$\left(1 - \frac{a_1 a_2}{16}\right).$$

By the use of the calculus, the reduction factor can be determined in the form of a series.[1] The simple reduction factor given in (1) above, is all that is required for experimental work, where the arc of oscillation a is always small.

To obtain an idea of the value of the reduction factor in definite circumstances, let $a = 10^\circ = 0{\cdot}174$ radian. Then—

$$1 - \frac{a^2}{16} = 1 - 0{\cdot}0019.$$

That is, the observed period of oscillation must be diminished by about 0·2 per cent. in order to obtain the period for an infinitely small arc of oscillation.

The same reduction factor must be applied to the period of oscillation of a magnet in a magnetic field. In the case of the bifilar suspension, a further correction would be required, owing to the fact that the ends of the suspending fibres attached to the oscillating body move on the surface of a cylinder, instead of in a plane (p. 107).

Observations of torsional oscillations require no reduction factor, so long as the oscillations do not strain the wire beyond its limits of elasticity; and if they do, a reduction factor, based on the elastic properties of the wire, is required.

Correction for buoyancy and inertia of the air.—If a pendulum is swung in air, it loses part of its weight owing to the buoyancy of the air; this loss is equal to the weight of air displaced (p. 35). Hence, **if m is the mass of air displaced by the pendulum, an upward force mg will act on the pendulum at a point occupying the position of the centre of gravity of the air displaced**, that is, at the centre of figure of the pendulum. The centre of figure will not coincide with the centre of gravity, unless the pendulum is homogeneous throughout.

When a body moves through a fluid, the fluid must flow away from it in front, and up to it behind. Hence the mass of the body is only part of the total mass which is moving. If the fluid is frictionless, no energy will be dissipated owing to its motion; when the velocity of the body increases, extra energy is conferred on the fluid, and this energy is returned to the body as its velocity slows down. Hence **the dynamical effect of the surrounding fluid is to increase the inertia of the body** (p. 20).

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[1] Gibson's "Elementary Treatise on the Calculus" (Macmillan), pp. 402 and 432.

Sir G. Stokes has shown that a sphere moving through a fluid has an effective inertia equal to its own mass, *plus* half the mass of the fluid that would fill the space occupied by the sphere. Thus, a spherical bubble of air rising through water possesses considerable inertia, although the actual mass of the air in the bubble is small. Let the spherical bob of a pendulum be of brass (density = 8·0), and let it swing in air when the density of the latter is equal to 0·0012. Then if V is the volume of the bob, its inertia will be equal to—

$$8V + 0{\cdot}0012 \,.\, \frac{V}{2} = V(8{\cdot}0006).$$

Thus the surrounding air increases the inertia of the brass sphere by about 1 part in 10,000.

It is only in a few special cases that the increase of inertia due to the surrounding fluid can be calculated with accuracy.

The corrections for buoyancy and inertia, to be applied to the reversible pendulum, were first investigated by Bessel; these corrections will now be discussed.

Let m be the mass of the pendulum, and m_1 the mass of air displaced by it. When the pendulum is in the erect position, let its centre of gravity be at a distance l_1, and its centre of figure at a distance s_1, from the supporting knife-edge. Then an angular displacement α calls into play a restoring torque equal to

$$mgl_1 \sin \alpha - m_1 g s_1 \sin \alpha = g(ml_1 - m_1 s_1) \sin \alpha\ ;$$

and when α is small, this reduces to $g(ml_1 - m_1 s_1)\alpha$, so that the restoring torque per unit twist $= g(ml_1 - m_1 s_1)$.

The air carried along with the pendulum increases its moment of inertia by a small amount, say i_1; thus the total moment of inertia of the pendulum $= \{m(l_1^2 + k^2) + i\}$. To determine the **period of oscillation T_1 in the erect position,** we have—

$$T_1 = 2\pi \sqrt{\frac{m(l_1^2 + k^2) + i_1}{g(ml_1 - m_1 s_1)}},$$

$$\therefore \frac{gT_1^2}{(2\pi)^2} = \frac{l_1 + k^2 + \frac{i_1}{m}}{l_1 - \frac{m_1}{m}s_1} = \frac{l_1^2 + k^2 + \frac{i_1}{m}}{l_1\left(1 - \frac{m_1}{m}\frac{s_1}{l_1}\right)}$$

$$= \frac{l_1^2 + k^2 + \frac{i_1}{m}}{l_1}\left(1 + \frac{m_1}{m}\frac{s_1}{l_1} + \ldots\ldots\right)$$

$$= \frac{l_1^2 + k^2}{l_1} + \frac{l_1^2 + k^2}{l_1}\cdot\frac{m_1 s_1}{ml_1} + \frac{i_1}{ml_1} + \quad \ldots\ldots \quad (1)$$

Here, terms containing products of m_1/m and i_1, and powers of m_1/m higher than the first, have been neglected, since m_1/m and i_1 are very small quantities.

When the pendulum is swung in the inverted position, let its centre of gravity be at a distance l_2, and its centre of figure at a distance s_2, from the supporting knife-edge. Then if the air carried along by the pendulum increases the moment of inertia of the latter by i_2, the **period of oscillation T_2 in the inverted position** will be given by—

$$\frac{gT_2^2}{(2\pi)^2} = \frac{l_2^2 + k^2}{l_2} + \frac{l_2^2 + k^2}{l_2}\cdot\frac{m_1 s_2}{ml_2} + \frac{i_2}{ml_2} \quad \ldots \quad (2)$$

Multiply (1) through by l_1, and (2) through by l_2, and subtract (2) from (1); then we obtain—

$$\frac{g}{(2\pi)^2}(T_1^2 l_1 - T_2^2 l_2) = l_1^2 - l_2^2 + (l_1^2 + k^2)\frac{m_1 s_1}{ml_1} - (l_2^2 + k^2)\frac{m_1 s_2}{ml_2}$$
$$+ \frac{i_1 - i_2}{m} \quad \ldots\ldots\ldots\ldots \quad (3)$$

Each of the quantities $(l_1^2 + k^2)/l_1$ and $(l_2^2 + k^2)/l_2$, which are found in the second and third terms on the right-hand side of (3), is multiplied by the small quantity m_1/m, which is of the order of magnitude 1/10,000 (p. 150); hence the magnitudes of these quantities need not be known accurately. The periods of oscillation T_1 and T_2 are nearly equal, and the approximate value of the length λ of the equivalent simple pendulum is equal to $(l_1 + l_2)$ (p. 101). Thus, to a first approximation—

$$\lambda = \frac{gT_1^2}{(2\pi)^2} = \frac{l_1^2 + k^2}{l_1} = \frac{l_2^2 + k^2}{l_2}.$$

Substituting λ for $(l_1^2 + k^2)/l_1$ and $(l_2^2 + k^2)/l_2$ in the correcting terms on the right-hand side of (3), we have—

$$\frac{g}{(2\pi)^2}(T_1^2 l_1 - T_2^2 l_2) = l_1^2 - l_2^2 + \frac{m_1\lambda}{m}(s_1 - s_2) + \frac{i_1 - i_2}{m}.$$

Dividing through by $(l_1 - l_2)$, we have—

$$\frac{g}{(2\pi)^2}\frac{T_1^2 l_1 - T_2^2 l_2}{l_1 - l_2} = l_1 + l_2 + \frac{m_1\lambda}{m} \cdot \frac{s_1 - s_2}{l_1 - l_2} + \frac{i_1 - i_2}{m(l_1 - l_2)} \quad . \quad . \quad . \quad (4)$$

The method used in manipulating the terms on the left-hand side of this equation was explained on p. 102.

The position of the centre of figure can be determined geometrically with a sufficient degree of accuracy to give the value of $(s_1 - s_2)$; and m_1/m is merely the ratio of the density of the air to the average density of the pendulum. Hence, the value of the first correction term on the right can be found. The value of the second correction term on the right can be obtained by using another pendulum identical in shape and dimensions with the first, but of different mass; in this case $(i_1 - i_2)$ will have the same value for both pendulums, but the other quantities will be different, and we obtain two simultaneous equations from which the unknown quantity g can be eliminated and $(i_1 - i_2)$ determined. The gross value of the two correcting terms on the right can be obtained also by swinging a single pendulum, first in air, and then in a vacuum; then subsequent experiments made in air can be corrected, remembering that the gross value of the two correcting terms will vary with the density of the air, and therefore with the temperature and the barometric pressure.

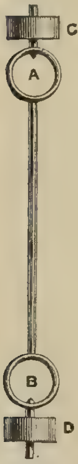

Fig. 60.—Repsold's pendulum.

Equation (4) above indicates that if the pendulum is made symmetrical in form about its middle point, the two correcting terms on the right of (4) are eliminated, since in that case $s_1 = s_2$, and $i_1 = i_2$. This principle was utilised by Repsold in constructing his pendulum (Fig. 60). The two knife-edges A and B were placed at equal distances from the middle of the pendulum, and one of the cylindrical portions, D, was made solid, while the other, C, was hollow.

Correction for curvature of knife-edges.—If the knife-edge on which a pendulum is supported were quite sharp (that is, if the two faces of the knife-edge met in a geometrical line), the pendulum would rotate about an axis coinciding with the knife-edge. Such ideal conditions are impossible of attainment, and therefore a correction becomes necessary for the imperfect sharpness of the knife-edges. Laplace pointed out that, to a first approximation, the knife-edges might be supposed to be cylindrical surfaces of finite radius; later, Bessel showed how error, due to imperfect sharpness of the knife-edges, could be eliminated.

Let AB (Fig. 61) represent the axis of the pendulum, and DEF the section of the cylindrical surface of the knife-edge, the radius of curvature r_1 being exaggerated very much. As the pendulum oscillates, the cylinder DEF rolls to and fro on the plane surface which supports it. This will not affect the moment of inertia to any appreciable extent, for the instantaneous centre of rotation is the point E where the cylinder rests on the plane, and the distance from C, the centre of gravity of the pendulum, to E, is not sensibly different from the distance CA; this can be seen if we imagine a small circular arc, with C as centre and CA as radius, to be drawn through A; it will pass approximately through C, since it is perpendicular to CA, and so is the small arc AE.

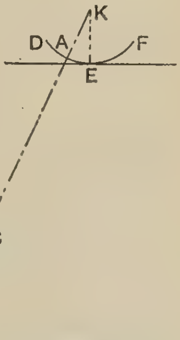

FIG. 61.—The effects of imperfect sharpness of the knife-edges of a pendulum.

Draw a vertical line through E, and produce CA to cut this line in K. Then K is the centre of curvature of the knife-edge; as the pendulum oscillates, the point K moves to and fro along a horizontal line. The restoring torque called into play by a twist α is found by multiplying mg, the force of gravity acting at C, into its perpendicular distance from the point E, and this distance is equal to $(l_1+r_1)\sin\alpha=(l_1+r_1)\alpha$ when α is small. Hence, the restoring torque per unit twist $=mg(l_1+r_1)$. Thus for **the period of oscillation** T_1 **in the erect position**, we have—

$$\frac{gT_1^2}{(2\pi)^2}=\frac{l_1^2+k^2}{l_1+r_1}=\frac{l_1^2+k^2}{l_1\left(1+\frac{r_1}{l_1}\right)}=\frac{l_1^2+k^2}{l_1}\left(1-\frac{r_1}{l_1}+\ldots\right) \quad . \quad . \quad (5)$$

Let T_2 be **the period of oscillation in the inverted position**, and let r_2 be the radius of curvature of the knife-edge on which the pendulum is supported in this position. Then—

$$\frac{gT_2^2}{(2\pi)^2}=\frac{l_2^2+k^2}{l_2}\left(1-\frac{r_2}{l_2}+\ldots\right). \quad . \quad . \quad . \quad (6)$$

Multiplying (5) through by l_1, and (6) through by l_2, and subtracting (6) from (5), we have—

$$\frac{g}{(2\pi)^2}(T_1^2 l_1 - T_2^2 l_2) = l_1^2 - l_2^2 + \lambda(r_2 - r_1),$$

where λ is written for $(l_1^2 + k^2)/l_1$ and $(l_2^2 + k^2)/l_2$ in the correcting term (compare pp. 101 and 151).

$$\therefore \frac{g}{(2\pi)^2}\frac{T_1^2 l_1 - T_2^2 l_2}{l_1 - l_2} = l_1 + l_2 + \lambda\frac{r_2 - r_1}{l_1 - l_2} \quad . \quad . \quad . \quad (7)$$

Now interchange the knife-edges, without altering the values of l_1 and l_2. If T_1' and T_2' are the new periods in the erect and inverted positions, we have—

$$\frac{g}{(2\pi)^2}\frac{T_1'^2 l_1 - T_2'^2 l_2}{l_1 - l_2} = l_1 + l_2 + \lambda\frac{r_1 - r_2}{l_1 - l_2} \quad . \quad . \quad . \quad . \quad (8)$$

On adding (7) and (8) together, the correcting terms involving r_1 and r_2 disappear, and the effect of imperfect sharpness of the knife-edges is eliminated.

Hence most modern pendulums for the determination of g have been made with interchangeable knife-edges. The pendulum constructed for the United States Coast and Geodetic Survey had agate planes attached to it, and these were in turn supported on a single knife-edge, so that only a single knife-edge was used, and $r_1 = r_2$.

Vibration of point of support.—If a force be applied to the point from which a pendulum is supported, *some* displacement will be produced, and, in general, the displacement will be proportional to the applied force, and will become equal to zero when the force ceases to act ; hence, in general, the support is capable of vibrating under the control of the elastic force called into play by its displacement. The period of vibration will be equal to $2\pi\sqrt{M/f_1}$, where M denotes the inertia of the matter which vibrates with the point of support, and f_1 is the restoring force called into play by unit linear displacement (compare pp. 20 and 94).

Let P (Fig. 62) be the position of the point of support when the simple pendulum PA hangs in its position of equilibrium. When the pendulum is displaced on either side of its position of equilibrium, so that the supporting filament makes an angle ϕ with the vertical, the tension of the filament is equal to $mg \cos \phi$ (p. 89), and the horizontal force acting on the point of support is $mg \tan \phi$, which is equal to $mg\phi$ when ϕ is small. If the pendulum were maintained in its displaced position, the point of support would be shifted under the action of the tension of the filament, and its new position of equilibrium, P′, would be such that the displacement $PP' = mg\phi/f_1$. Hence, when the pendulum oscillates about A, the position of equilibrium of the point of support shifts from P to P′, while the bob moves from A to A′.

The actual motion of the point of support will depend on its period of vibration; for, when its position of equilibrium is shifted, it will commence to move towards its new position of equilibrium, but it will not reach this position until a time equal to a quarter of its period of vibration has elapsed. **When the period of vibration of the support is small in comparison with the period of oscillation of the pendulum, the support will follow the motion of the pendulum**; that is, the displacement of the point of support at each instant will be proportional to the force exerted on it by the pendulum. When the pendulum makes a small angle ϕ with the vertical, it exerts a horizontal force equal to $mg\phi$ on the point of support; and since a horizontal force f_1 applied to the point of support produces unit displacement, the displacement produced by the horizontal force $mg\phi$ is equal to $mg\phi\, f_1$.

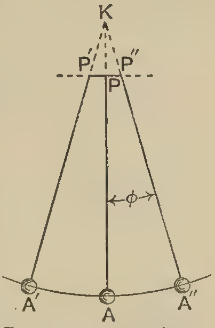

FIG. 62.—Pendulum swinging from an elastic support, the natural period of the support being less than that of the pendulum.

Produce AP and AP′ to meet in K; then if $\angle AKA' = \phi$, the displacement of the point of support $= PP' = (mg/f_1) \times \angle AKA'$. Thus, since $\angle AKA' = PP'/KP$—

$$PP' = \frac{mg}{f_1} \cdot \frac{PP'}{KP},$$

$$\therefore \; KP = mg/f_1 = \delta \text{ (say).}$$

But the pendulum now virtually swings about the point K; and if l is the length of the supporting filament, the period of oscillation T of the pendulum is given by the equation—

$$T = 2\pi \sqrt{\frac{l+\delta}{g}}.$$

Hence, **when the natural period of vibration of the support is less than the natural period of oscillation of the pendulum, the period of oscillation of the pendulum is lengthened.**

Now let us suppose that the period of vibration of the point of support is greater than the period of oscillation of the pendulum. While the pendulum bob is to the left of its position of equilibrium A, it pulls the point of support toward the left, and therefore the velocity of the point of support in that direction increases; that is, while the pendulum bob moves from A to A′ and back to A (Fig. 63), the point of support moves with increasing velocity toward the left, its maximum velocity being acquired as the bob moves from left to right through the position A. But an oscillating body acquires its maximum velocity as it moves through its position of equilibrium; hence **as the pendulum bob swings from left to right through its position of equilibrium** A, **the point of support moves from right to left through its position of equilibrium** P.

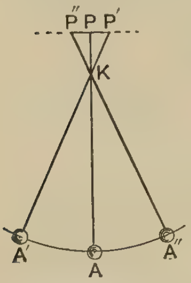

FIG. 63.—Pendulum swinging from an elastic support, the natural period of the support being greater than that of the pendulum.

As the bob moves from A to A″ the force exerted by it diminishes the velocity of the point of support, and as the bob reaches A″ the point of support comes to rest at P″, a point to the left of P. As the bob moves back from A″ to A, the pull of the filament on the point of support causes this to move with increasing velocity toward the right; and **as the bob moves from right to left through** A, **the point of support moves with its maximum velocity from left to right through** P. Hence the pendulum now virtually oscillates about the point of intersection K of A′P′ and A″P″; and if PK = δ, the period of oscillation T of the pendulum is given by the equation—

$$T = 2\pi\sqrt{\frac{l-\delta}{g}}.$$

Hence, **when the natural period of vibration of the support is greater than the natural period of oscillation of the pendulum, the period of oscillation of the pendulum is diminished.** The pendulum bob and the point of support both execute simple harmonic motions of equal periods, but their phases differ by π.

Oscillations of a coupled system.—When two bodies are connected together in such a manner that the displacement of one alters the position of equilibrium of the other, they may be said to be coupled, and to constitute a coupled system. Thus a pendulum and its support constitute a coupled system, if the support is not perfectly rigid. The oscillations of such a system will now be studied in detail.

We must first determine the acceleration corresponding to the maximum value of a simple harmonic displacement.

It has been proved (p. 86) that a simple harmonic displacement may be represented by the equation—

$$y = a \sin \theta,$$

where θ is the instantaneous phase-angle. The corresponding velocity is given by the equation—

$$v = a \,.\, \frac{2\pi}{T} \,.\, \cos \theta.$$

The maximum displacement corresponds to a phase-angle of $\pi/2$ or $3\pi/2$, and the velocity corresponding to either of these phase-angles is equal to zero. The acceleration is the rate of change of the velocity (p. 10). At a given instant, let the phase-angle be equal to $\left(\frac{\pi}{2} - \frac{\beta}{2}\right)$, where β is very small; and after a time t' let the phase-angle be equal to $\left(\frac{\pi}{2} + \frac{\beta}{2}\right)$. During the time t' the phase-angle has increased by β; since the phase-angle increases at a uniform rate of $(2\pi/T)$ per second, it follows that $\beta = (2\pi t'/T)$. Meanwhile, the velocity has increased from—

$$a\left(\frac{2\pi}{T}\right) \cos\left(\frac{\pi}{2} - \frac{\beta}{2}\right) = a\left(\frac{2\pi}{T}\right) \sin \frac{\beta}{2}$$

to

$$a\left(\frac{2\pi}{T}\right) \cos\left(\frac{\pi}{2} + \frac{\beta}{2}\right) = -a\left(\frac{2\pi}{T}\right) \sin \frac{\beta}{2},$$

and therefore the increase in the velocity $= -2a\left(\frac{2\pi}{T}\right) \sin \frac{\beta}{2}$

$$= -a\frac{2\pi}{T} \,.\, \beta, \text{ since } \beta \text{ is small.}$$

$\therefore$ Rate of increase of velocity $= -\left(a \,.\, \frac{2\pi}{T} \,.\, \frac{2\pi t'}{T}\right) \div t' = -a\left(\frac{2\pi}{T}\right)^2.$

The negative sign indicates that the acceleration is toward the origin.

Thus **the numerical value of the acceleration at the extremity of a simple harmonic oscillation is found by multiplying the linear amplitude by** $(2\pi/T)^2$. (Compare p. 92.)

Let us now investigate the oscillations which can be performed by a simple pendulum and its support. Let the inertia of the support = M, while the restoring force called into play by unit displacement of the support $= f_1$. Then the natural period of vibration of the support $= T_s = 2\pi\sqrt{M/f_1}$.

Let the bob of the simple pendulum have a mass m, and let the distance, from the centre of gravity of the bob to the point at which the filament is attached to the support, be equal to l. Then the natural period of oscillation of the pendulum $= T_p = 2\pi\sqrt{l/g}$.

Let the positions of the pendulum bob and the point of support be represented A′ and P′ (Fig. 62), when the bob is at its position of maximum displacement; and let the period of oscillation common to the pendulum and its support be denoted by T. Then if the pendulum makes an angle α with the vertical, and $PK = \delta$ as before, the linear amplitude of the vibration performed by the point of support $= PP' = \delta\alpha$, and the acceleration of the point of support is in the direction from P′ to P, and is equal to—

$$\left(\frac{2\pi}{T}\right)^2 \delta\alpha.$$

The inertia of a moving body, multiplied by its acceleration, gives the resultant force acting on the body (p. 20). Now, the forces acting on the support are (1) the restoring force $f_1\delta\alpha$, acting in the direction P′P; and (2) the horizontal component of the tension of the pendulum filament, equal to $mg\alpha$, acting in the direction PP′. Hence, equating the product of the mass and the acceleration of the support, to the resultant force acting on it, we have—

$$M\left(\frac{2\pi}{T}\right)^2 \delta\alpha = f_1\delta\alpha - mg\alpha,$$

$$\therefore\ M\left(\frac{2\pi}{T}\right)^2 \delta = f_1\delta - mg \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The linear amplitude of the oscillation executed by the pendulum bob $= AA' = (l+\delta)\alpha$. Hence, equating the product of the mass and the acceleration of the bob, to the horizontal component of the tension of the filament, we have—

$$m\left(\frac{2\pi}{T}\right)^2 (l+\delta)\alpha = mg\alpha,$$

$$\therefore\ \left(\frac{2\pi}{T}\right)^2 (l+\delta) = g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1)—

$$\delta = -\frac{mg}{M\left(\frac{2\pi}{T}\right)^2 - f_1}.$$

Substituting the value of δ in (2), we have—

$$\left(\frac{2\pi}{T}\right)^2\left\{l - \frac{mg}{M\left(\frac{2\pi}{T}\right)^2 - f_1}\right\} = g,$$

$$\left(\frac{2\pi}{T}\right)^2\left\{\frac{l}{g} - \frac{\frac{m}{M}}{\left(\frac{2\pi}{T}\right)^2 - \frac{f_1}{M}}\right\} = 1.$$

In this equation, substitute $l/g = (T_p/2\pi)^2$, and $f_1/M = (2\pi/T_s)^2$. Then—

$$\left(\frac{2\pi}{T}\right)^2\left\{\left(\frac{T_p}{2\pi}\right)^2 - \frac{\frac{m}{M}}{\left(\frac{2\pi}{T}\right)^2 - \left(\frac{2\pi}{T_s}\right)^2}\right\} = 1.$$

$$\therefore \left(\frac{2\pi}{T}\right)^4 - \left(\frac{2\pi}{T}\right)^2\left\{\left(\frac{2\pi}{T_p}\right)^2 + \left(\frac{2\pi}{T_s}\right)^2 + \frac{m}{M}\left(\frac{2\pi}{T_p}\right)^2\right\} + \left(\frac{2\pi}{T_p}\right)^2\left(\frac{2\pi}{T_s}\right)^2 = 0.$$

Cancel 2π, and substitute—

$$n = (1/T),\ n_p = 1/T_p,\ \text{and}\ n_s = 1/T_s.$$

Here, n, n_p, and n_s are the **frequencies** (p. 84) corresponding to the periods T, T_p, and T_s. Then—

$$n^4 - n^2\left(n_p^2 + n_s^2 + \frac{m}{M}n_p^2\right) + n_p^2 n_s^2 = 0. \quad . \quad . \quad . \quad (3)$$

Equation (3) is a biquadratic in n, the frequency of the oscillations possible to the pendulum when coupled to the support; it may also be looked upon as a quadratic equation in n^2.

If a quadratic equation has two positive roots, say a and b, it can be written in the form—

$$(x-a)(x-b) = 0;$$

for this equation is satisfied by putting $x=a$, or $x=b$, and by no other values of x. Then—

$$x^2 - (a+b)x + ab = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is an equation of the same general form as (3), if n^2 is treated as the unknown quantity. Hence, there will be two positive values of n^2 which will satisfy (3); let these values be n_1^2 and n_2^2. Then n_1^2 and n_2^2 in (3) correspond to a and b in (4), and therefore—

$$n_1^2 n_2^2 = n_p^2 n_s^2;$$

$$\therefore n_1 n_2 = n_p n_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Negative values of n_1 and n_2 are excluded, as being irrelevant to the problem in hand. Also—

$$n_1^2+n_2^2=n_p^2+n_s^2+\frac{m}{M}n_p^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Adding $2n_1n_2$ to the left-hand side, and the equal value $2n_pn_s$ to the right-hand side of (6), we have—

$$(n_1+n_2)^2=(n_p+n_s)^2+\frac{m}{M}n_p^2,$$

$$\therefore\ n_1+n_2=\sqrt{(n_p+n_s)^2+\frac{m}{M}n_p^2} \quad . \quad . \quad . \quad . \quad (7)$$

Subtracting $2n_1n_2$ from the left-hand side, and the equal value $2n_pn_s$ from the right-hand side of (6), we have—

$$n_1-n_2=\sqrt{(n_p-n_s)^2+\frac{m}{M}n_p^2} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equations (7) and (8) suffice to determine the values of n_1 and n_2. The other two roots of the biquadratic (3) are merely $(-n_1)$ and $(-n_2)$, and these negative values are discarded as being irrelevant to the problems in hand.

Now, we have found two values, n_1 and n_2, for the frequency of the oscillations executed by the coupled system. The meaning of this result is as follows. The pendulum drives the support, and they both oscillate with a frequency n_1, which is greater or less than the natural frequency of the pendulum, according as the natural frequency of the support is less or greater than that of the pendulum (p. 155). But the support also drives the pendulum, and they both oscillate with a frequency n_2, which is greater or less than the natural frequency of the support, according as the natural frequency of the pendulum is less or greater than that of the support. Hence n_1 may be called the modified frequency of the pendulum, and n_2 the modified frequency of the support.

The following graphical method of determining n_1 and n_2 from (5) and (7) will make the nature of the solution clearer. Let AB (Fig. 64) $=n_p$, the natural frequency of the pendulum. Produce AB to C, and make BC $=n_s$, the natural frequency of the support. On AC as diameter, draw the circle ADC and through B draw BD perpendicular to AB. From A draw AF perpendicular to AC, and equal to $\sqrt{m/M}\times$AB.

Join FC; and with D as centre, and (FC/2) as radius, describe an arc cutting AC in a point (not marked in Fig. 64) which is used as centre in describing the circle A′DC′ through the point D; then $n_1 = A'B$, and $n_2 = BC'$.

For $(A'B) \times (BC') = (BD)^2 = AB \times BC$, which satisfies (5). Also

$$A'C' = FC = \sqrt{(AC)^2 + \frac{m}{M}(AB)^2},$$

which satisfies (7). In Fig. 64, $AB = n_p$ is greater than $BC = n_s$; that is, the natural frequency of the pendulum is greater than that of the point of support. Accordingly, A′B, the modified frequency of the pendulum, is greater than AB, its natural frequency; and BC′, the modified frequency of the support, is less than BC, its natural frequency.

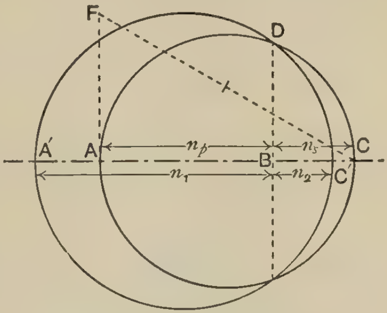

Fig. 64.—Graphical method of determining the oscillation frequencies of a coupled system.

It remains only to determine the **amplitudes of the oscillations** simultaneously executed by the pendulum. Let the oscillations be started by pulling the pendulum aside through a distance A from its position of equilibrium, and then releasing it; at the moment of release, both oscillations of the pendulum must be at their positions of maximum displacement.

Therefore, if a_1 and a_2 are the respective amptitudes of the oscillations of the pendulum, performed with frequencies n_1 and n_2, we have—

$$a_1 + a_2 = A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Let A′ be the initial displacement of the point of support; at the instant when the pendulum is released, the restoring force, $f_1 A'$, called into play by the displacement A′ of the support, is in equilibrium with the horizontal component of the tension of the filament, $mg\phi$, where ϕ is the small inclination of the filament to the vertical. Then $\phi = (A - A')/l$, and—

$$f_1 A' = mg \cdot \frac{A - A'}{l}.$$

$$\therefore\ A'\left(f_1 + \frac{mg}{l}\right) = \frac{mg}{l}A. \quad . \quad . \quad . \quad . \qquad (10)$$

Since the frequency is equal to the reciprocal of the period of oscillation, the acceleration corresponding to the maximum displacement a_1 of the oscillation of frequency n_1 is equal to $(2\pi n_1)^2 a_1$; the corresponding acceleration for the oscillation of frequency $n_2 = (2\pi n_2)^2 a_2$. The mass of the pendulum bob, multiplied by the sum of these accelerations, gives the horizontal force acting on the bob at the instant when it is released. Thus—

$$m\{(2\pi n_1)^2 a_1 + (2\pi n_2)^2\} a_2 = \frac{mg}{l}(A - A') = \frac{mg}{l}A\left(1 - \frac{\frac{mg}{l}}{f_1 + \frac{mg}{l}}\right),$$

from (10);

$$\therefore\ (2\pi n_1)^2 a_1 + (2\pi n_2)^2 a_2 = A\frac{g}{l} \cdot \frac{f_1}{f_1 + \frac{mg}{l}} = A\frac{g}{l} \cdot \frac{1}{1 + \frac{m}{f_1} \cdot \frac{g}{l}}.$$

Substitute $g/l = (2\pi n_p)^2$, and $\frac{M}{f_1} = 1/(2\pi n_s)^2$, and we obtain—

$$(2\pi n_1)^2 a_1 + (2\pi n_2)^2 a_2 = A \frac{(2\pi n_p)^2}{1 + \frac{m}{M}\left(\frac{n_p}{n_s}\right)^2} \quad . \quad . \qquad (11)$$

Also, as already proved—

$$a_1 + a_2 = A.$$

Solving these simultaneous equations for a_1 and a_2, we obtain—

$$\{(2\pi n_1)^2 - (2\pi n_2)^2\} a_1 = A\left\{\frac{(2\pi n_p)^2}{1 + \frac{m}{M}\left(\frac{n_p}{n_s}\right)^2} - (2\pi n_2)^2\right\} \quad . \qquad (12)$$

$$\{(2\pi n_1)^2 - (2\pi n_2)^2\} a_2 = A\left\{(2\pi n_1)^2 - \frac{(2\pi n_p)^2}{1 + \frac{m}{M}\left(\frac{n_p}{n_s}\right)^2}\right\} \quad . \qquad (13)$$

Experimental study of the oscillations of a coupled system.—A very simple experiment will suffice to illustrate the results obtained in the foregoing investigation. A boxwood metre scale AB, Fig. 65, is clamped in a vice, in a vertical position. The upper end of the scale is weighted with two circular half-kilogram weights, W, in the manner shown in detail to the right. A cork C, that fits into the cylindrical hole in the middle of the weights, is slotted so as to fit on the end of the scale; two wooden wedges clamp the weights to the scale. A pin, P, fixed in the cork, supports a simple pendulum consisting of a small mass, M, hung from a piece of cotton. The weighted metre scale forms an elastic support, which can be adjusted to vibrate in any desired period by varying the point at which it is clamped in the vice.

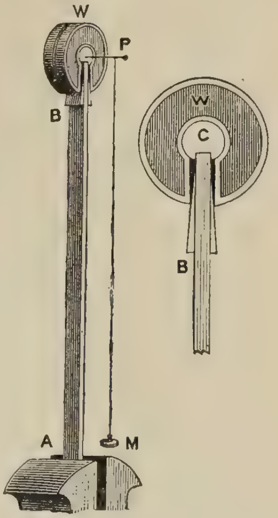

FIG. 65.—Experimental arrangement for the study of the oscillations of a coupled system.

EXPT. 15.—For the simple pendulum, use a ten gram weight, supported by a piece of cotton a metre in length. Support it on a firm retort stand, and adjust the weighted metre scale so that its period of vibration is equal to the period of the pendulum. Hang the pendulum from the pin P, pull it aside, and then release it.

The character of the oscillations of the coupled system can be derived with ease from the foregoing investigation Assuming that the inertia of the support is sensibly equal to the mass of the two half-kilogram weights, we have $m/M = 1/100$. Let $\mathbf{n}$ be the natural frequency of oscillation common to the pendulum and its support; then in equations (7) and (8), p. 160, $n_p = n_s = \mathbf{n}$. Solving these equations for n_1 and n_2, we find that—

$$n_1 + n_2 = \sqrt{(2\mathbf{n})^2 + \frac{\mathbf{n}^2}{100}} = 2{\cdot}002n,$$

$$n_1 - n_2 = \sqrt{\frac{\mathbf{n}^2}{100}} = 0{\cdot}1n.$$

$$\therefore\ n_1 = 1{\cdot}05\mathbf{n}$$
$$n_2 = 0{\cdot}95\mathbf{n}.$$

Substituting these values in equations (12) and (13), p. 162, we find that—

$$\frac{a_1}{a_2}=\frac{\frac{1}{1+\frac{1}{100}}-(0{\cdot}95)^2}{(1{\cdot}05)^2-\frac{1}{1+\frac{1}{100}}}=\frac{0{\cdot}09}{0{\cdot}11}.$$

Hence, the pendulum executes two simple harmonic oscillations simultaneously, the frequency of the quicker oscillation being about 5 per cent. greater, and that of the slower oscillation being about 5 per cent. less, than the natural frequency of the pendulum. The amplitudes of the two oscillations are nearly equal, the quicker oscillation having an amplitude slightly smaller than that of the slower oscillation.

On pulling the simple pendulum aside and releasing it, the two oscillations executed by it start in the same phase. But the phase of the slower gradually falls behind that of the quicker oscillation, and when five oscillations have been completed the difference of phase amounts to π; that is, the displacement due to the slower oscillation is in the opposite direction to that due to the quicker oscillation, and as the two amplitudes are nearly equal, the pendulum practically comes to rest.

Let us now turn our attention to the support. On releasing the pendulum, the displacement of the support is small. The two oscillations executed by the support start in opposite phases: the slower oscillation of the pendulum drags the support after it in the manner indicated by Fig. 62, p. 155, and therefore the slower oscillations of the pendulum and support are in the same phase; but the quicker oscillations of the pendulum and support are performed in the manner indicated in Fig. 63, p. 156, and therefore they are in opposite phases. Hence, when the oscillations of the pendulum neutralise each other, and the pendulum comes to rest, the oscillations of the support just come into phase, and the oscillatory displacement of the support acquires its maximum value.

In the next stage, the slower oscillation of the pendulum continues to lag more and more behind its quicker oscillation, until their phase difference amounts to 2π, when the pendulum again acquires its maximum arc of swing; simultaneously, the support comes to rest, its two oscillations being once more in opposite phases.

Hence the two oscillations of the pendulum cause it to come to rest periodically; this phenomenon, which is essentially similar to that known as *beats* in connection with experiments on sound, can be easily observed by the aid of the device depicted in Fig. 65. In the absence of friction there would be no loss of energy: when the pendulum comes to rest, its energy has been communicated to the support; and when the pendulum acquires its maximum energy, the support is stationary.

Let us assume that the two oscillations, executed simultaneously by the pendulum, are equal in amplitude, that is, let $a_1 = a_2 = a$. Then the displacement of the pendulum, at a time t after the commencement of the experiment, is given by the equation—

$$y = a \sin 2\pi n_1 t + a \sin 2\pi n_2 t$$

$$= a\left\{\sin\left(2\pi\frac{n_1+n_2}{2}t + 2\pi\frac{n_1-n_2}{2}t\right) + \sin\left(2\pi\frac{n_1+n_2}{2}t - 2\pi\frac{n_1-n_2}{2}t\right)\right\}$$

$$= 2a\sin\left(2\pi\frac{n_1+n_2}{2}t\right)\cos\left(2\pi\frac{n_1-n_2}{2}t\right)$$

$$= \left\{2a\cos\left(2\pi\frac{n_1-n_2}{2}t\right)\right\}\sin\left(2\pi\frac{n_1+n_2}{2}t\right) \quad . \quad . \quad (14)$$

Equation (14) may be interpreted as representing an oscillation of frequency $(n_1 + n_2)/2$, the amplitude being variable and equal to

$$2a\cos\left(2\pi\frac{n_1-n_2}{2}t\right).$$

When $$2\pi\frac{n_1-n_2}{2}t = \frac{\pi}{2}, \text{ or } \frac{3\pi}{2}, \frac{5\pi}{2}, \text{ \&c.},$$

$$\text{that is, when } t = \frac{1}{2(n_1-n_2)}, \frac{3}{2(n_1-n_2)}, \frac{5}{2(n_1-n_2)}, \text{ \&c.},$$

the amplitude becomes equal to zero. When the oscillation frequencies of the pendulum and its support are equal (that is, $n_p = n_s = \mathbf{n}$) the value of $(n_1 - n_2)$ is equal to $\mathbf{n}\sqrt{(m/M)}$ from equation (8), p. 160. In the experiment at present under discussion, $(m/M) = 1/100$, and therefore $n_1 - n_2 = \mathbf{n}/10$. Also, in this case $(n_1 + n_2)$ is very nearly equal to $2\mathbf{n}$. Therefore, the displacement of the pendulum, at the time t, may be represented to a close approximation by the equation—

$$y = \left\{2a\cos\left(2\pi\frac{\mathbf{n}}{20}t\right)\right\}\sin 2\pi\mathbf{n}t.$$

Consequently, the amplitude waxes and wanes, passing through a complete cycle while the pendulum executes 20 oscillations; thus the pendulum comes to rest periodically at intervals equal to ten times its period of oscillation. The oscillations executed are represented graphically in Fig. 66. The dotted wave curve represents the variable amplitude; the actual oscillations of the pendulum are represented by the full line curve. An attentive examination of the diagram reveals the fact that at every "beat" the pendulum loses half an oscillation; this loss can easily be observed, using the apparatus described above. Were it not for this loss of half an oscillation at each beat, the oscillations of the pendulum would be indistinguishable from a

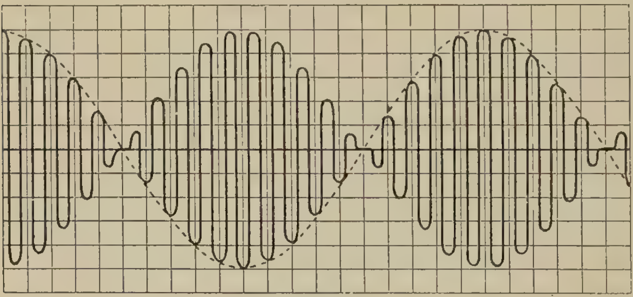

FIG. 66.—Graph of the oscillations executed by a member of a coupled system.

simple harmonic motion of which the amplitude periodically waxes and wanes.

Oscillations of an ordinary pendulum.—The results deduced from the foregoing investigation, so far as they refer to the conditions under which, in general, the oscillations of a pendulum are observed, may be summarised briefly as follows:—

1. **The pendulum will simultaneously execute two simple harmonic motions of different frequencies.** In general, the natural frequency of the support will be much greater than the natural frequency of the pendulum: thus, the quicker oscillation of the pendulum will have a frequency somewhat greater than the natural

frequency of the support, and the slower oscillation will have a frequency less than the natural frequency of the pendulum. This can be proved readily by the aid of Fig. 64.

2. **The smaller the ratio** (m/M), **the more nearly do the oscillation frequencies of the pendulum agree with the natural frequencies of the pendulum and the support.** This, also, can be proved by the aid of Fig. 64.

3. **The amplitude of the oscillation of greater frequency will be small in comparison with the amplitude of the oscillation of smaller frequency**; in other words, the oscillation which is executed with a frequency approximately equal to the natural frequency of the pendulum will have a much larger amplitude than that which is executed with a frequency approximately equal to the natural frequency of the support.

In equations (12) and (13), let m/M be very small; that is, let the mass of the pendulum bob be very small in comparison with the inertia of the support. Further, let n_s, the frequency of vibration of the support, be very great in comparison with n_p, the frequency of the pendulum. Then—

$$\frac{m}{\mathrm{M}}\left(\frac{n_p}{n_s}\right)^2$$

will be negligibly small in comparison with unity.

Let a_1 be the amplitude of the oscillation executed with the frequency n_1, agreeing approximately with the natural frequency n_p of the pendulum; and let a_2 be the amplitude of the oscillation executed with the much greater frequency n_2, agreeing approximately with the natural frequency n_s of the support. Then from (12) and (13) (p. 162)—

$$\frac{a_1}{a_2}=\frac{(n_p)^2-(n_2)^2}{(n_1)^2-(n_p)^2}$$

n_1 is slightly smaller than n_p, so that the denominator of the fraction on the right-hand side is a very small negative quantity. Also, n_2 is very much greater than n_p, so that the numerator of the fraction is a very large negative quantity. Hence a_1/a_2 is a very large positive quantity.

We conclude, therefore, that the inertia and the frequency of vibration of the support of a pendulum should both be as great as possible; in other words, **the support should be massive and rigidly constructed.** In these circumstances, the quicker oscillation executed by the pendulum will have an amplitude which is negligibly small, and the slower oscillation will be executed in a period only slightly longer than the true period of the pendulum.

Compound pendulum with elastic support.—Having determined the conditions which must be complied with by the support of a pendulum, in order that the oscillations of the pendulum may be interfered with as little as possible, it now becomes necessary to determine the correction that must be applied to the experimental results obtained, in order that the effects produced by the yielding of the support may be eliminated.

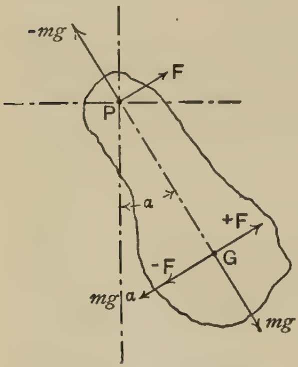

FIG. 67.—Compound pendulum swung from an elastic support.

Let Fig. 67 represent a body oscillating about the point P. The oscillating body obviously exerts a variable force on the support at P; this force is equal and opposite to that which the support exerts on the body. Let us determine the value of the force exerted on the body at the instant when it is at the extremity of an oscillation of angular amplitude α.

Let G be the centre of gravity of the body, and let the line PG be called the axis. If m is the mass of the body, the force of gravity, acting vertically downwards at G, is equal to mg; and the components of this force, perpendicular and parallel to the axis, are respectively equal to $mg\sin\alpha$ and $mg\cos\alpha$, where α is the inclination of the axis to the vertical. Since α is small, the force perpendicular to the axis has the approximate value $mg\alpha$ and that parallel to the axis has the approximate value mg. The centrifugal force acting on the support at P is small, since it is proportional to the square of the angular velocity of the body, and the angular velocity is small.

The support must exert on the body a force equal to $-mg$, in the direction of the axis GP produced. This, however, is not the only force exerted on the body; for as the body oscillates to and fro, its motion may be resolved into a rotation about the centre of gravity G, and a linear motion of the centre of gravity (compare p. 55). When the body is at the extremity of an oscillation, its position being that represented in

Fig. 67, it has just ceased to rotate about G in an anticlockwise sense, and is just about to commence to rotate about G in a clockwise sense; hence a torque must act in a clockwise sense about G. Let F be the force exerted by the support on the body at P, in a direction perpendicular to the axis; then if $PG = l$, the force F produces a torque Fl about G, and this is equal to the rate at which the moment of momentum about G is increasing (p. 61). But the moment of momentum is equal to the moment of inertia mk^2 about G, multiplied by the angular velocity; and the rate of increase of the moment of momentum is equal to mk^2 multiplied by the angular acceleration, and, at the end of an oscillation of angular amplitude α, the angular acceleration is equal to $(2\pi/T)^2\alpha$ (p. 157). Hence—

$$Fl = mk^2\alpha\left(\frac{2\pi}{T}\right)^2.$$

At G apply two opposite forces, perpendicular to the axis, each numerically equal to F. The force F acting at P, and the equal but opposite force (– F) acting at G, merely change the angular velocity about G without affecting the linear acceleration of G (p. 40). The linear acceleration of G is due to the resultant of $mg\alpha$, acting from right to left, and the force F acting from left to right; that is, to $(mg\alpha - F)$. Hence, since the linear acceleration of G is equal to the angular acceleration multiplied by l, we have—

$$mg\alpha - F = ml\alpha\left(\frac{2\pi}{T}\right)^2,$$

$$\therefore\ \alpha\left(\frac{2\pi}{T}\right)^2 = \frac{mg\alpha - F}{ml},$$

and

$$Fl = mk^2 . \frac{mg\alpha - F}{ml},$$

$$\therefore\ F(l^2 + k^2) = k^2 . mg\alpha,$$

$$\therefore\ F = mg\alpha\frac{k^2}{l^2 + k^2}.$$

Now, the resultant horizontal force exerted on the body by the support is equal to the resultant of the horizontal components of mg (the force parallel to the axis) and of F (the force perpendicular to the axis). Hence the resultant force R exerted on the support by the body is given by the equation—

$$R = mg\sin\alpha - F\cos\alpha$$
$$= mg\alpha - F,$$

since α is small

$$\therefore\ R = mg\alpha\left(1 - \frac{k^2}{k^2 + l^2}\right) = mg\alpha\frac{l^2}{k^2 + l^2}.$$

If λ is the length of the equivalent simple pendulum—

$$\lambda = \frac{l}{l^2 + k^2} \text{ (p. 101).}$$

Hence
$$R = mga\frac{l}{\lambda}.$$

We can now use this result to determine the correction which must be applied to observations made with a reversible pendulum, in order to eliminate error due to the yielding of the support Let P (Fig. 68) be the position of the point of support when the pendulum hangs vertically downwards; and let P be the position of the point of support when the pendulum is at the extremity of an oscillation of angular amplitude α. Let the axis GP′ of the pendulum, when produced, cut the vertical through P′ in the point K; then K is the stationary point on the axis of the pendulum, and **the oscillations are executed as if the point K on the axis were fixed.** Let $PK = \delta_1$.

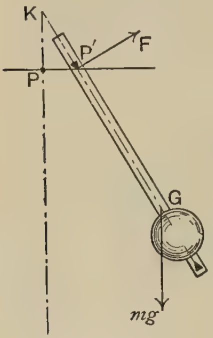

FIG. 68.—Reversible pendulum swung from an elastic support.

If unit force applied to the support produces a displacement d, the displacement PP′ produced by the horizontal force R is equal to Rd, and therefore—

$$PP' = mgda\frac{l_1}{\lambda},$$

where l_1 is the distance from P to the centre of gravity G of the pendulum in its "erect" position (p. 103).

Then $\alpha = \frac{PP'}{\delta_1}$, and therefore—

$$\delta_1 = \frac{PP'}{\alpha} = mgd\frac{l_1}{\lambda}.$$

When the pendulum is swung in the "inverted" position, the stationary point on the axis will be at a distance δ_2 above P, where—

$$\delta_2 = mgd\frac{l_2}{\lambda}.$$

If the periods of oscillation in the "erect" and "inverted" positions are not exactly equal, the values of λ, in the formulæ for δ_1 and δ_2, will differ slightly; but it is unnecessary to determine δ_1 and δ_2 with a very great percentage accuracy, since both of these quantities are small, and are finally added, as corrections, to much larger quantities. Hence in both cases we may write $\lambda = l_1 + l_2$.

Let
$$L_1 = l_1 + \delta_1 = l_1\left(1 + \frac{\delta_1}{l_1}\right) = l_1\left(1 + \frac{mgd}{\lambda}\right)$$
$$L_2 = l_2 + \delta_2 = l_2\left(1 + \frac{mgd}{\lambda}\right).$$

In determining the potential energy of the pendulum at the extremity of a swing (p. 97), we must remember that the centre of gravity moves in an arc of a circle of radius L_1 or L_2; also L_1 and L_2 must be substituted for l_1 and l_2 in the expressions for the moment of inertia of the pendulum in its erect and inverted positions. Hence, finally, we obtain the following formulæ for the periods T_1 and T_2 in the erect and inverted position—

$$g\frac{T_1^2}{(2\pi)^2} = \frac{L_1^2 + k_2}{L_1},$$
$$g\frac{T_2^2}{(2\pi)^2} = \frac{L_2^2 + k^2}{L_2}.$$

Dealing with these equations in the manner explained on p. 102, we obtain—

$$\frac{4\pi^2}{g} = \frac{T_1^2 + T_2^2}{2(L_1 + L_2)} + \frac{T_1^2 - T_2^2}{2(L_1 - L_2)}$$

In the first term on the right-hand side of this equation—

$$L_1 + L_2 = (l_1 + l_2)\left(1 + \frac{mgd}{\lambda}\right)$$
$$= l_1 + l_2 + mgd,$$

since $\lambda = l_1 + l_2$, to a first approximation. Now, d is the displacement of the point of support due to the application of a horizontal force of unit value; hence mgd is the displacement produced by a horizontal force equal to the weight of the pendulum. To determine this, the pendulum may be hung from one end of a cord which passes over a pulley, the other end of the cord being attached to the support so as to pull this in a horizontal direction; the resulting displacement of the point of support can be measured with a microscope.

In evaluating the term $(T_1^2 - T_2^2)/2(L_1 - L_2)$, which will be very small, since T_1 and T_2 are nearly equal, we may write $L_1 - L_2 = l_1 - l_2$, these quantities being measured in the manner explained on p. 103.

Damped oscillations.—In the foregoing investigations, it has been assumed that oscillations are performed under the action of an attracting force directed towards the position of equilibrium of the oscillating body. In this case there is no progressive diminution in the amplitude of the oscillations; consequently such oscillations, when once started, would persist for ever. This, of course, is at variance with experience; any oscillating body, if left to itself, comes to rest in a longer or shorter interval of time, since frictional forces—that is, forces which *always tend to diminish the velocity of motion*—cannot be eliminated entirely. Oscillations which progressively diminish in amplitude are said to be **damped.**

When a body moves through air at a velocity which does not exceed a few feet per second, it is acted on by a frictional opposing force which is proportional to the velocity. The air is pushed away from the body in front and flows up to the body behind (p. 149), with the result that contiguous layers of air move with different velocities; the air being viscous (p. 2) or endowed with a property which opposes the relative motion of its parts, energy is dissipated in the form of heat, and therefore the body is acted upon by a force which opposes its motion. Even if a body oscillates in a vacuum, frictional forces are generally called into play; thus if the body executes torsional oscillations about a wire to which it is attached, the twisting and untwisting of the wire produce relative motions in its parts, and the viscosity of the material of which the wire is composed causes a continual dissipation of energy.

It is obvious that the effect of a force, which always opposes the motion of an oscillating body, is to cause the amplitude to diminish progressively. As the body moves from its position of equilibrium to the extremity of a swing, some energy is dissipated, and therefore the potential energy possessed by the body at the extremity of the swing is less than the kinetic energy possessed by it as it passed through its position of equilibrium (compare p. 94). Thus, **a very small frictional force will cause the amplitude to diminish at a perceptible rate.** On the other hand, **a small frictional force produces but a small alteration in the period of oscillation.** As the body moves from its position of equilibrium to the end of a swing, its motion is opposed, not only by the central force proportional to the displacement, but also by the frictional force ; hence it reaches its turning point (that is, the position at which it is stationary for an instant) in a smaller time than would have been required had the frictional force been absent. As the body returns from the extremity of a swing to its position of equilibrium, its motion is accelerated by the central force and retarded by the frictional force ; hence the time occupied in this portion of the oscillation is longer than it would have been had the frictional force been absent. Thus, in going from, and returning to its position of equilibrium, the time required for the outward journey is diminished, and that required for the return journey is increased ; hence, since the diminution and increase are both small, the net result will be a very small alteration in the time required for each half oscillation.

The properties of oscillations, controlled by a force proportional to the displacement from a fixed point, have been deduced from the revolution of a body under the action of a central force proportional to the distances from the fixed point (p. 92). It follows that the properties of oscillations damped by a frictional force proportional to the velocity can be deduced from the revolution of a body about a fixed point, under the joint action of a central force proportional to the displacement from that point, and an opposing force proportional to the velocity. The characteristics of such revolutions have been explained previously (pp. 74 to 82) ; we can now apply the results obtained to the study of damped oscillations.

Let a body P (Fig. 69) revolve about the fixed point O under the action of a central force directed toward O and equal to $f_1 \times \text{OP}$; together with an opposing force proportional to the velocity, and equal to κ when the velocity has unit value ; then the body will traverse an equiangular spiral ABCDE, with a uniform angular velocity ω given by the equation—

$$\omega = \sin \alpha \sqrt{\frac{f_1}{m}},$$

where m is the mass of the body ; α is the angle of the spiral, and is given by the equation—

$$\cos \alpha = \frac{\kappa}{2} \sqrt{\frac{1}{f_1 m}}.$$

Let the motion of the body be resolved parallel and perpendicular to the axis JOH ; then the motion parallel to JOH is controlled only by the forces parallel to JOH. The component of the central force

$f_1 \times OP$, resolved parallel to JOH, is equal to $f_1 \times OQ$, where Q is the projection of P on JOH; this is a force proportional to the displacement OQ of the point Q from O. Let the velocity of the body P be equal to v, and let the direction of this velocity make an angle ϕ with JOH. Then the opposing force acting on the body is equal to κv, and the component of this force, resolved parallel to JOH, is equal to $\kappa v \cos \phi$, and $v \cos \phi$ is the velocity of the point Q. Hence the motion of the body parallel to JOH is controlled by a force directed toward O, and equal to $f_1 \times OQ$, together with an opposing force equal to ($\kappa \times$ velocity of Q).

Hence **if P moves along the equiangular spiral ABCDE with a uniform angular velocity ω, the motion of its projection Q will determine the oscillations performed by a body, controlled by a force directed toward O and equal to $f_1 \times$ OQ, and opposed by a frictional force proportional to the velocity.** The curve to the right of Fig. 69 shows the displacement at each instant during the course of these oscillations. It is evident that the amplitude continually decreases as the time increases. It was proved on p. 79, that when κ is small, the angular velocity with which P revolves about O is practically the same as if the frictional force were absent; and since a complete revolution of P corresponds to a complete damped oscillation, it follows that the period of oscillation is scarcely affected by the frictional force when κ is small.

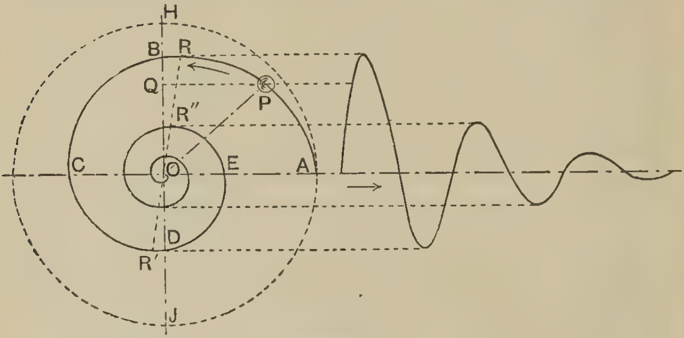

FIG. 69.—Graphical representation of damped oscillations.

Correction of period of oscillation for damping.—A pendulum, when suitably constructed and mounted, will continue to oscillate for a very long time, with the consequence that the period of oscillation can be determined with very great accuracy. Although the damping of the oscillation produces a very small effect on the period of oscillation, this effect may yet exceed the unavoidable experimental error in determining the period. It therefore becomes necessary to determine the correction due to damping; that is, to determine, from the observed period T, the period T′ that would have been observed if damping had been absent.

The angular velocity ω of P (Fig. 69) in its spiral path, is obviously equal to $2\pi/T$, where T is the period of one revolution. Hence—

$$T = \frac{1}{\sin \alpha} \cdot 2\pi \sqrt{\frac{m}{f_1}}$$

$$= \frac{T'}{\sin \alpha},$$

where T′ is the period of revolution under the action of the central force alone. Then, if T represents the observed period of the damped oscillation, T′ will represent the period of the same oscillations if damping were removed, and—

$$T' = T \sin \alpha.$$

Thus, the correction coefficient for damping is equal to $\sin \alpha$, where α is the angle of the spiral.

The value of α must be determined by observing the rate at which the amplitude decreases. The extremity of a linear oscillation will be reached as P passes through a point R on the spiral, such that the tangent to the spiral at R is perpendicular to OH (Fig. 69). Since the tangent makes a constant angle α with the radius vector OR (p. 75), it follows that the projection of R on OH is equal to OR $\sin \alpha$.

The next extremity of a swing *on the same side of* O will be the projection on OH of the point R″ of the spiral; R″ lies on the line OR, since the angle of the spiral is constant. The amplitude of this oscillation is equal to OR″ $\sin \alpha$. The time which elapses between the instants when P passes though R and R″ is obviously equal to T, since the radius vector turns through an angle equal to 2π in this time.

Now, let OR $= r_0$, while OR″ $= r_1$. Then if a_0 is the amplitude when P passes through R, and a_1 is the amplitude when P passes through R″, we have—

$$\frac{a_0}{a_1} = \frac{r_0 \sin \alpha}{r_1 \sin \alpha} = \frac{r_0}{r_1}.$$

a_0 and a_1 are two consecutive amplitudes on the same side of the point of equilibrium O. Let a_n be the amplitude after n complete oscillations, and let r_n be the corresponding length of the radius vector of the spiral. Then—

$$\frac{a_0}{a_n} = \frac{r_0}{r_n}.$$

From equation (7), p. 82, we see that the radius vector r_t measured at the end of any interval of time t, is related to the radius vector r_0 at the beginning of that interval, according to the equation—

$$r_t = r_0 \cdot \epsilon^{-\frac{\kappa t}{2m}}.$$

To determine r_n, notice that this radius vector is measured n complete periods, each equal to T, later than r_0. Thus $t = n\mathrm{T}$, and—

$$\frac{a_0}{a_n} = \frac{r_0}{r_n} = \epsilon^{\frac{\kappa n \mathrm{T}}{2m}}$$

$$\therefore \frac{\kappa n \mathrm{T}}{2m} = \log_\epsilon a_o - \log_\epsilon a_n,$$

$$= 2{\cdot}30\,(\log_{10} a_o - \log_{10} a_n).$$

$$\therefore \frac{\kappa \mathrm{T}}{2m} = \frac{2{\cdot}30}{n}(\log_{10} a_o - \log_{10} a_n).$$

Now, from p. 79, $\cos \alpha = \frac{\kappa}{2}\sqrt{\frac{1}{f_1 m}} = \frac{\kappa \sin \alpha}{2m}\left(\frac{1}{\sin \alpha}\sqrt{\frac{m}{f_1}}\right)$

$$= \frac{\kappa \sin \alpha}{2m} \cdot \frac{\mathrm{T}}{2\pi};$$

$$\therefore \cot \alpha = \frac{1}{2\pi} \cdot \left(\frac{\kappa \mathrm{T}}{2m}\right)$$

$$= \frac{1}{2\pi} \cdot \frac{2{\cdot}30}{n} \cdot (\log_{10} a_o - \log_{10} a_n).$$

This suffices to determine the angle α of the spiral, and the correction coefficient for damping is equal to $\sin \alpha$.

Problem.—*In one of Captain Kater's experiments, the arc of swing decreased from* 1·41° *to* 1·18° *in* 500 *seconds. Determine the correction coefficient of the period of oscillation, for damping.*

Here $a_0/a_n = 1{\cdot}41/1{\cdot}18$.

The number n of swings is not given, but we know that $n = \frac{500}{\text{T}}$, where T is the observed time of swing.

Then
$$\cot\alpha = \frac{1}{6{\cdot}28}\cdot\frac{2{\cdot}30\times\text{T}}{500}\cdot(0{\cdot}1492 - 0{\cdot}0719)$$
$$= 0{\cdot}000056\text{T},$$
$$\sin\alpha = \frac{1}{\sqrt{1+\cot^2\alpha}} = 1 - \tfrac{1}{2}\cot^2\alpha + \ldots$$
$$= 1 - (\text{T}^2\times 1{\cdot}5\times 10^{-9}).$$

The period of oscillation was about 2 seconds, so that the correction coefficient was equal to $1 - 6\times 10^{-9}$. This coefficient was so nearly equal to unity that the error incurred in neglecting the difference was exceeded by the unavoidable errors of observation.

The quantity $\frac{2{\cdot}30}{n}(\log_{10} a_0 - \log_{10} a_n)$ is called **the logarithmic decrement.** It is equal to the logarithm of the ratio of the amplitudes of two consecutive swings on the same side of the position of equilibrium. Notice that the logarithm of the amplitude decreases proportionately with the time, and therefore—

$$\frac{1}{n}\log_e\left(\frac{a_0}{a_n}\right) = \log_e\left(\frac{a_0}{a_1}\right).$$

Some authors use the term logarithmic decrement to denote the logarithm of the ratio of two consecutive amplitudes on opposite sides of the zero; obviously, this value will be half of that obtained according to the above definition.

The results obtained above apply, not only to pendulum oscillations, but to oscillations of all kinds.

Problem.—*A suspended-coil galvanometer has an observed period of five seconds. When the suspended system is set oscillating, the spot of light on the scale moves from its position of equilibrium through* 150 *scale divisions; the next displacement of the spot of light on the same side of its position of equilibrium is equal to* 100 *scale divisions. Determine the period of oscillation of the suspended system, corrected for damping.*

Assume that the scale is so far from the mirror attached to the suspended system, that the deflections of the spot of light are proportional to the angular displacements of the suspended system.

Here $n=1$, and $a_0/a_1=150/100$; then—

$$\cot\alpha = \frac{2\cdot30}{2\pi}\log_{10}1\cdot5$$
$$= \frac{2\cdot30}{6\cdot28}\times 0\cdot1761 = 0\cdot0645,$$

$$\sin\alpha = \frac{1}{\sqrt{1+\cot^2\alpha}} = 1-\tfrac{1}{2}\cot^2\alpha = 1-0\cdot002.$$

Therefore the period of an undamped oscillation $= 5(1-0\cdot002)$
$= 4\cdot99$ sec.

Even in this case, where the damping is considerable, the period is only increased by two-tenths of 1 per cent.

The ballistic galvanometer.—In certain circumstances it is necessary to determine the value of a quantity of electricity, suddenly discharged through a galvanometer, in terms of the "throw" of the needle (that is, the angle through which the needle swings from its position of equilibrium). Observations of the kind are called **ballistic,** and a galvanometer which can be used in making such observations is called a **ballistic galvanometer.** In general, instruments of this character are provided with a mirror which is attached to the moving system; a beam of light, reflected from the mirror, is focussed on a horizontal scale, on which it forms a bright spot. When the moving system turns through an angle θ, the beam of light turns through twice this angle, or 2θ. The distance through which the spot of light moves over the scale, divided by the distance of the scale from the mirror, gives $\tan 2\theta$. In general, 2θ is so small that $\tan 2\theta$ may be taken as equal to the circular measure of 2θ, and hence θ is found readily. In order that the displacement of the spot of light may be observed accurately, it is necessary that the period of oscillation of the suspended system shall be long; it generally lies between 5 sec. and 20 sec.

Ballistic galvanometers are of two types. In suspended magnet instruments, the moving system consists of a magnet hung by a torsionless fibre, and controlled by the earth's horizontal magnetic field or the field of a fixed magnet; the current is sent through a fixed coil of which the axis is perpendicular to the direction of the magnet in its position of equilibrium. In suspended coil instruments, the current is sent through a coil which is hung between the poles of a fixed magnet, the plane of the coil being parallel to the field of the magnet; the controlling force acting on the coil is due, either to a thin metal strip from which the coil is hung, or to a bifilar suspension consisting of two thin metal strips. In any case, the period of oscillation T of the suspended system is given by the equation—

$$T = 2\pi \sqrt{\frac{I}{\tau_1}},$$

where I is the moment of inertia of the suspended system about the axis of rotation, and τ_1 is the restoring torque called into play by unit angular displacement or twist. In order to make the period of oscillation sufficiently long, the moment of inertia of the suspended system may be made large, or the torque per unit twist may be made small; a small torque per unit twist will ensure a large deflection for very small currents.

When a steady electrical current is transmitted through a galvanometer of this type, the deflection, if this is small, is practically proportional to the current. Thus the steady deflection ϕ, and the current C which produces it, are connected by the equation—

$$\phi = GC,$$

where G is a constant, depending on the construction and adjustment of the instrument. It is clear that G represents the steady angular deflection that would be produced by unit current.

In a ballistic experiment, a quantity of electricity Q is transmitted through the galvanometer; during the transmission the current rises from zero to its maximum value, and then sinks to zero; **it is essential that the transmission shall be completed before the suspended system has moved appreciably from its position of equilibrium.** The current exerts a torque on the suspended system, and this swings away from its position of equilibrium, its maximum angular deflection being (say) θ. If the suspended system were undamped, it would continue to oscillate through an angle θ on either side of its position of equilibrium; thus θ denotes the angular amplitude of the oscillations executed by the suspended system. The effect of damping will be explained later; for the present, let us assume that the oscillations of the suspended system are undamped.

At the instant when the electrical discharge is completed, the suspended system has acquired an angular velocity equal to $(2\pi\theta/\mathrm{T})$; for it is moving from its position of equilibrium with an angular velocity equal to that which it will possess at each subsequent passage through the same position. Let the discharge be completed in a time t; then the moment of momentum acquired during the time t is equal to $\mathrm{I}\omega = \mathrm{I} \times (2\pi\theta/\mathrm{T})$ (p. 61), and this is equal to the product of the average torque and the time t during which it has acted. Now a steady current C would produce a steady angular deflection equal to GC, and it must therefore produce a torque equal to $\tau_1\mathrm{GC}$, since τ_1 denotes the torque per unit twist. Hence, the torque acting on the suspended system is proportional to the current, and therefore the average torque during the discharge must have been proportional to the average current Q/t. Thus, the average torque acting during the time $t = \tau_1\mathrm{G}(\mathrm{Q}/t)$, and this value, multiplied by the time t, gives the moment of momentum acquired by the moving system.

$$\tau_1\mathrm{GQ} = \mathrm{I}\,\frac{2\pi\theta}{\mathrm{T}},$$

$$\mathrm{Q} = \frac{1}{\mathrm{G}} \cdot \frac{\mathrm{I}}{\tau_1} \cdot \frac{2\pi\theta}{\mathrm{T}}$$

$$= \frac{1}{\mathrm{G}}\left(\frac{\mathrm{T}}{2\pi}\right)^2 \cdot \frac{2\pi\theta}{\mathrm{T}} = \frac{\mathrm{T}}{2\pi\mathrm{G}}\,\theta.$$

The constant G can be obtained by sending a steady current of known value through the instrument, and observing the steady deflection produced. Q can then be obtained in terms of T and θ, both of which can be observed.

Correction of ballistic observations for damping.—The value of θ used in the preceding investigation, is the angular amplitude of the oscillations that would have been generated in the suspended system if there had been no damping. The damping of a ballistic galvanometer should be small, but some damping is certain to occur; thus the observed amplitude will be somewhat smaller than it would have been in the absence of damping. The method of determining the undamped angular deflection θ, from observations of the slightly damped oscillations actually produced, must now be explained.

The effect of damping on linear oscillations has been investigated already. In passing from linear oscillations to angular oscillations, the restoring force per unit displacement, f_1, must be replaced by the restoring torque per unit twist, τ_1; and the mass of the body, m, must be replaced by I, the moment of inertia about the axis of rotation. Similarly, κ, the opposing frictional force per unit linear velocity, must be replaced by the opposing frictional torque per unit angular velocity; the latter may still be denoted by κ. If the amplitude θ of an undamped angular oscillation is represented by the radius of the circle AHJ (Fig. 69, p. 174) and a point moves round AHJ with uniform velocity in the time T, the projection of this point on JOH will give the instantaneous value of the angular displacement. When damping is present, the tracing-point must move along the equiangular spiral ABCDE with uniform angular velocity; it has been proved already that the angular velocity is scarcely affected by the damping.

Now, the angle of the spiral will be small when the damping is small; and in this case, if we suppose the tracing-point to start from A at the instant when the discharge is completed, the first turning-point will occur practically at B, and OB is the amplitude of the first of the damped oscillations. The next amplitude will be practically equal to OD, and so on.

Now, as proved on p. 81—

$$\frac{\mathrm{OA}}{\mathrm{OB}} = \frac{\mathrm{OB}}{\mathrm{OC}} = \frac{\mathrm{OC}}{\mathrm{OD}},$$

$$\therefore\ (\mathrm{OC})^2 = \mathrm{OB} \times \mathrm{OD},$$

and

$$\mathrm{OA} = \mathrm{OB}\frac{\mathrm{OB}}{\mathrm{OC}} = \mathrm{OB}\frac{\mathrm{OB}}{\sqrt{\mathrm{OB} \times \mathrm{OD}}} = \mathrm{OB}\sqrt{\frac{\mathrm{OB}}{\mathrm{OD}}}.$$

Let $\mathrm{OB} = \theta_1$, the observed amplitude of the first angular oscillation of the suspended system; and let $\mathrm{OD} = \theta_2$, the next observed amplitude on the opposite side of the zero. Then the amplitude $\theta = \mathrm{OH} = \mathrm{OA}$ that would have been observed if damping had been absent, is given by the equation—

$$\theta = \theta_1\sqrt{\frac{\theta_1}{\theta_2}}.$$

If a_1 and a_2 denote the displacements of the spot of light corresponding to the angular oscillations θ_1 and θ_2, and a is the displacement that would have been observed in the absence of damping, we have—

$$a = a_1 \sqrt{\frac{a_1}{a_2}}.$$

The results obtained on p. 176 show that a_1/a_2 is a constant ratio so long as the damping remains constant; therefore, in order to determine a, the value of a_1 must be multiplied by a factor which can be calculated once for all, and used so long as there is no alteration in the conditions under which the galvanometer is used.

Many text-books of physics give a more complicated correction for damping, depending on the calculation of the logarithmic decrement. **The method explained above gives the undamped amplitude with an accuracy as great as that with which the damped amplitude can be observed**; any further refinements are obviously unnecessary.

When the damping is very small, a further simplification may be made.

Let $a_2 = a_1 - \delta$, so that δ is the difference between the amplitudes of the first two oscillations on opposite sides of the zero. Then—

$$a = a_1 \sqrt{\frac{a_1}{a_1 - \delta}} = a_1 \frac{1}{\left(1 - \frac{\delta}{a_1}\right)^{\frac{1}{2}}} = a_1\left(1 + \tfrac{1}{2}\frac{\delta}{a_1} + \ldots\right) = a_1 + \frac{\delta}{2}\ .$$

This result will be correct to within less than one per cent., if δ/a_1 is not greater than 0·1. Consequently, this correction is sufficiently accurate for ordinary experimental work.

To determine the zero position, from observations made on an oscillating system.—It frequently happens that much time can be saved by determining the position of equilibrium of an oscillating system without waiting for the system to come to rest. The approximation obtained at the end of the last section shows that if the amplitude decreases by δ in half a period, then it will decrease by $\delta/2$ in a quarter period; and we may conclude that in a whole period it will decrease by 2δ. Hence two consecutive oscillations on the same side of the zero will differ in amplitude by 2δ, and the point midway between the extremities of these oscillations will differ from the extremity of the first oscillation by δ. Also the intermediate oscillation on the opposite side of the zero will have an amplitude less than that of the first oscillation by δ. Hence the following rule, which is extremely useful in determining the position of equilibrium of the pointer of a balance, and in other similar cases.

Let a, b, c, be three consecutive turning-points, a and c being on one side, and b on the other side of the position of equilibrium. It is best to measure a, b, and c from one end of the scale; by this means negative values are avoided. Then the position of equilibrium is midway between $(a+c)/2$ and b, and its value is therefore $(a+c+2b)/4$.

Aperiodic and dead-beat motion.—The angle α of the equiangular spiral, from which a series of damped oscillations may be derived, is determined by the equation (p. 79)—

$$\cos\alpha = \frac{\kappa}{2}\sqrt{\frac{1}{f_1 m}}.$$

Now if $(\kappa/2) = \sqrt{(f_1 m)}$, we shall have $\alpha = 0$, and the spiral will make zero angle with the radius. In this case, if the spiral commences at B (Fig. 69, p. 174), it will coincide with the radius BO, and the tracing-point will never pass beyond O. Consequently, the body will move directly up to its position of equilibrium and remain there. This kind of motion is called **dead-beat**; it is brought to considerable perfection in many well-designed ammeters and voltmeters. If κ is greater than $2\sqrt{1/(f_1 m)}$, the motion toward the position of equilibrium will be of the same character, but it will be much slower; in this case the motion is said to be **aperiodic.**

Methods of varying the damping of a galvanometer.—The damping of a galvanometer may be *increased* in several ways according to the construction of the instrument.

1. (*a*) In a **suspended-magnet galvanometer,** the moving system may be enclosed in a small chamber, the clearance allowed being just enough to permit of the required motion of the system. In this case, the friction between the walls of the chamber, and the air which is set in motion by the moving system, is considerable.

(*b*) Vanes attached to the moving system may be immersed in oil of suitable viscosity.

(*c*) The moving magnet may be placed very close to a disc of copper.

As the magnet moves, its lines of force cut the copper and produce induced currents in it; the energy dissipated as heat by these currents is derived from the moving system, and consequently this soon comes to rest.

2. In a **suspended-coil galvanometer**, the damping may be increased by joining the terminals of the instrument by a wire of low resistance; or by winding the coil on a copper frame, round which the currents induced by cutting lines of force can flow.

The damping of a galvanometer may be *decreased* by increasing the moment of inertia of the suspended system. If a_0 and a_1 are two consecutive amplitudes on the same side of the zero, then—

$$\frac{\kappa T}{2I} = \log_e \frac{a_0}{a_1} \text{ (compare p. 176).}$$

If the moment of inertia, I, of the suspended system is quadrupled, T will be doubled; and therefore if κ remains constant, log (a_0/a_1) will be halved. Thus if the original values of a_0 and a_1 were 120 and 100, quadrupling the moment of inertia of the suspended system would change a_0 and a_1 to 120 and 110, or practically halve the damping. At the same time, the sensitiveness of the instrument would be halved; for (p. 180)—

$$Q = \frac{T}{2\pi G}\theta,$$

and therefore for a given value of Q, the value of the product $T\theta$ is constant; and quadrupling the moment of inertia doubles T, and therefore halves θ.

Forced oscillations.—The oscillations of a body or a system of bodies, which is set in motion and then left to itself, have been considered in the previous investigations. Such oscillations are said to be *free*. Attention must now be directed to the case where a body, capable of oscillation, is acted upon by an external force which varies periodically. In this case the body is constrained to oscillate in a period equal to that of the periodic force, and the resulting oscillations are said to be *forced*.

The general nature of forced oscillations may be studied by the aid of Fig. 70; this diagram represents a pendulum attached to a point of support which itself executes a s.h.m. On starting, the motion of the pendulum is apparently irregular, due to the simultaneous execution of free and forced oscillations. But after a time the free oscillations die down, and the pendulum executes a s.h.m. in a period equal to that of the point of support.

Let A′A be the actual length l_1 of the simple pendulum. Then if T_1 is the period of its free oscillation, $T_1=2\pi(l_1/g)^{\frac{1}{2}}$. Let T be the period of the s.h.m. executed by the point of support. A pendulum of length l would complete an oscillation in T seconds, if $T=2\pi(l/g)^{\frac{1}{2}}$. Thus, the pendulum must oscillate as if it were supported from a point O, by means of a fibre of length l. Consequently, if $T>T_1$, we must have $l>l_1$, and the point O will be on the side of A′ remote from A. In this case the phases of the point of support and the pendulum bob will be equal. As the point of support moves from A′ to B′, the pendulum bob moves from A to B (I, Fig. 70). If $T<T_1$, we must have $l<l_1$, and the point O lies between A′ and A. In this case the phases of the point of support and the pendulum bob differ by π. As the point of support moves from A′ to B′ (III, Fig. 70), the pendulum bob moves from A to B; at any instant the point of support and the pendulum bob will be moving in opposite directions.

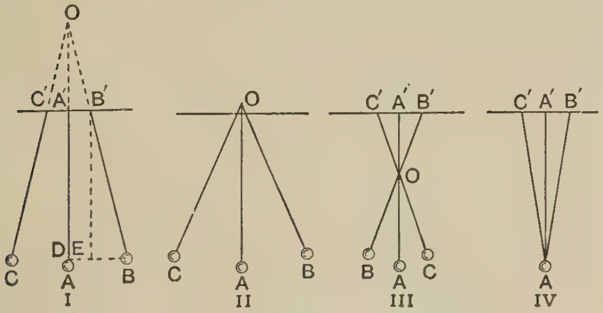

Fig. 70.—The characteristics of forced oscillations.

With a given amplitude of the point of support, it is obvious that the arc AB described by the pendulum bob will increase as the point O approaches A′. When O is situated at A′, $l=l_1$, and the period of oscillation of the point of support is equal to the free period of the pendulum. In this case an infinitesimal periodic displacement of the point of support will produce an indefinitely great amplitude of swing in the pendulum bob (II, Fig. 70). This is an instance of the **sympathetic communication of oscillations.** It will be remembered that an ordinary swing can be caused to oscillate over a considerable arc by small impulses, each of which is applied so as to increase the velocity of the swing as it passes through its position of equilibrium.

Lastly, when T is excessively small in comparison with T_1, l will be very small in comparison with l_1, and the point O will coincide approximately with A. The pendulum bob will then be unable to move appreciably during the time required for the point of support to complete a vibration. Consequently, in this case, the pendulum bob will remain stationary (IV, Fig. 70).

Let α and a be the amplitudes of the oscillations respectively executed by the pendulum bob and the point of support (Fig. 70, I.) Draw BD perpendicular to A′A. Then $DB = \alpha$ and $A'B' = a$. From the similar triangles BDO and B′A′O, we have—

$$\frac{DB}{OD} = \frac{A'B'}{OA'}, \text{ or } \frac{\alpha}{l} = \frac{a}{l - l_1};$$

$$\therefore \alpha = a \frac{l}{l - l_1}.$$

Also
$$l = g\left(\frac{T}{2\pi}\right)^2, \text{ and } l_1 = g\left(\frac{T_1}{2\pi}\right)^2;$$

$$\therefore \alpha = a\frac{T^2}{T^2 - T_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When $T > T_1$, α has the same sign as a. As T approaches the value T_1, the value of α increases, and becomes equal to infinity when $T = T_1$. When $T < T_1$, α and a have opposite signs, indicating a difference of π in the phases of the corresponding vibrations. Finally, when $T = 0$, $\alpha = 0$.

From B′ draw B′E perpendicular to A′B′, cutting DB in E. Then $EB = (\alpha - a)$. As the point of support is displaced from A′ to B′, the position of equilibrium of the pendulum bob is displaced from A to E; we may term A and E its positions of *absolute* and *relative equilibrium*. Similarly, we may term EB, or $(\alpha - a)$, the *relative displacement* of the pendulum bob. From reasoning similar to that employed on p. 90, it follows that the restoring force acting on the pendulum bob when at B is proportional to EB, or $(\alpha - a)$. If f_1 is equal to the restoring force called into play by unit displacement of the pendulum bob when the point of support is fixed at A′, then $f_1(\alpha - a)$ will be equal to the restoring force when the bob is at B and the point of support is displaced to B′. Let **F**$=f_1(\alpha-a)$. When **F** is positive, it will act from B toward D. The tension of the suspending filament will exert a reaction equal to **F** on the point of support; when **F** is positive, this reaction will tend to increase the displacement of the latter.

From the similar triangles BEB′ and B′A′O, we have—

$$\frac{\mathrm{EB}}{\mathrm{B'B}}=\frac{\mathrm{A'B'}}{\mathrm{OB''}}, \text{ or } \frac{\alpha-a}{l_1}=\frac{a}{l-l_1};$$

$$\therefore\ \alpha-a=a\frac{l_1}{l-l_1}=a\frac{\mathrm{T}_1^2}{\mathrm{T}^2-\mathrm{T}_1^2}.$$

Consequently—

$$\mathbf{F}=f_1(\alpha-a)=f_1 a\frac{\mathrm{T}_1^2}{\mathrm{T}^2-\mathrm{T}_1^2} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The reaction due to the pendulum increases from zero to **F** as the pendulum bob moves from A to B. It tends to increase or decrease the displacement of the point of support, according as T is greater or less than T_1. As the value of T approximates to T_1, the value of **F** approaches $\pm\infty$. The work W, performed by the pendulum bob on the point of support, during the displacement of the latter from A′ to B′, is equal to the average value of the reaction (*i.e.* **F**/2) multiplied by the distance A′B′. Thus—

$$\mathrm{W}=\frac{f_1 a^2}{2}\,\frac{\mathrm{T}_1^2}{\mathrm{T}^2-\mathrm{T}_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If $\mathrm{T}>\mathrm{T}_1$, part of the kinetic energy possessed by the pendulum bob when it passes through its position of equilibrium, A, is afterwards used up, during the displacement from A to B, in producing an increased displacement of the point of support. When $\mathrm{T}<\mathrm{T}_1$, the point of support does work on the pendulum, the energy of the latter being greater at the extremity of an oscillation than when passing through the point A.

Equations (1), (2), and (3) comprise no magnitudes relating merely to a pendulum. They will apply equally well to any body attracted toward a point with a force proportional to the displacement, the point itself being constrained to execute a s.h.m.

Resultant of two mutually perpendicular s.h.m.'s which agree in period and phase.—Let a tracing-point move with uniform velocity around the circle ABCD (Fig. 71), and let its instantaneous position be projected on the axis X′OX, thus defining a s.h.m. in that axis. Further, let a second tracing-point move around another circle EFGH, its instantaneous positions being projected on the axis YOY′, thus defining a s.h.m. in that axis. If the two tracing-points complete their circular paths in equal times, the periods of the s.h.m.'s will be equal; and if one tracing-point passes through A at the instant when the other passes through D, the phases of the s.h.m.'s will be equal. Divide each circle into sixteen equal parts, and number these as in the diagram. When the tracing-points are at A and D, the resultant of the two displacements is equal to OK, where K is the intersection of perpendiculars to OX and OY drawn through the points A and E respectively. When the

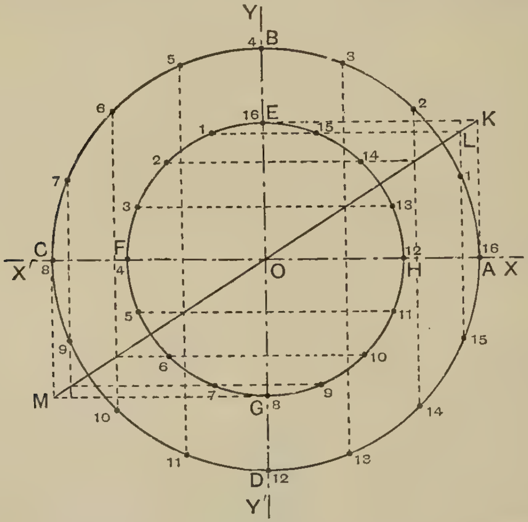

FIG. 71.—Graphical method of determining the resultant of two mutually perpendicular s.h.m.'s which agree in period and phase.

tracing-points pass through the points numbered 1 on the two circles, the resultant of the two displacements is equal to OL, where L is the intersection of perpendiculars to OX and OY drawn through the points numbered 1 on the circles. Proceeding in this manner, the resultant of the two s.h.m.'s is found to be a s.h.m. of amplitude equal to OK, executed in the straight line KOM, inclined to the axis XOX' at an angle of which the tangent is equal to (AK/EK) or (OE/OA).

The same result can be obtained analytically. Let the s.h.m. parallel to XOX satisfy the equation—

$$x = a_1 \cos \frac{2\pi t}{T},$$

while the s.h.m. parallel to YOY′ satisfies the equation—

$$y = a_2 \cos \frac{2\pi t}{T},$$

the period T being the same for both. Then at any instant the resultant displacement is equal to—

$$\sqrt{x^2+y^2} = \sqrt{a_1{}^2+a_2{}^2} \,.\, \cos \frac{2\pi t}{T},$$

which is the equation to a s.h.m., of period $=$ T, and amplitude $= \sqrt{a_1{}^2+a_2{}^2}$, executed in a straight line inclined to XOX′ at an angle of which the tangent is equal to $y/x = a_2/a_1$.

Consequently, to change the direction in which a s.h.m. is executed, we must compound it with another s.h.m. in a perpendicular direction, the two s.h.m.'s agreeing in period and phase.

Foucault's pendulum.—If we wish to set a simple pendulum oscillating in its own free period, by moving the point of attachment of the supporting filament, the point of attachment must be caused to execute a s.h.m. agreeing in period with the pendulum (p. 185). A uniform motion of the point of support corresponds to a s.h.m. of infinitely great period, and such a motion would not set the pendulum oscillating in its free period (p. 186). Hence, we conclude that a uniform motion of the point of support cannot alter the plane in which a pendulum swings. For, in order to alter the plane of oscillation, we must produce another oscillation in the free period of the pendulum, and in a direction perpendicular to that in which the pendulum is oscillating; and a uniform motion of the point of support cannot do this.

Now, when a simple pendulum is set swinging, its point of support moves with the earth; but the motion is uniform, and therefore the pendulum continues to swing in a plane parallel to that in which its oscillations were started.

By reasoning similar to that used on p. 67, it can be proved that at a point on the earth in latitude θ, a horizontal N and S line is rotating with an angular velocity $\omega \sin \theta$, where ω is the angular velocity of the earth about its axis. Hence, any horizontal line in latitude θ rotates with an angular velocity $\omega \sin \theta$.

If we set a simple pendulum oscillating above any horizontal line on the earth, the line rotates with an angular velocity $\omega \sin \theta$, but the directions in which the oscillations of the pendulum are performed remains unchanged. Hence, the line on the earth rotates beneath the pendulum, and to an observer who considers the line to be stationary in space, the direction of the oscillations of the pendulum appears to rotate with an angular velocity $\omega \sin \theta$. This phenomenon was observed by Foucault; it furnishes a direct proof of the rotation of the earth, which would be at our disposal even if we had no means of determining it by observing the apparent motion of the stars.

EXPT. 16.—Hang a simple pendulum from a horizontal arm attached to a retort stand, and observe that the plane of oscillation of the pendulum remains unchanged if the retort stand is rotated uniformly about a vertical axis.

QUESTIONS ON CHAPTER V

1. A watch is placed, face upwards, on a scale pan which is hung from a vertical wire; how would you expect the time indicated by the watch to be affected, (*a*) if the period of torsional oscillation of the watch and scale pan, about the wire as axis, is greater than the period of the balance wheel of the watch; and (*b*), if the period of the torsional oscillations is less than that of the balance wheel? What would happen if the period of the torsional oscillations were equal to that of the balance wheel?

2. The bob of a pendulum is observed to swing to a distance of 5 cm. on either side of its position of equilibrium, and 5 minutes afterwards the amplitude of swing has diminished to 3 cm. What time must elapse before the amplitude has diminished to 1 mm.?

3. A pendulum consists of a spherical bob of lead, weighing five kilograms, hung from a long thin wire; the period of oscillation of the pendulum is equal to 5 seconds. When the bob is at rest, a bullet weighing 5 gm. is fired into it, along a horizontal line passing through the centre of the bob, and the pendulum is thereby set in oscillation over an arc of 5°. Calculate the velocity of the bullet.

CHAPTER VI

GRAVITATION

Constancy of weight of matter.—If a piece of tinfoil be weighed, first when it is spread out flat, and afterwards when it is rolled tightly into a ball, it will be found that the weight is the same in both cases. Hence we conclude that the weight of a given quantity of matter at a given place is independent of the way in which the matter is distributed, and we may infer that the downward pull exerted on each particle is the same, whether the particle is isolated or is surrounded by other particles.

Landolt has carried out some very delicate tests in order to determine whether a chemical reaction affects the weight of matter. The principle on which his experiments were designed can be understood from the following description of an ideal arrangement. Let the limbs of an inverted U-tube be filled, one with a solution of ferrous sulphate, and the other with a solution of silver sulphate, the extremities of the limbs being sealed. Let the tube be cleaned and dried externally, and then let it be weighed carefully, first before the solutions have been allowed to mix, and subsequently after the solutions have been mixed by shaking the U-tube. When the solutions mix, metallic silver is deposited, and ferric sulphate is formed. Since the tube is sealed, none of the matter can escape; and any change of weight which could not be accounted for as due to moisture deposited on, or evaporated from, the outside of the glass, or to a change of temperature[1] of the U-tube, or to some similar cause, would indicate that the chemical reaction produced a change in the weight of the reacting substances. Landolt observed that the weight of the vessel and its contents was not exactly the same, before and after the reaction ; but his latest researches have shown that the differences observed were due to the deposition of moisture, and to a slight rise in temperature due to the reaction, and when allowance was made for these sources of error, the weight of the reacting substances was found to be unchanged by the reaction.

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[1] When a body, warmer than the surrounding air, is weighed on an ordinary balance, the convection currents of air rising from the warm body impinge on the balance arm above, and so produce an apparent loss of weight in the body.

Hence we conclude that **the downward pull of gravity on each atom of a substance is independent of the presence or absence of neighbouring atoms, and the resultant pull is the sum of the pulls exerted on all the atoms.** This gives the most conclusive proof of the law that the attraction of gravity is proportional to the mass of the attracted body. (Compare p. 21.)

Gravitational attraction.—The earth itself is merely a large aggregate of matter. Since the earth attracts every particle of a material body, we may infer that a material body attracts every particle of the earth. Further, we may infer that any two material bodies attract each other ; and if the distance between the bodies is constant, and very large in comparison with the dimensions of either, the attraction will be doubled if we double the mass of one body, and quadrupled if we double the masses of both. In general, if m_1 and m_2 are the masses of two small bodies, separated by a constant distance d, then the force f exerted by one on the other varies as m_1m_2, or $f \propto m_1m_2$.

In order to explain the motions of the planets around the sun, Newton assumed that the force exerted by one body on another varies inversely as the square of the distance between them. Combining this with the result obtained above, we have

$$f \propto \frac{m_1m_2}{d^2},$$

or

$$f = G\frac{m_1m_2}{d^2},$$

where G is a constant called the **Newtonian constant of gravitation.** The value of G is equal to the force exerted by one unit mass, on another unit mass placed at unit distance, provided that each mass of matter is so small that any particle of one is equidistant from *all* particles of the other.

Newton's law of gravitation cannot be verified directly, since it applies to the attraction between bodies which are very small in comparison with the distance separating them ; hence, laboratory experiments, intended to test this law, would have to be performed with such small bodies that the forces called into play would be unmeasurable. When experiments are performed with bodies of finite dimensions, each particle of one body must be assumed to attract every particle of the other, according to the law given above ; then, by employing suitable mathematical devices, the total attraction exerted by one body on the other can be calculated, and the result compared with that obtained experimentally. Some of the mathematical devices referred to will now be explained.

POTENTIAL

Gravitational potential.—Let a body of unit mass (its dimensions being so small that the Newtonian law applies directly) be placed at a distance d_a from a particle of mass m ; then the force f exerted on the unit mass is equal to $G(m \times 1)/d_a^2$. Now let the unit mass be moved in a straight line away from the particle, until the distance between the two is equal to d_b ; then if d_b is only slightly greater than d_a, the average force exerted on the unit mass, as it moves over the distance $(d_b - d_a)$, will be sensibly equal to $Gm/d_a d_b$, since this value is less than Gm/d_a^2, the force exerted at the first point, and greater than Gm/d_b^2, the force exerted at the second point. The work done during the displacement of the unit mass is called the **difference of potential** between the points. This will be equal to—

$$G \frac{m}{d_a d_b} (d_b - d_a) = Gm \left(\frac{1}{d_a} - \frac{1}{d_b} \right).$$

Now suppose that we start with the unit mass at a distance d_1 from a mass m, and move it away from m through points at distances d_2, d_3, d_4, ... d_n. The work done in the displacement from d_1 to d_2, and that from d_2 to d_3, and from d_3 to d_4, &c., can be written down directly. The difference of potential between the starting and stopping points is equal to the total work done during the displacement, and this is found by adding the following quantities :—

$$\mathrm{G}m\left(\frac{1}{d_1}-\frac{1}{d_2}\right)$$

$$\mathrm{G}m\left(\frac{1}{d_2}-\frac{1}{d_3}\right)$$

$$\mathrm{G}m\left(\frac{1}{a_3}-\frac{1}{d_4}\right)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\mathrm{G}m\left(\frac{1}{d_{n-2}}-\frac{1}{d_{n-1}}\right)$$

$$\mathrm{G}m\left(\frac{1}{d_{n-1}}-\frac{1}{d_n}\right).$$

Adding these quantities together, we find that their sum is equal to—

$$\mathrm{G}m\left(\frac{1}{d_1}-\frac{1}{d_n}\right).$$

If d_n is infinitely great in comparison with d_1, we may neglect $1/d_n$. Hence, **the work done in moving unit mass from a point at a distance d_1, to another point at an infinite distance from a mass m, is equal to** $\mathrm{G}m/d_1$.

In moving the unit mass away from the attracting particle, work must be performed by an external agent; but if the unit mass moves towards the attracting particle, work is performed on the external agent. In the first case, the work performed has a positive sign, in the second case, it has a negative sign; but if the path over which the unit mass is carried is the same in both cases, the numerical value of the work is the same in both cases. Hence, if unit mass is carried from an infinite distance to a point at a distance d_1 from an attracting particle of mass m, the work done is equal to $(-\mathrm{G}m/d_1)$; this quantity is called **the gravitational potential** at the given point. Thus, **the gravitational potential at a given point is equal to the work done in bringing unit mass to the point from an infinite distance.**

Thus, at each point in the neighbourhood of an attracting mass the gravitational potential will have a definite negative value; at an infinite distance the potential will be equal to zero.

All points at a uniform distance r from a single attracting particle, and therefore lying on an imaginary sphere concentric with that particle, will be at a uniform potential $(-\mathrm{G}m/r)$; for in moving unit mass from one point to another on the sphere, the motion will be at right-angles to the force acting towards the attracting particle at the centre; therefore no work is done, and there will be no difference of potential between points on the spherical surface. In general, a surface which passes through all points possessing a certain definite potential, is called an **equipotential surface.**

If the distances, measured from a given point to attracting particles of masses m_1, m_2, m_3, m_4, &c., be r_1, r_2, r_3, r_4, &c., then the potential at the given point is equal to—

$$-G\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}+\frac{m_3}{r_3}+\frac{m_4}{r_4}+\ldots\right).$$

For, the forces exerted by the various attracting particles are independent one of another, and therefore the work done in moving unit mass from infinity to the given point is equal to the sum—

work done by attracting force due to m_1, + *work done by attracting force due to* m_2, + *work done by attracting force due to* m_3, + *&c.*

If the potentials at two neighbouring points A and B are known, the force per unit mass, acting along the direction AB, can be calculated. For, let V_1 be the potential at A, while V_2 is the potential at B. If f is the force per unit mass acting along the line AB, then $f \times AB$ = work done in moving unit mass from A to B, and this is equal to the difference of potential between A and B.

$$\therefore f \times AB = V_2 - V_1$$

and

$$f = \frac{V_2 - V_1}{AB}.$$

If V_2 is algebraically greater than V_1, the force acts from B to A; if V_2 is algebraically less than V_1, the force acts from A to B.

The resultant force per unit mass at any point, will be perpendicular to the equipotential surface passing through that point; and the direction of the resultant force will be towards the neighbouring equipotential surface of lower potential.

The potential at any point in the neighbourhood of a number of attracting masses can be obtained, in many cases, without much trouble, since it is merely the sum of a number of terms each of which is a scalar (or undirected) quantity (p. 7); and when the potentials at two neighbouring points are known, the force acting from one of these points to the other can be calculated. If we attempted to calculate the force directly, we should have to find the sum of the component forces due to the various attracting masses, resolved in the given direction; and this is a much more difficult task than that of calculating the potential. In short, a potential, being an undirected quantity, is determined more easily than a force, which cannot be known without a specification of both magnitude and direction.

Potential at a point outside a thin uniform spherical shell.—Let ABDE (Fig. 72) be the section of a uniform spherical shell of radius r, by a plane passing through the centre C. Let it be required to determine the potential at an external point P, due to the gravitational attraction of the shell.

Let the straight line joining P to C cut the shell in the points A and D. Divide the shell into narrow annular strips by means of planes perpendicular to ACD, and let FH be the section of one of these strips. Join F to C, and F and H to P; and from F drop the perpendicular FK on to CA. Then, if the width FH of the strip is very small, the area of the strip will be equal to $2\pi \times \text{FK} \times \text{FH}$.

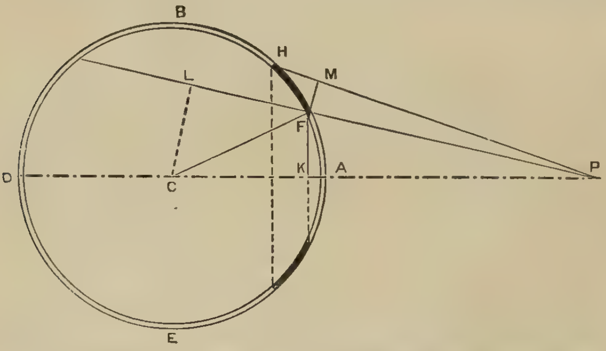

FIG. 72.—Calculation of the potential at a point outside a uniform spherical shell.

Let m be the total mass of the shell, while m_1 is its mass per unit area. Then $m = 4\pi r^2 m_1$. The mass of the strip, of which FH is the section, will be equal to $m_1 \times 2\pi \times \text{FK} \times \text{FH}$, and since the whole mass of the strip is at an approximately uniform distance PF from P, the potential at P due to the strip will be equal to—

$$-G \cdot \frac{2\pi m_1 \times \text{FK} \times \text{FH}}{\text{PF}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From C draw CL perpendicular to PF produced; and with P as centre, and PF as radius, describe the circular arc FM cutting PH in M. Then FHM approximates to a triangle with a right angle at M. Also ∠ FHM = ∠ FCL; for, if we suppose that FH is fixed rigidly at right-angles to CF, and that CF is rotated about C until it lies over CL, then FH will become parallel to PL; and by making FH small enough, we can make PH as nearly parallel to PL as we please. Thus, the triangles FHM and FCL are both right-angled triangles with the angles FHM and FCL equal; hence these triangles are similar, and—

$$\frac{\mathrm{FH}}{\mathrm{HM}} = \frac{\mathrm{FC}}{\mathrm{CL}} = \frac{r}{\mathrm{CL}} \quad \therefore \mathrm{FH} = \mathrm{HM}\frac{r}{\mathrm{CL}}.$$

Further, since the triangles FPK and CPL are similar,

$$\frac{\mathrm{CL}}{\mathrm{CP}} = \frac{\mathrm{FK}}{\mathrm{PF}} \quad \therefore \mathrm{CL} = \mathrm{CP}\frac{\mathrm{FK}}{\mathrm{PF}},$$

and

$$\mathrm{FH} = \frac{\mathrm{HM} \times r \times \mathrm{PF}}{\mathrm{CP} \times \mathrm{FK}}.$$

Therefore in (1)

$$\frac{\mathrm{FK} \times \mathrm{FH}}{\mathrm{PF}} = \frac{\mathrm{FK}}{\mathrm{PF}} \times \frac{\mathrm{HM} \times r \times \mathrm{PF}}{\mathrm{CP} \times \mathrm{FK}}$$

$$= \frac{r}{\mathrm{CP}} \cdot \mathrm{HM}.$$

Let CP = R; thus R denotes the distance from P to the centre of the shell. Also let HM = δ. Then, substituting in (1), the potential at P due to the annular strip of which FM is the section, is found to be equal to

$$-2\pi m_1 \mathrm{G} \cdot \frac{r}{\mathrm{R}} \cdot \delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Now, to determine the potential at P due to the whole of the shell, we must find an expression similar to (2) for each of the annular strips into which the shell has been divided, and then add all of these expressions together. The quantity $2\pi m_1 \mathrm{G}r/\mathrm{R}$ is constant for all these strips, and δ represents MH, which is the increase in the distance from P, due to crossing from one edge of the strip to the other. Hence the sum of the values of δ for the various strips will be equal to the increase in the distance from P due to crossing all the strips, that is, due to passing around the semicircle ABD from A to D, and this is equal to DP − AP = DA = $2r$.

Hence, the potential, V, at P, due to the whole of the spherical shell, is given by the equation—

$$V = -2\pi m_1 G \frac{r}{R} \cdot 2r = -G \frac{4\pi r^2 m_1}{R} = -G \frac{m}{R},$$

and this is the value of the potential at P if the whole mass m of the shell were concentrated at its centre C.

Thus, **at any point outside a uniform spherical shell, the potential has the same value as if the whole of the mass were concentrated at the centre of the shell.**

From this we conclude that **at any point outside a uniform spherical shell, the force per unit mass has the same value as if the mass of the shell were concentrated at its centre.**

Potential at a point outside a solid sphere. If a sphere can be divided, in imagination, into spherical shells each of which is uniform throughout, then the potential due to each shell has the same value as if the whole mass of the shell were concentrated at its centre, and the potential of the solid sphere is equal to the sum of the potentials of the shells. Hence we arrive at the following important law:—

Let the density of a solid sphere be uniform at a constant distance from the centre, although it may vary in any manner as the distance from the centre increases or decreases; then, the potential at any external point, and therefore the force per unit mass at any external point, will have the same value as if the whole mass of the sphere were concentrated at its centre.

As we shall find later, experiments on gravitation are performed generally by observing the attraction exerted by one sphere on another, since the whole mass of each sphere can be considered to be concentrated at its centre, and therefore the Newtonian equation (p. 192) can be applied without difficulty. If we were justified in assuming the earth to be a sphere, its density varying only with the distance from the centre, then the force exerted on unit mass at any external point would have the same value as if the whole mass of the earth were concentrated at its centre. It is often convenient to make the above assumptions, although they are only approximately true.

Force per unit mass inside a uniform spherical shell. Let P (Fig. 73) be a point inside a spherical shell of which ABD is a section through the centre C; it is required to determine the resultant force exerted on unit mass placed at P.

From E and F, any two neighbouring points in AB, draw the straight lines EPG and FPH, cutting the shell, on the side of P remote from E and F, in the points G and H. Let the circle ABD rotate through a small angle ϕ about the diameter ACD; then if $PF=d_1$, while $PG=d_2$, and $\angle FPA=\theta$, the small arc FE will sweep out an element of area equal to $FE \times (d_1 \sin \theta)\phi$, and GH will sweep out an element of area equal to $GH \times (d_2 \sin \theta)\phi$. Also, the angle PFE is approximately equal to the angle PGH. (These angles can be made as nearly equal as we please by making the angle FPE small enough.) Hence, since the angles FPE and GPH are equal, the triangles PFE and PGH are similar, and—

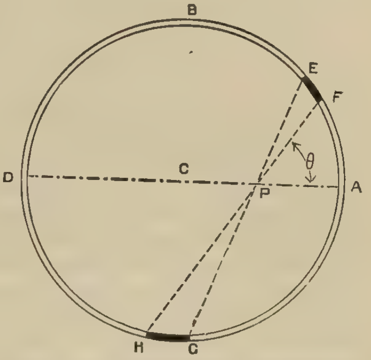

FIG. 73.—To prove that the gravitational attraction of a uniform spherical shell is equal to zero at all internal points.

$$\frac{GH}{PG} = \frac{FE}{PF},$$

$$\therefore \frac{GH}{FE} = \frac{PG}{PF} = \frac{d_2}{d_1}.$$

If the mass per unit area of the shell is equal to m_1, the mass of the element swept out by FE is equal to $m_1 \times FE \times (d_1 \sin \theta)\phi$, and the force exerted by this element on unit mass at P is equal to—

$$\frac{m_1(d_1 \sin \theta)\phi \times FE}{d_1^2},$$

and the force exerted on unit mass at P by the element swept out by GH is equal to—

$$\frac{m_1(d_2 \sin \theta)\phi \times GH}{d_2^2}.$$

The elements swept out by FE and GH exert oppositely directed forces on unit mass at P; hence the resultant force due to the two elements is equal to—

$$\frac{m_1(d_1 \sin\theta)\phi \times \mathrm{FE}}{d_1^2} - \frac{m_1(d_2 \sin\theta)\phi \times \mathrm{GH}}{d_2^2}$$

$$= \frac{m_1(d_1 \sin\theta)\phi}{d_1^2} \times \mathrm{FE} - \frac{m_1(d_2 \sin\theta)\phi}{d_2^2} \times \frac{d_2}{d_1}\mathrm{FE}$$

$$= m_1\mathrm{FE}\left(\frac{\phi \times \sin\theta}{d_1} - \frac{\phi \times \sin\theta}{d_1}\right) = 0.$$

Thus, the elements swept out by FE and GH exert equal and opposite forces on unit mass at P; and the whole of the spherical shell can be divided into pairs of elements similar to the pair swept out by FE and GH, and each pair exerts zero force on unit mass at P. Hence, the **resultant force exerted by the whole shell on unit mass at P is equal to zero.**

Since there is no resultant force acting on unit mass when placed inside the shell, no work is done when unit mass is moved from one point to another inside the shell; hence, **all points within the shell are at the same potential as a point on the surface of the shell.**

Now let us suppose that a very narrow straight tunnel is bored from the surface to the centre of a uniform sphere; and let it be required to determine the force exerted by the sphere on unit mass placed at various positions in the tunnel, and at external points in the same straight line. Let R be the radius of the sphere. At the centre of the sphere the force per unit mass is equal to zero: the whole sphere can be divided into uniform spherical shells, each of which exerts zero force at the centre. At a distance r from the centre, no force is exerted by the shells with radii which exceed r, so that the force per unit mass is equal to the attractions of the shells with radii equal to and less than r; that is, to—

$$\mathrm{G}\frac{\frac{4}{3}\pi r^3}{r^2} = \tfrac{4}{3}\mathrm{G}\pi r.$$

Hence, in proceeding from the centre to the surface of the sphere, the force per unit mass is proportional to the distance from the centre.

At an external point at a distance D from the centre of the sphere the force per unit mass is equal to—

$$\tfrac{4}{3}\pi\mathrm{G}\frac{\mathrm{R}^3}{\mathrm{D}^2}.$$

The variation in the force for points in the tunnel, and at external points in the same straight line, is exhibited graphically in Fig. 74.

THE MASS OF THE EARTH.

Airy's determination of the mass of the earth.—The results obtained above were utilised by Airy in an attempt to determine the mass of the earth by means of a pendulum. It has been explained (p. 21) that the gravitational force acting on unit mass at any place, is equal to the acceleration of a body falling freely at that place, and the value common to these quantities is denoted by g. Further, it has been explained how the value of g may be determined with accuracy by the aid of a reversible pendulum (p. 102). Let the value of g be determined at the bottom of a deep mine shaft; if the point at which the determination is made is at a distance R from the centre of the earth (which may be considered to be roughly spherical in shape), and if M is the mass of the sphere of which R is the radius, then the value g_1 of the force per unit mass is given by the equation—

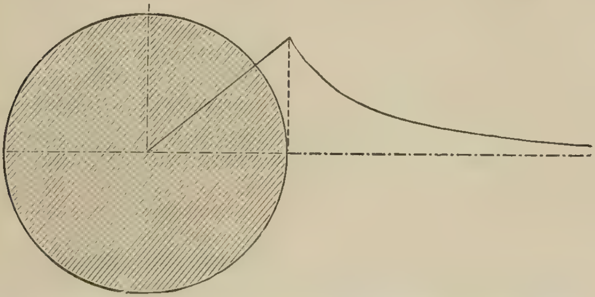

FIG. 74.—Graph showing the gravitational force at various distances from the centre of a uniform solid sphere.

$$g_1 = G\frac{M}{R^2}.$$

Now let the value g_2 of the force per unit mass be determined at the surface of the earth, at a height h above the point at which the value g_1 was determined. Let it be assumed that the matter, comprised in the shell lying between the spherical surface of radius R and the surface of the earth, is of uniform density and of mass m. Then the value of g_2 is given by the equation—

$$g_2 = G\frac{M+m}{(R+h)^2}$$

$$\therefore \frac{g_2}{g_1} = \frac{M+m}{M} \cdot \left(\frac{R}{R+h}\right) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

After several abortive attempts, the values of g_1 and g_2 were determined by Airy at the top and bottom of the Harton coal pit, near Sunderland, in 1854. The depth h of the pit was measured, and R was known from astronomical and geodetic surveys. Airy assumed that the mean density of the shell of thickness h was 2·5, and hence calculated the value of m. Then M remained the only unknown quantity in (2), and this was evaluated. Knowing the dimensions, and therefore the volume of the earth, the mean density of the earth was calculated ; Airy found this to be equal to 6·5. Methods such as this are untrustworthy, since the assumption that the shell of thickness h is uniform in density is untrue ; part of the shell consists of water of density about 1, and the rest of rocks, &c., of density about 2·5. If the shell is of variable density, the assumption which underlies the whole method is inadmissible. Hence, only a rough estimate of the mass and density of the earth can be obtained by such experiments.

Comparison of the mass of the earth with the mass of a mountain.—If a plumb bob is hung near the side of a mountain, a horizontal force due to the attraction of the mountain will pull the bob towards one side, and the attraction due to the remainder of the earth will pull it vertically downwards. The supporting filament will lie along the resultant of these two forces, and if the filament is inclined at an angle θ to the vertical, we shall have—

$$\tan\theta = \frac{f}{F},$$

where f is the horizontal force due to the attraction of the mountain, and F is the vertical force due to the attraction of the remainder of the earth. If the disposition of mass in the mountain were known with accuracy, its attraction on the plumb bob could be calculated. Let us suppose that the calculated attraction of the mountain is equivalent to that of a mass m at a distance

d; then, if M is the mass of the remainder of the earth, the centre of the earth being at a distance R from the plumb bob, we shall have—

$$\tan\theta = \frac{f}{F} = \frac{\frac{m}{d^2}}{\frac{M}{R^2}} = \frac{m}{M}\frac{R^2}{d^2}.$$

In experiments of this kind, the chief difficulty is to determine the distribution of mass in the mountain; for obvious reasons, this distribution cannot be inferred with any high degree of accuracy.

The first comparison between the mass of the earth and that of a mountain was made by Bouguer in 1740. The mountain chosen was Chimborazo, in the Andes. Bouguer encountered serious trouble from snow and wind storms; the cold was intense, and the levelling screws of the instruments could not be turned until they were heated by fire. In spite of these difficulties Bouguer obtained a series of observations, which led to the conclusion that the earth as a whole is about twelve times as dense as the mountain. As might be expected, the result is not very accurate; but it sufficed to prove that the earth is not merely a shell, empty as some had contended, or full of water as others had maintained.

In 1774 Maskelyne performed a similar experiment on the sides of Schiehallion, a mountain in Perthshire. A telescope, pivoted about an east and west axis near to the object glass end, and called a zenith sector, was used. A graduated arc was attached to the telescope near to its eye-piece end, and over this a plumb line hung. The telescope was mounted first on the southern slopes of the mountain, and was pointed to a star near to the zenith at the instant when it crossed the meridian. Then the graduation of the arc over which the plumb line hung was observed. Next, the telescope was removed to the northern slopes of the mountain, and a similar observation was made of the same star as it crossed the meridian once more. The stars are so far away from the earth that rays from one of them reached the two observing stations in sensibly parallel directions; hence, the axis of the telescope had the same direction during the two experiments, but the plumb bob was pulled to the north in the first, and to the south in the second experiment, by the attraction of the mountain (Fig. 75). Hence, the difference between the two readings of the graduated arc where it was crossed by the plumb line, gave the value of 2θ. The mountain was surveyed carefully, and the final result obtained was that the earth is 4·5 times as dense as water. A more accurate survey of the mountain, made thirty years later, led to the value 5 for the mean density of the earth.

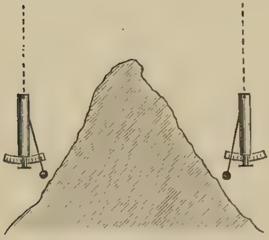

Fig. 75. — Maskelyne's experimental comparison of the mass of the earth with the mass of a mountain.

The Newtonian Constant of Gravitation.

If we can measure the force f exerted by a sphere of mass m_1, on another sphere of mass m_2, the distance between the centres of the spheres being equal to d, the Newtonian constant of gravitation, G, can be calculated from the equation—

$$f = G\frac{m_1 m_2}{d^2}.$$

Apparatus to effect these measurements was designed and made by the Rev. John Michell toward the close of the eighteenth century. Michell died before he had had an opportunity of effecting any measurements, but his apparatus was given to Henry Cavendish, who carried out the experiment in 1797–8 on lines closely resembling those laid down by Michell.

The Cavendish experiment.—Two lead balls, A and B (Fig. 76), each 2 inches in diameter, were suspended by short wires from the ends of a thin deal rod 6 ft. long. In order to strengthen the rod without unduly increasing its mass, its ends were tied by a wire to a short strut, fixed at right angles to the rod at its middle point. The rod was suspended in a horizontal position from a vertical wire attached to the upper extremity of the strut. Two large lead balls, C and D, each 8 inches in diameter, were placed on opposite sides of the rod with their centres in the same horizontal plane as the centres of A and B, while the line joining the centres of A and C, and that joining the centres of B and D, were equal in length, both being perpendicular to the vertical plane in which the rod lay. Thus the pull exerted by C on A, and that exerted by D on B, both tended to twist the rod in one direction about the wire as axis. A horizontal scale, divided into twentieths of an inch, was placed near one end of the rod and at right angles to its length; the end of the rod carried a vernier which moved over the scale without touching it, and thus any change in the angular position of the rod could be measured. The apparatus was protected from air currents by a case (not shown in Fig. 76) and was set up in a closed room; the scale and vernier were read from outside the room by the aid of a telescope.

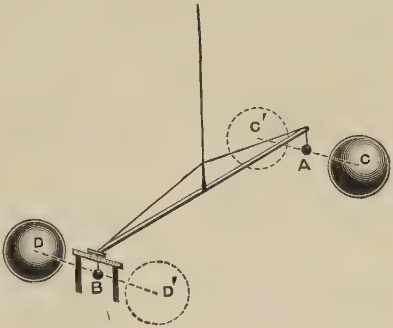

Fig. 76.—The Cavendish experiment.

An observation was made with the attracting balls in the positions C and D shown in Fig. 76. They were then moved to C′ and D′, points in the straight lines CA and DB produced, such that CA = AC′, and DB = BD′. The attractions now exerted on A and B tended to rotate the rod about the wire as axis, in a direction opposite to that of the rotation produced in the first experiment. The angle through which the rod turned between the two experiments gave twice the angle θ through which it was rotated from its position of equilibrium in either experiment. It was found to be best not to attempt to bring the rod to rest when an observation was made, but to observe three consecutive turning points during its oscillations, and from these to infer the position of rest in the manner explained on p. 183.

Let M denote the mass of either of the attracting balls, while m denotes the mass of either of the attracted balls. If d denotes the distance between A and C, or that between D and B, then the force f exerted on either A or B is given by the equation—

$$f = G\frac{Mm}{d^2}.$$

Let l be the distance between the centre of the attracted balls A and B; then, since the force acting on A was equal and oppositely directed to that acting on B, and both forces were perpendicular to the line joining the centres of A and B, it follows that the suspended system was acted on by a torque equal to—

$$fl = Gl\frac{Mm}{d^2}.$$

Let τ_1 be the torque which would produce unit angular deflection of the suspended system; then since the rod was twisted through an angle θ by the attractions exerted on the balls at its ends, the torque acting on it must have been equal to $\tau_1\theta$; thus—

$$Gl\frac{Mm}{d^2} = \tau_1\theta.$$

The torque τ_1 necessary to twist the rod through unit angle was determined in the manner explained on p. 112. Hence, finally—

$$G = \frac{\tau_1\theta d^2}{Mml}.$$

From twenty-nine results obtained by Cavendish, the mean value of G is found to be given by the equation

$$G = 6\cdot562 \times 10^{-8}.$$

The meaning of the result is, that if two spherical gram weights were placed with their centres one centimetre apart, each would exert on the other an attractive force equal to $6\cdot56 \times 10^{-8}$ of a dyne.

A knowledge of the value of G enables us to determine the mean density of the earth. For, at the surface of the earth the force per unit mass is equal to 981 dynes per gram. Let r be the radius of the earth, in centimetres, and let ρ be the mean density of the earth in grams per c.c. Then the mass M of the earth is given by the equation

$$M = \tfrac{4}{3}\pi\rho r^3,$$

and

$$981 = G\frac{M \times 1}{r^2} = \tfrac{4}{3}\pi\rho G r,$$

$$\therefore\ \rho = \frac{3 \times 981}{4\pi G r}.$$

Cavendish's value of G leads to the value 5·448 for the mean density of the earth.

Later experiments made with the torsion balance.—A piece of apparatus resembling that used by Cavendish is called a **torsion balance**, because the torque due to the forces to be measured is balanced against the restoring torque called into play by the torsion or twist of the suspending wire. Cavendish's experiment has been repeated, with various improvements, by Reich in Germany (1837), by Baily in England (1841), by Cornu and Baille in France (1870), by Boys in England (1895), by Braun and by Eötvos in Germany (1896). Of these repetitions, that due to Boys presents the greatest number of points of interest, and gave the result which is probably most nearly correct.

Most torsion balance experiments have been rendered inaccurate by the imperfect elasticity of the wire. When a wire is twisted, the angle of twist is not exactly proportional to the applied torque, and a "permanent set" is also acquired by the wire: that is, when the applied torque is removed the wire does not untwist completely. Prof. Boys found that if fused quartz (rock crystal) be drawn out into filaments, these are perfectly elastic and free from the defects just mentioned. Also they can be made very small in diameter, so as to exert a very small restoring torque when twisted; and they are very strong (stronger than a steel wire of the same diameter would be). Using a quartz fibre suspension, Prof. Boys was able to reduce the size of the apparatus, and thus to keep it at a more uniform temperature and to minimise the disturbance due to air currents.

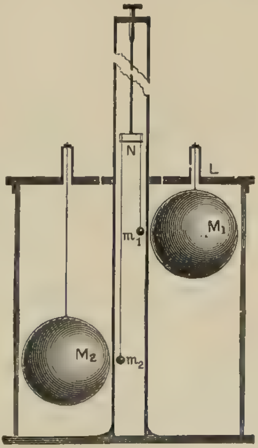

FIG. 77.—Boys's apparatus for determining the Newtonian constant of gravitation.

The design of Boys's torsion balance will be understood on reference to Fig. 77, which represents the apparatus at first experimented with. The two attracted balls, m_1 and m_2, were of gold, and were each about 0·2 inch in diameter. They were hung by quartz fibres from the ends of a small mirror N, 0·9 inch in length; the mirror was itself suspended by a torsion fibre of quartz, about 17 inches long. The attracting masses were lead balls M_1 and M_2, each about 4·5 inches in diameter. In order to minimise the attraction of M_1 for m_2, and of M_2 for m_1, the balls M_1 and m_1 were supported with their centres in a horizontal plane which was six inches above the plane in which the centres of the balls m_2 and M_2 lay. The attracting balls M_1 and M_2 were suspended from a lid L which could be rotated into a position such that the greatest possible torque was exerted on the moving system. The deflections produced were measured by observing the reflection of a scale, graduated in fiftieths of an inch, in the mirror N, by the aid of a telescope. The period of oscillation of the suspended system was a little over three minutes. The results obtained were—

$$G = 6{\cdot}6576 \times 10^{-8}$$
$$\rho = 5{\cdot}5270.$$

Determinations of G by the aid of the ordinary balance have been made by von Jolly in 1878, by Poynting in 1893, and by Richarz and Krigar-Menzel in 1898. In Poynting's experiment two spheres of lead, each weighing about 50 lb., were hung, one from each end of the beam of a balance. These were counterpoised, and then a lead ball weighing about 350 lb. was brought, first under one of the suspended spheres, and then under the other, with the result that the beam was deflected first in one, and then in the opposite direction. The deflection θ having been observed in each case, a centigram rider was moved about an inch along the beam, and the deflection ϕ produced by this was observed. Let M and m denote the masses of the attracting and attracted balls, placed with their centres at a distance d apart; and let us suppose that the centigram rider was removed from a point at a distance l_1 to another at a distance l_2 from the axis of rotation of the beam. Then if the length of the beam of the balance $= 2l$.

$$\frac{G\dfrac{Mm}{d^2}l}{0{\cdot}01 \times g(l_2 - l)} = \frac{\theta}{\phi}.$$

Poynting's results are—

$$G = 6{\cdot}6984 \times 10^{-8}$$
$$\rho = 5{\cdot}4934.$$

Richarz and Krigar-Menzel used a balance which had two scale pans, one a little more than two metres above the other, hung from each end of the beam. A parallelepiped of lead, 2 metres high and 2·1 metres square, was built up in such a position that the lower scale pans of the balance were below it, and the upper pans above it. An experiment was carried out by placing a kilogram weight in the upper left-hand pan, and another equal weight in the lower right-hand pan; the attraction of the lead parallelepiped pulled the first weight down and the second one up, and so depressed the left-hand end of the beam, producing a deflection of the pointer, which was observed. The weight was then transferred from the upper to the lower left-hand pan, and the other weight was transferred from the lower to the upper right-hand pan, when an equal and opposite deflection of the beam was produced. This deflection was compared with that produced by moving a rider of known mass through a definite distance along the beam. Next, the lead parallelepiped was removed, and a similar series of observations were made in its absence. It was found that the weight of a kilogram is increased by more than half a milligram, merely by lowering it towards the earth through two metres.

The values obtained by Richarz and Krigar-Menzel are—

$$G = 6{\cdot}685 \times 10^{-8}$$
$$\rho = 5{\cdot}505$$

PROPERTIES OF GRAVITATION.

The "inverse square law."—The determinations of G, by independent experimenters working under different conditions, afford a means of testing the inverse square law experimentally. In determining the value of G, all experimenters have assumed the inverse square law, and the concordance between their results affords a very satisfactory proof of the truth of that law. The following table gives the approximate masses of the attracting and attracted bodies, together with the distance separating them and the value of G obtained, in a number of the most accurate experiments. The values of d are only approximate, but they suffice to indicate the range of distances over which gravitational attraction has been measured.

Name of experimenter	Attracted mass m	Attracting mass M	Mean distance d	G
Boys	3 grams (about)	7,400 grams	6·5 cm.	$6·6576 \times 10^{-8}$
Braun	54 grams	9,150 grams	10 cm.	$6·6578 \times 10^{-8}$
Cavendish ...	520 grams	155,000 grams	22 cm.	$6·562 \times 10^{-8}$
Poynting ...	21,000 grams	153,000 grams	30 cm.	$6·6984 \times 10^{-8}$
Richarz and Krigar-Menzel	1,000 grams	88,000,000 grams	100 cm.	$6·685 \times 10^{-8}$

Thus, laboratory experiments confirm the truth of the inverse square law for distances lying between 6 cm. and 100 cm.

Measurements of the gravitational attraction exerted by the earth, at points near its surface, afford some additional confirmation. For instance, von Jolly found that the diminution in the value of g, per unit increase in the height above the surface of the earth, is equal to about $2·96 \times 10^{-6}$ dyne per gram per cm.; the calculated value of the same quantity would be $3·08 \times 10^{-6}$ dyne per gram per cm. if the whole mass of the earth were concentrated at its centre. For the attraction of the earth to be exactly the same as if its mass were concentrated at its centre, the density of the earth should be uniform at a constant distance from its centre (p. 198), and we know that this is not the case in the outer crust of the earth, and we have no reason to suppose that it is so in the interior of the earth.

Variations in the density of the earth would have little or no effect on its gravitational attraction at very distant points. Thus, at the moon, the gravitational attraction due to the earth should be $(1/60·3)^2$ times the value of g at the surface of the earth, since the radius of the moon's orbit is practically 60·3 times the radius of the earth. The orbit of the moon is nearly circular; and for a body to revolve in a circular orbit, it must be acted upon by a force directed toward the centre of the circle, and equal to $r\omega^2$ for each unit of mass of the body, where ω is the angular velocity of the body and r the radius of its circular orbit (p. 74). Now a revolution of the moon occupies 27·32 days, or $2·360 \times 10^6$ seconds; and the radius of the moon's orbit is 240 thousand miles, or $3·86 \times 10^{0}$ cm. Hence, for the moon—

$$r\omega^2 = 3·86 \times 10^{10} \times \left(\frac{2\pi}{2·36 \times 10^6}\right)^2 = 0·273 \text{ cm.}/(\text{sec.})^2,$$

and the mean value of g near to the surface of the earth = 980 cm /(sec.)^2, so that—

$$\frac{g}{(60{\cdot}3)^2} = \frac{980}{(60{\cdot}3)^2} = 0{\cdot}269 \text{ cm./(sec.)}^2.$$

These results agree as well as might be expected when we remember that the force of gravity at the surface of the earth is not exactly the same as if the mass of the earth were concentrated at its centre.

The motions of the planets are exactly in accordance with the inverse square law. Hence, finally, we have satisfactory evidence as to the truth of the inverse square law for distances varying between $2{\cdot}78 \times 10^9$ miles (the radius of the orbit of Neptune) and about 2 inches (the distance between the attracting masses in Boys's experiment).

Gravitational attraction is independent of the medium in which the attracting bodies are placed, and is unaffected by interposing matter of any kind between them.

Gravitational attraction is propagated with a velocity so great that it may be considered to be infinite. To understand this statement, let it be supposed that a body A is attracted by another body B. While the bodies are stationary, an equal attraction is exerted on B by A. If A suddenly moves towards B, the attraction on A increases ; but if gravitational force were propagated with a finite velocity, an increased attraction would not be felt at once by B. If such an action as this occurred, it would affect the motions of the planets, and no effects, which could be explained as due to this cause, have been observed.

Variations of terrestrial gravitation.—The relative values of g at various points on the earth's surface may be determined by swinging an invariable pendulum (that is, a compound pendulum provided with only one supporting knife edge) successively at the given stations. Kater and Sabine carried out numerous experiments with an invariable pendulum beating seconds ; later, von Sterneck used an invariable pendulum which beat half seconds. The dimensions of a half seconds pendulum need be only one-quarter those of a seconds pendulum ; hence the former is only about ten inches in length, and it may be enclosed in a case which can be exhausted, so that the correction for the effects of the surrounding atmosphere (p. 152) can be determined with accuracy.

It has been found that, owing to the approximately spheroidal shape of the earth, gravity varies with the latitude. If ϕ is the latitude at any station at the sea level, the latest experiments have shown that the value of g at that station will be given by the equation—

$$g = 978{\cdot}00\,(1 + 0{\cdot}005310 \sin^2 \phi)\ \text{cm.}/(\text{sec.})^2.$$

At a station above the sea level two extra corrections must be applied. In the first place, neglecting the attraction of the ground above sea level, g may be assumed to vary inversely as the square of the distance from the centre of the earth ; in the second place, the attraction due to the ground above the sea level may be allowed for. Values of g obtained experimentally at various heights above the sea level do not agree very well with those calculated according to the above rule. It appears probable that mountain ranges, and high grounds generally, are parts of huge masses of matter which are less dense than that generally found in the interior of the earth, and may be compared with icebergs floating partly submerged in the denser sea water. Hence, we may regard the total mass of matter between any station and the centre of the earth as being independent of the height above sea level.

ATTEMPTS TO EXPLAIN GRAVITATION.

A number of hypotheses as to the cause of gravitational attraction have been advanced by various philosophers and men of science : of these hypotheses, two will now be discussed. Of course, if we are content to rest satisfied with action at a distance as the cause of phenomena, no hypothesis is needed to explain gravitation. But it may be argued, with justice, that Newton's law of gravitation describes the phenomena produced by gravitation, without giving us any idea how one body attracts another. To explain gravitation, we are obliged to suppose that the force exerted by one body on another is due to some kind of connecting mechanism ; the simplest kind of mechanism would be a medium, such as the ether, which produces the apparent attraction of bodies by pushing them towards each other with a force which varies inversely as the square of the distance separating them.

Le Sage's hypothesis.—Le Sage supposed that space is traversed by innumerable small particles endowed with inertia, which are moving in various directions with great velocities. When these particles strike against matter they are brought to rest. A single body would experience no resultant force due to the stoppage of the particles which strike against it; for the change of momentum on one side of the body would be equal to that on the opposite side. But two bodies would be urged towards each other, for the space between them would be poorer in moving particles than the remainder of the surrounding space, and therefore the impacts of particles moving in such directions that their stoppage tends to force the bodies towards each other would exceed those of particles the stoppage of which would tend to force the bodies apart. It should be noticed that the particles must be brought to rest by striking against the bodies; if they merely rebounded from the bodies, we should have the same conditions as in a gas, according to the kinetic theory,[1] and in this case no apparent attraction is produced. But if these particles are brought to rest when they strike against a body, their energy must be communicated to the body in the form of heat; and Maxwell calculated that the energy communicated to a body in this manner would render it red hot in the course of a few seconds. Hence, Le Sage's hypothesis must be abandoned.

Osborne Reynolds's hypothesis.—Dilatancy.—Prof. Osborne Reynolds has proposed an explanation of gravitation in terms of an interesting property of granular media, which he discovered. This is a statical hypothesis: that is, it does not depend essentially on motion of the surrounding medium.

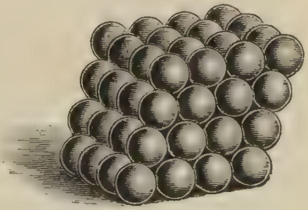
FIG. 78.—Close, or normal piling.

If spherical bodies are piled in such a manner that each fits as far as possible into the spaces between its neighbours, their arrangement will resemble that depicted in Fig. 78; such an arrangement is called *normal piling*. If the bodies are displaced from their positions in normal piling, the empty spaces between them will be increased in size, and the volume occupied by the aggregate will increase (Fig. 79). Hence, if we suppose a space to be filled with spheres in normal piling, any external forces which act on the aggregate, and produce a change in the relative positions of the spheres, will increase the volume of the aggregate. In other words, a granular medium has the property of expanding when it is compressed laterally: this property is called **dilatancy.**

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[1] *Heat for Advanced Students*, by E. Edser (Macmillan), pp. 286–292.

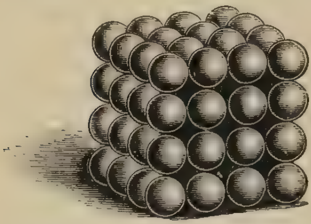
FIG. 79.—Loose, or cubical piling.

EXPT. 17. – Obtain an indiarubber balloon, such as is made to be inflated and used as a toy by children. Pour boiling water on sand which has been sifted; by this means a mixture of sand and water, free from air, is obtained. Introduce the sand into the balloon by the aid of a funnel; the sand may be washed into the balloon by means of water, superfluous water being expelled occasionally. When the balloon is full of the wet sand, fix a piece of glass tube into its mouth, and pour water in until it stands at some height in the tube. On squeezing the sides of the balloon, water is not expelled as might have been expected, but is sucked into the balloon; thus showing that the overall volume of the sand increases under lateral pressure.

EXPT. 18.—Having filled a balloon with wet sand as in the previous experiment, close its mouth with a ligature of cotton, taking care that there is a small superfluity of water, but no air, above the sand. Carefully knead the mass until it assumes a shape like that of a tea-cake. On removing the hands, it will be found that the indiarubber contracts, displacing the sand particles until the superfluous water is sucked into their interstices; but as soon as the superfluous water has disappeared, no further displacement of the sand grains can occur without a vacuum being formed; hence the arrangement becomes quite rigid, and, when it is stood on its edge, can support half a hundredweight.

The explanations of the following phenomena will suggest themselves readily.

The sand on the sea-shore, left wet by the ebbing tide, becomes firm and apparently dry under the pressure of a foot; but when the foot is removed, the sand is found to be surcharged with water. Similarly, cement and mortar become hard under the pressure of the trowel, and regain fluidity when the trowel is removed.

If a measuring vessel is filled with wheat or rice, pressure applied to the grain will cause it to overflow; subsequent shaking will cause the grain to sink down, until about a tenth part of the vessel is left empty.

According to Prof. Osborne Reynolds, the ether may be considered to be a granular medium to which a uniform pressure is applied at the confines of the universe. Let it be supposed that many, comparatively large spheres are embedded in this medium. If the piling of the medium were originally normal, and subsequently the spheres expanded slightly, a displacement of the grains near each sphere would be produced, and each sphere would be surrounded by a space in which the piling is comparatively loose. This loosely-piled space is the seat of potential energy, since the expansion, which accompanied the change from normal to loose piling, took place against the external pressure applied to the medium. Thus, any two of the spheres will appear to attract each other; for a system always tends to assume such a condition that its potential energy is a minimum, and when two spheres are very near to each other the volume of the loosely-piled space surrounding both is smaller than the volumes of the corresponding spaces surrounding the spheres when they are far apart. (Fig. 80.) Prof. Osborne Reynolds has found that the attraction would vary inversely as the square of the distance which separates the spheres.

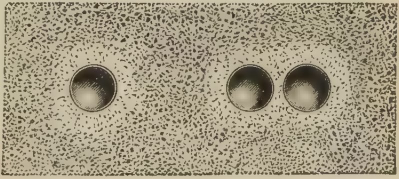

FIG. 80.—Osborne Reynolds's hypothesis to account for gravitation.

QUESTIONS ON CHAPTER VI

1. A, B, and C are the angles of an equilateral triangle, the length of a side of which is equal to l. A mass m is placed at B, and an equal mass is placed at C; determine the resultant gravitational attraction exerted at the point A.

2. A very small spherical body revolves in a circle just outside the surface of a large sphere of density ρ, the plane of the circle passing

through the centre of the sphere. Determine the period of a revolution, if the centripetal force acting on the revolving body is due to the gravitational attraction exerted upon it by the sphere.

3. A narrow straight tunnel is bored from the surface to the centre of a sphere of radius r and density ρ. Determine the work done in carrying a small mass of matter, m, from the centre to the surface of the sphere.

4. Determine the force exerted by the sun on unit mass of the earth, by the method indicated in question 10, p. 45; hence, assuming the value of G, in the c.g.s. system, to be equal to $6{\cdot}66 \times 10^{-8}$, calculate the mass of the sun.

5. Assuming the mean radius of the earth to be equal to $6{\cdot}37 \times 10^6$ metres, and that $g=980$ dynes per gram, while $G=66{\cdot}6 \times 10^{-8}$ in c.g.s. units; calculate the mass of the earth, on the assumption that it is a sphere consisting of concentric shells, each shell being of uniform density.

6. Assuming the orbits of the earth and the moon to be approximately circular, calculate the relative masses of the earth and the sun; given, that the moon completes 13 revolutions in a year, and that the distance from the sun to the earth is 390 times as great as the distance from the earth to the moon.

7. A sphere of platinum, of radius 10 cm., and density 21·5 gm. per c.c., is placed so that it is just immersed in water, on the surface of the earth. Prove that the water will be heaped up over the sphere, and determine the radius of curvature of the water just above the sphere.

8. Let r denote the radius of a uniform solid sphere, while g denotes the gravitational attraction exerted on a gram of matter placed just outside the surface of the sphere. Prove that the gravitational attraction diminishes at the rate of $2g/r$ per centimetre displacement normal to the surface of the sphere.

9. Von Jolly found that, when a mass of 5 kilograms is displaced through 21 metres normal to the surface of the earth, its weight is diminished by an amount equal to the weight of 31·7 milligrams (p. 210). From this result, calculate the diminution of gravitational attraction on a gram, due to a displacement of 1 cm. normal to the surface of the earth; and compare the value obtained with that calculated theoretically, if $g=980$ dynes per gram, and $r=6{\cdot}37 \times 10^6$ metres.

10. A point is chosen near to a uniform plane lamina of small thickness; and when the lamina is viewed from this point, it apparently extends to infinity on all sides. Calculate the gravitational attraction exerted by the lamina on a gram of matter placed at the chosen point, if the mass per unit area of the lamina is equal to m.

CHAPTER VII

ELASTICITY

Stress and strain.—The shape of a solid can be altered only by the application of a suitable system of forces. If the modified shape is retained by the solid only so long as the forces act, and the original shape is regained when they cease to act, the solid is said to possess **shape elasticity**; if the modified shape is retained after the forces cease to act, the solid is said to be **plastic**. Fluids, including liquids and gases, possess no definite shape, and therefore cannot possess shape elasticity. The volume of matter, whether this matter be solid or fluid, can be altered by the application of suitable forces, and in most cases the original volume is regained when the forces cease to act; hence, all matter possesses **volume elasticity**.

Any alteration produced in the shape or volume of matter is called a **strain**. The forces which must be applied in order to produce the strain are called **stresses**. These forces are always distributed over surfaces, and the stresses are measured in units of force per unit area. The study of the elastic properties of matter resolves itself into an examination of the relation between various types of strain and the stresses which produce them.

A body is **homogeneous** if all of its particles possess similar properties. A body is **isotropic** if its properties in any one direction are the same as in all other directions. Fluids are generally both homogeneous and isotropic, but it is difficult to find a solid which complies with these conditions. Fibrous substances (such as wood) will not be isotropic; metals are crystalline in structure, and they will not be isotropic if the crystals are arranged regularly. The following investigations of the elastic properties of matter are carried out on the assumption that the matter is both homogeneous and isotropic; since solids seldom comply exactly with these conditions, the results obtained will be only approximately true, although in many cases the approximation is fairly close.

Compressive and linear strains.—Let a volume V of matter be compressed so that the diminution of volume is equal to v; if the compression is uniform (that is, if all elements of volume which were originally equal are equally diminished), the diminution of volume per unit volume is equal to v/V: this is the quantitative expression for the **compressive strain.** If the strain is not uniform, we can determine its value at any point by finding the diminution of volume v' experienced by a small element of volume V′ immediately surrounding the point, and then calculating the value of v'/V'. A **linear strain** is defined as the change of length per unit length in any line. Thus, if the original length of the line was L, and the change of length is equal to l, the linear strain is equal to l/L. If a positive sign is prefixed to a diminution of length or volume, then a negative sign must be prefixed to an increase of length or volume, and *vice versâ.*

It can be proved easily that **a small uniform compressive strain, equal to δ per unit volume, is equivalent to linear strains equal to $(\delta/3)$ per unit length, in any three mutually perpendicular directions.** For, let a unit cube suffer a diminution of volume equal to δ, and, as a consequence, let each edge of the cube be diminished in length by x. Then the final volume of the cube is equal to $(1-x)^3$, and the diminution of volume, δ, is given by the equation—

$$\delta = 1 - (1-x)^3 = 3x - 3x^2 + x^3.$$

When the strain is very small, x^2 and x^3 will be negligibly small in comparison with x; therefore $\delta = 3x$, and $x = \delta/3$. In the following investigations it will be assumed that all strains dealt with are so small that their squares and higher powers may be neglected in comparison with the strains themselves.

Compressive stress.—A cube of matter can be compressed uniformly by applying equal and uniform pressures to the faces of the cube. In this case, the cube as a whole will be in equilibrium under the action of the forces applied to its faces; for, if the pressure is equal to f_1, the force acting normally on one of the faces of area a is equal to f_1a, and an equal force will act in the opposite direction on the opposite face of the cube.

It will now be proved that **if the faces of a cube are subjected to a uniform pressure f_1, then a pressure f_1 will act across any plane within the cube.**

Let the plane cut the edges of the cube which terminate in O (Fig. 81) in the points A, B, and C. The cube as a whole is in equilibrium under the action of forces, each equal to f_1 per unit area, applied normally to its faces. The part OABC of the cube cut off by the plane ABC must be in equilibrium under the action of the forces exerted on the faces OAC, OBC, OAB, and ABC. Through OA draw an imaginary plane AOD perpendicular to BC, and cutting that line in D; then AD and OD are both perpendicular to BC, since they lie in a plane which is perpendicular to BC. Thus AD and OD are the altitudes of the triangles ABC and OBC, which possess the common base BC. Let $\angle$ ADO $= \theta$; then the normal to the plane ABC will be inclined to AO, the normal to OBC, at an angle θ; for if ABC were rotated about BC until AD coincided with OD, the normal to ABC would become parallel to OA, while the normal and the line AD would both have been turned through the same angle θ.

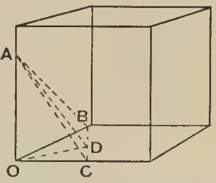

FIG. 81.—Internal stress due to a uniform external pressure.

Let x be the pressure exerted normally on ABC; then since the area ABC $= \frac{1}{2}$. BC . AD, the total force acting normally to ABC is equal to $\frac{1}{2}$. BC . AD . x, and the component of this force, resolved parallel to AO, is equal to $\frac{1}{2}$. BC . AD . $x \cos \theta$. The total force acting normally to OBC is equal to $\frac{1}{2}$. BC . OD . f_1. In order that the portion AOBC of the cube may be in equilibrium—

$$\tfrac{1}{2}\,\mathrm{BC}\,.\,\mathrm{AD}\,.\,x\cos\theta = \tfrac{1}{2}\,\mathrm{BC}\,.\,\mathrm{OD}\,.\,f_1,$$

and since AD $\cos \theta =$ OD, it follows that—

$$x = f_1.$$

Thus, the matter on either side of the plane ABC is subjected to a compressive stress f_1 by the matter on the opposite side of the plane.

If we imagine a spherical surface to be described within the cube, a uniform compressive stress f_1 will act over its surface. For a spherical surface may be imagined to consist of an infinite number of small plane elements of area, and each of these elements will be acted upon by a compressive stress equal to f_1 per unit area.

Bulk modulus of elasticity.—The compressive stresses applied to the surfaces of a cube will produce a certain strain; if we measure the strains corresponding to a series of stresses, we can plot a curve which exhibits the relation between the stresses and the strains they produce. A very small element of this curve differs but inappreciably from a straight line. The curve must start at the origin, for zero stress produces zero strain; and the first part of the curve may be considered to be straight, and therefore the ratio of the stress to the strain for this part of the curve must have a constant value. **The ratio of the stress to the resulting strain is called the modulus of elasticity**, provided the strain is so small that the ratio is constant. Thus the **bulk modulus of elasticity**, denoted by k, is given by the equation—

$$k = \frac{\textit{compressive stress}}{\textit{compressive strain}}.$$

The strain is measured by the ratio of the diminution of volume to the original volume, and therefore it has no dimensions; in other words, its value is independent of the unit in terms of which the volumes are measured. Thus, k is expressed as a certain number of units of force per unit area; preferably, as so many dynes per sq. cm. The dimensions of k are—

$$\frac{\mathrm{ML}}{\mathrm{T}^2} \div \mathrm{L}^2 = \frac{\mathrm{M}}{\mathrm{LT}^2}.$$

The compressive strain produced by a stress f_1 is equal to f_1/k; the corresponding linear strain is one-third of the compressive strain (p. 218), that is, it is equal to $f_1/3k$.

Young's modulus.—Let a cylinder, or a prism, be subjected to equal and opposite stresses applied to its ends, *its sides meanwhile being free from stress*; then the ratio of the stress applied to either end, to the consequent elongation per unit length, is called Young's modulus.

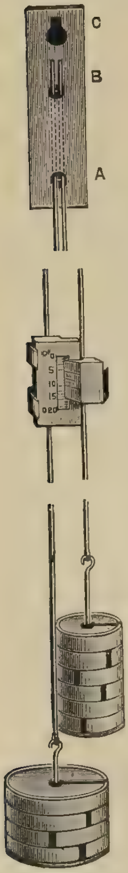

FIG. 82.—Apparatus for the determination of Young's modulus.

Experiment shows that for metals such as steel, brass, etc., the ratio is constant so long as the elongation does not exceed 0·001 of the original length of the cylinder or prism.

Young's modulus will be expressed as a certain number of units of force per unit area. Its dimensions take the same form as those of the bulk modulus.

EXPT. 19.—To determine the value of Young's modulus for the material of which a wire is composed.

When a wire is stretched by equal and opposite forces applied to its ends, the **stress** is equal to the force applied to an end divided by the area of the wire. The **strain** is equal to the elongation divided by the original length of the wire. Since the elongation is necessarily very small, special precautions are necessary in order that it may be measured accurately.

Two precisely similar wires of the material to be tested are threaded through holes A and B (Fig. 82) in a brass plate, and the wires are then soldered to the plate. The plate may be supported from a beam in the ceiling of the laboratory by means of a slotted hole C, which can be slipped over the head of a strong screw.

The lower ends of the wires are attached to rods which terminate in plates, on which circular slotted weights can be supported. One wire carries a scale, graduated in half-millimetres, at a height of about four feet from the floor; a vernier, reading to a tenth of a scale division, is attached to the other wire. The wire to which the scale is attached carries a constant load, sufficient to keep it straight; the load on the other wire must be varied during an experiment. If the varying load produces any displacement of the point of support of the wires, this will entail no relative motion of the scale and vernier, since the upper ends of both wires are fastened together. Any expansion due to a change of temperature will also affect both wires equally. Hence, the strain produced by a known addition to the load can be measured with accuracy.

Enter your observations in a table such as the following. The original load of the wire to be tested is denoted by X; the load must be sufficient to straighten the wire without stretching it very much.

The load may be increased by half a kilogram at a time, but the total increase of load must not be sufficient to stretch the wire beyond the point at which it becomes permanently elongated to an appreciable extent.

Load.	Vernier readings.		Mean.	Elongation.
X	...	...	a	
X+1	...	...	b	
X+2	...	...	c	
X+3	...	↑ ...	d	
X+4	...	...	e	
X+5	↓ ...	...	f	$f-a$
X+6	...	...	g	$g-b$
X+7	...	...	h	$h-c$
X+8	...	...	k	$k-d$
X+9	...	...	l	$l-e$

The vernier readings are recorded, first as the load is increased from X to X+9, and then as the load is once more decreased to X. The mean of the two readings corresponding to each load is entered in the fourth column; let these means be equal to a, b, c, d, &c. Now $(b-a)$, $(c-b)$, $(d-c)$ . . . $(l-k)$ are the extensions due to the addition of one of the weights; but if we attempt to obtain the mean extension by adding $(b-a)$, $(c-b)$ . . . $(l-k)$, we obtain $(l-a)/9$, so that only the first and last observations are utilised. *To avoid wasting the observations* b, c, d . . . k, *we must use no observation more than once.* Thus if we obtain the mean of $(f-a)$, $(g-b)$ . . . $(l-e)$, this will give the mean extension due to the addition of five of the weights, *and all the observations will be utilised in obtaining this mean.* If each weight is half a kilogram, we obtain the mean extension due to 2·5 kilo.; converting the kilograms into dynes, and dividing by the sectional area of the wire, we obtain the stress which produces the observed elongation. To obtain the strain, the mean elongation must be divided by the original length of the wire.

If a longitudinal stress equal to f_1 produces a longitudinal strain equal to e per unit length, then Young's modulus E is given by the equation—

$$\mathrm{E} = \frac{f_1}{e},$$

$$\therefore \quad e = \frac{f_1}{\mathrm{E}}.$$

Poisson's ratio.—When an elongation is produced by longitudinal stresses, a change is produced in the lateral dimensions of the strained substance. Thus, when a wire is stretched, its diameter diminishes; and when the longitudinal strain is small, the lateral strain is proportional to it. **The ratio of the lateral strain to the longitudinal strain is called Poisson's ratio.** Thus, if the original length and diameter of a wire are denoted by L and D, while the increase in length due to a given longitudinal stress is denoted by l, and the diminution of diameter by d, then Poisson's ratio σ is given by the equation—

$$\sigma = \frac{\frac{d}{D}}{\frac{l}{L}}.$$

Methods of determining Poisson's ratio will be described later.

Elastic properties of matter.—If the load carried by a wire is increased progressively, and the elongation corresponding to each load is observed, the relation between the loads and the consequent elongations can be exhibited in the form of a curve such as OABC, Fig. 83. From O to A the curve is approximately straight, that is, the elongation is proportional to the load. But at A the elongation commences to increase more quickly than the load; A is called the **yield point**, and after A has been passed the elastic limits of the wire have been exceeded. Near B the curve becomes practically horizontal, thus indicating that the wire continues to elongate without any increase of load. To prevent the wire from being broken the load must be diminished; thus the curve slopes downwards from B to C.

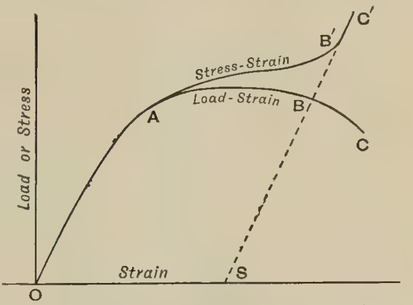

FIG. 83.—Graph exhibiting the relation between load, stress, and strain.

As the elongation of the wire increases, its sectional area diminishes; hence, the curve OABC does not represent the relation between the stress (load per unit area) and the elongation. If we divide each load by the corresponding sectional area of the wire, and plot the results against the elongations, we obtain a curve such as OAB′C′, which represents the relation between the stress and the elongation. It will be observed that the elasticity of the wire breaks down at A, and is recovered near B′; between B′ and C′ the elongation is approximately proportional to the stress, and the ratio of the stress to the strain may even have a greater value than between O and A. If the load is diminished progressively after reaching C′, the curve C′B′AO is not re-traversed, but the elongation diminishes along C′S. At S all stress has been removed, but the wire has not returned to its original length; a permanent elongation OS has been produced. Thus, **if a wire is strained beyond its elastic limits, the yield point is raised, and a permanent elongation (called a permanent set) is produced.** These properties are very noticeable in connection with copper wire; a piece of annealed copper wire is practically plastic; but after it has been stretched it becomes highly elastic. Thus "hard drawn" copper wire (that is, copper wire that has been drawn forcibly through a hole smaller in diameter than the wire) yields under a stress of about 4×10^9 dynes per sq. cm., while pianoforte steel wire yields under a stress of 24×10^9 dynes per sq. cm. The value of Young's modulus for steel wire is about 2×10^{12} dynes per sq. cm.: for hard drawn copper wire it is about $1{\cdot}2 \times 10^{12}$ dynes per sq. cm.; while for annealed copper wire it is about $1{\cdot}0 \times 10^{12}$ dynes per sq. cm. Hammering or rolling a metal produces a similar extension of the elastic limit.

If a polished section of a metal is etched with dilute nitric acid, and then examined under a microscope, its appearance suggests that the metal is made up of numerous aggregates of crystals. If the metal has been strained so as to produce a permanent set, it is observed that relative motions have been produced amongst the crystals; the motion occurs along definite planes of cleavage, and the effect of overstraining, rolling, or hammering the metal is to distribute the planes of cleavage indifferently in all directions, thus decreasing the tendency to yield in any particular direction. If the metal is heated and then allowed to cool slowly, it is said to have been annealed; microscopic examination now shows that the crystals have coalesced into large aggregates, with cleavage planes distributed in definite directions ; hence the metal has again become soft or plastic.

Impurities may increase or decrease the elasticity of a metal. The impurity forms a sort of cement which unites the crystal aggregates ; if this cement is plastic, the elastic limits will be diminished, while if the cement is more rigid than the metal, the elastic limit will be extended. Thus Roberts-Austen found that the addition of 2 per cent. of potassium to gold diminished the strength in the ratio 1 : 12. When carbon is added to molten iron, a carbide Fe_3C is formed ; if there is not much of this carbide present, it acts as a hard cement which unites the crystal aggregates of the iron, and thus the steel formed by the union is much more elastic than the pure iron.

Substances which are elastic at ordinary temperatures become plastic when the temperature is raised sufficiently. The carbon filament of an electric glow lamp is highly elastic at ordinary temperatures, as may be inferred from its continuous quivering ; but when the filament is heated by an electric current, so as to glow, it becomes almost plastic, and can be distorted by placing a pole of a magnet near the lamp. When lead, which is plastic at ordinary temperatures, is cooled in liquid air, it becomes elastic, and rings like steel when struck. Generally speaking, the value of Young's modulus is diminished by a rise of temperature ; but it has been found that by alloying nickel with steel, a substance can be produced for which Young's modulus remain constant, or even increases with the temperature. These nickel steel alloys have proved most valuable in connection with the hair springs of watches and chronometers.

Shearing strain.—When a substance is strained in such a manner that all planes parallel to a certain datum plane remain undistorted and unaltered in dimensions, while they are displaced tangentially, relatively to each other, the substance is said to be sheared. When the shear is uniform, the relative displacement of any two planes is directly proportional to the perpendicular distance between them. Let a number of rectangular sheets of cardboard be piled so as to form a rectangular parallelepiped (Fig. 84) ; then if these sheets are displaced so that all faces of the resulting solid remain plane, the result is a uniform shear of the rectangular parallelepiped.

Let ABCF (Fig. 85) represent a solid cube, and let this suffer a uniform shear, so that it is distorted into the solid A′B′CF. Every point originally in the plane ABCD remains in that plane; any plane parallel to ABCD is called the **plane of the shear.** The value of the shear is measured by the angle A′DA; when the shear is small, the circular measure of the angle will be equal to its tangent, and **the shear θ is equal to** AA′/AD.

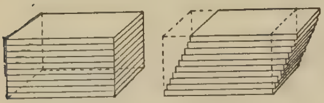

FIG. 84.—Shearing strain.

An important property of a sheared solid may be understood by examining a square piece of wire gauze of about one-tenth inch mesh, before and after it has been sheared (Fig. 86.) It will be observed that each mesh, which was square in the unsheared gauze, is distorted by the shear into a parallelogram with equal sides. Further, the diagonals of each mesh are altered in length. Each diagonal, originally parallel to the line joining the opposite corners D and B of the square, is lengthened; and each diagonal, originally parallel to AC, is shortened. **Hence, all lines originally parallel to BD are lengthened, and all lines parallel to AC are shortened.**

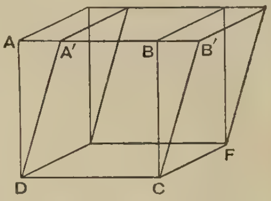

FIG. 85.—Shearing strain.

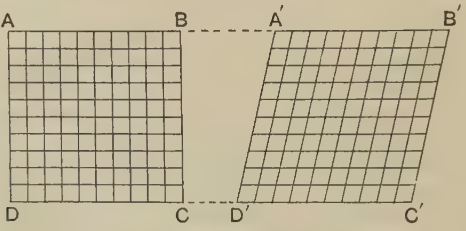

FIG. 86.—Wire gauze before and after shearing.

Join DB and DB′ (Fig, 87), and with D as centre and DB as radius, describe the circular arc BG cutting DB′ in G. Then GB′ is the elongation of the line DB, produced by the shear. If the shear is infinitesimal, the angle BB′G differs from 45° by an infinitesimal amount, and the arc BG differs infinitesimally from a straight line perpendicular to GB′. Hence, BB′G is an isosceles triangle with a right angle at G, and $GB' = BB'/\sqrt{2}$. Further, $DB = CB\sqrt{2}$. Hence, the elongation per unit length of DB is equal to—

$$GB'/DB = (BB'/\sqrt{2}) \div (CB\sqrt{2}) = (BB'/2CB) = \theta/2,$$

since the shear θ is measured by $BB'/CB = AA'/DA$. The student should find no difficulty in proving that the diminution of length per unit length of CA is also equal to $\theta/2$. Finally, from Fig. 86 we see that **all lines originally parallel to DB are equally elongated by $\theta/2$ per unit length, and all lines originally parallel to CA are equally diminished by $\theta/2$ per unit length.** Thus, a shear θ is equivalent to a linear extension $\theta/2$ per unit length combined with a linear compression $\theta/2$ per unit length, in directions which are mutually perpendicular, while either is inclined at an angle of 45° to the direction of the shearing displacement. Hence **we can produce a uniform shear by superposing a uniform elongation at right angles to an equal uniform linear compression.** This will be made clear by reference to Fig. 88.

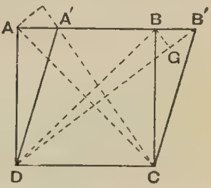

FIG. 87.—Linear strains produced by a shear.

The angle BDG (Fig. 87), through which the diagonal BD is rotated by the shear, is equal to—

$$BG/DB = (BB'/\sqrt{2}) \div (CB'\sqrt{2}) = BB'/2CB = \theta/2.$$

The diagonal CA is rotated through an equal angle in the same direction, so that the lines CA′ and DB′ are at right angles.

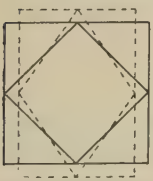
FIG. 88.—Uniform shear, produced by an elongation in one direction, and an equal compression in a perpendicular direction.

Shearing stresses.—If a substance can be sheared only by the application of finite forces, and if the shearing strain disappears when the forces that produced it cease to act, the substance is said to possess shear elasticity, or rigidity.

Let the cube depicted in Fig. 89 possess shear elasticity, and let it be required to determine the system of forces that will produce a uniform shear, the front face A and the back face B being parallel to the plane of the shear. The forces applied must be in equilibrium among themselves; that is, they must have no tendency to set the cube in motion, either linear or rotational. Let a tangential force f act on the face E in the direction of the arrow; and let an equal force act tangentially on the face F, in the opposite direction. The forces must be distributed over the surfaces E and F, so that if a is the area of a face of the cube, then f/a is the force per unit area. These forces obviously tend to produce a shear such as is depicted in Fig. 85, and they have no tendency to communicate linear motion to the cube. On the other hand, they constitute a torque, which tends to set the cube in rotation; and to produce equilibrium, an equal and opposite torque must be applied. Therefore, let forces, each numerically equal to f, act tangentially on the faces C and D in the directions of the arrows; the forces acting on the faces E and F, C, and D, leave the cube in equilibrium, and are sufficient to produce the required shear.

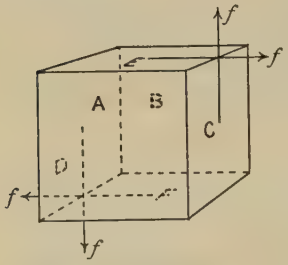

FIG. 89.—Shearing stresses.

Let the force per unit area acting tangentially on either of the faces E, F, C, or D, be denoted by f_1, so that $f_1 = f/a$; and let the shearing strain produced be equal to θ. Then if the shear is small, f_1/θ will be a constant for the substance (p. 220), and this constant is called the **coefficient of shear elasticity**, or **the simple rigidity**, of the substance. If we denote this constant by n, we have—

$$n = \frac{f_1}{\theta}.$$

The shearing strain has no dimensions, since it is the ratio of one length to another (p. 226). Hence the dimensions of n are equal to those of a force divided by an area, or—

$$\frac{\mathrm{ML}}{\mathrm{T}^2} \div \mathrm{L}^2 = \frac{\mathrm{M}}{\mathrm{LT}^2}.$$

The value of the simple rigidity is expressed in units of force per unit area; preferably, as so many dynes per square centimetre.

Let an imaginary plane be drawn between, and parallel to, the faces E and F. The part of the cube above the imaginary plane must be in equilibrium under the action of the forces applied to its surfaces, and thus its lower surface must be acted upon by a tangential force equal and opposite to the force acting on the face E. For similar reasons, the upper surface of the lower part of the cube must be acted upon by a tangential force equal and opposite to the force applied to the face F. Similar reasoning applies to the forces acting on any plane parallel to the faces C and D. These equal and opposite tangential forces, acting on opposite sides of any plane parallel to E or to C, are characteristic of a sheared solid.

Let an imaginary plane be drawn so as to cut both the faces E and C (Fig. 90) at an angle of 45°. The prismatic portion of the cube cut off by this plane must be in equilibrium under the action of the forces applied to its faces. If the areas of the mutually perpendicular faces are each equal to b, the tangential forces acting on these faces are each equal to f_1b, where f_1 is the tangential shearing stress (force per unit area). The resultant of these forces is equal to $f_1b\sqrt{2}$, and its direction is perpendicular to the slant face of the prism. Hence, the slant face of the prism must be acted upon by a normal force equal to $f_1b\sqrt{2}$. The area of the slant face is equal to $b\sqrt{2}$, and therefore the normal force per unit area acting on the slant face is equal to f_1. Of course, the direction of the normal force acting on the slant face is opposite to the direction of the resultant of the tangential forces; hence, **the material of the cube is subjected to a tensile or elongating stress f_1 perpendicular to any plane which cuts both the faces E and D at an angle of 45°.** It can be proved, in a similar manner, that **a pressure, or compressive stress,**

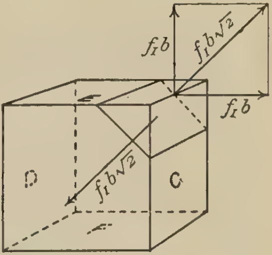

FIG. 90.—Tensile stress due to shearing stresses.

f_1 acts perpendicular to any plane which cuts both the faces E and D at an angle of 45°.

Now consider the cube depicted in Fig. 91. Let opposite tensile stresses, each equal to f_1 per unit area, act on the faces K and L; and let pressures, each equal to f_1 per unit area, act on the faces M and N. Let an imaginary plane cut both the faces K and M at an angle of 45°. Then, for the prismatic portion cut off by the plane to be in equilibrium, the resultant of the forces acting on its mutually perpendicular faces must be equal and opposite to the force acting on the slant face. If the area of each of the mutually perpendicular faces is equal to c, the force acting normally to each of these faces is equal to f_1c; the resultant of these forces is equal to $f_1c\sqrt{2}$, and the direction of this resultant makes an angle of 45° with either of the perpendicular faces (Fig. 92). Hence, the slant face must be acted upon by a tangential, or shearing, force equal to $f_1c\sqrt{2}$; and since the area of the slant face is equal to $c\sqrt{2}$, the shearing stress on the slant face is equal to f_1.

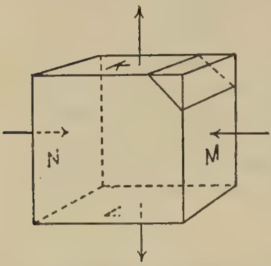

FIG. 91.—Mutually perpendicular compressive and tensile stresses.

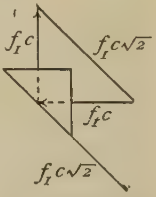

FIG. 92.—Shearing stress, due to mutually perpendicular compressive and tensile stresses.

Now let a substance be subjected to a pressure f_1 in any direction, and an equal tensile stress f_1 in a perpendicular direction. As a result, the substance is subjected to a shearing stress in a direction inclined at an angle of 45° to both the pressure and the tension. If the simple rigidity of the substance is equal to n, the corresponding shearing strain θ is equal to f_1/n (p. 87). Reference to Fig. 88 will show that the shear θ is equivalent to an elongation $\theta/2=f_1/2n$ per unit length parallel to the tension, and a linear compression equal to $\theta/2=f_1/2n$ per unit length parallel to the pressure.

Relation between elastic constants.—Definitions have now been given of four elastic constants—the bulk modulus k, Young's modulus E, Poisson's ratio σ, and the simple rigidity n. These constants are not all independent one of another; it will be proved that the four constants are connected by two equations, and therefore when two of the constants are known the other two can be determined by calculation.

In the following argument, it must be borne in mind that the stresses acting on a body determine the strains produced. Since the strain is either an expansion or a contraction per unit volume, an elongation or a shortening per unit length, or a shear, we can express the strains without reference to the actual dimensions of the body.

Let the rectangular parallelepiped depicted in Fig. 93 be subjected to a tensile stress of f_1 dynes per unit area, parallel to its length, its side faces being free from stress. As a consequence, a linear elongation will be produced in the direction of the stress, the value of the strain being f_1/E cm. per cm. length of the parallelepiped: let this be called the **longitudinal strain.**

The dimensions of the parallelepiped perpendicular to the length will be diminished, the strain being equal to σ times the longitudinal strain (p. 223), that is, to $\sigma f_1/\mathrm{E}$. Let this be called the **lateral strain.**

Let us now resolve the longitudinal tensile stress f_1 into three superposed tensile stresses, each equal to $f_1/3$. The strains produced will not be affected by this procedure. Also, let each side face be subjected to two opposite normal stresses, each equal to $f_1/3$; these stresses will not produce any strain, and thus the original strains remain unchanged. Now, we can group the stresses applied to the parallelepiped in any way we please, and the resultant strains must be identical with those produced by the tensile stresses applied to the end faces.

Each side face is subjected to a tensile stress $f_1/3$; grouping these stresses with the component tensile stresses $f_1/3$ applied to the end faces, we obtain a uniform dilatational stress $f_1/3$, which produces a uniform expansion equal to $f_1/3k$ c.c. per c.c. (p. 220); this is equivalent to a linear elongation equal to $f_1/9k$ cm. per cm. in all directions (p. 218). The result is a longitudinal expansion equal to $f_1/9k$ cm. per cm., and a lateral expansion equal to $f_1/9k$ cm. per cm.

The tensile stress $f_1/3$ acting on the faces A and B (Fig. 93), together with the compressive stresses $f_1/3$ acting on the faces C and D, are equivalent to a shearing stress $f_1/3$ parallel to the face E, and making an angle of 45° with the length of the parallelepiped. The resulting shear $f_1/3n$ is inclined at an angle of 45° to the length of the parallelepiped, and is equivalent to a longitudinal extension of $f_1/6n$ cm. per cm., together with a lateral compression of $f_1/6n$ cm. per cm. perpendicular to the side faces C and D. (Compare with Fig. 88.)

The remaining tensile stress $f_1/3$ acting on the end faces A and B, together with the compressive stress $f_1/3$ acting on the side faces E and F, produce a longitudinal extension equal to $f_1/6n$ cm. per cm., together with a lateral compression of $f_1/6n$ cm. per cm. perpendicular to the side faces E and F.

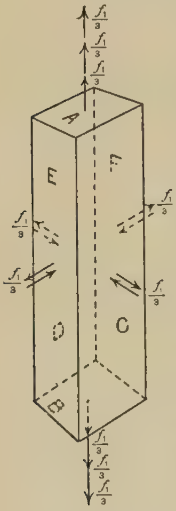

FIG. 93.—Rectangular parallelepiped subjected to tensile stresses.

Hence the total longitudinal extension is equal to—

$$\frac{f_1}{9k} + \frac{f_1}{6n} + \frac{f_1}{6n};$$

and since this can also be expressed, in terms of Young's modulus E, as f_1/E, we have—

$$\frac{f_1}{E} = \frac{f_1}{9k} + \frac{f_1}{3n},$$

$$\therefore \quad \frac{1}{E} = \frac{1}{9k} + \frac{1}{3n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The lateral compression perpendicular to the face C is equal to that perpendicular to the face E, and is equal to—

$$\frac{f_1}{6n} - \frac{f_1}{9k}.$$

Since this can also be expressed, in terms of Young's modulus E and Poisson's ratio σ, as $\sigma f_1/\mathrm{E}$, we have—

$$\frac{\sigma f_1}{\mathrm{E}} = \frac{f_1}{6n} - \frac{f_1}{9k},$$

$$\therefore \quad \frac{\sigma}{\mathrm{E}} = \frac{1}{6n} - \frac{1}{9k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Adding (1) and (2), we obtain the useful equation—

$$\frac{1+\sigma}{\mathrm{E}} = \frac{1}{2n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating E between (1) and (2), we obtain the equation—

$$\sigma = \frac{3k - 2n}{6k + 2n},$$

$$\therefore \quad 3k(1-2\sigma) = 2n(1+\sigma) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now, k and n are essentially positive quantities; a negative value of k, for instance, would mean that a uniform compressive stress produced an expansion instead of a contraction in volume.

If σ has a positive value, the right hand side of (4) is essentially positive, and for the left hand side to be positive $(1-2\sigma)$ must be positive. Hence **the value of σ must be less than** 1/2.

If σ has a negative value, the left hand side of (4) is essentially positive, and for the right hand side to be positive, $(1+\sigma)$ must be greater than zero. Hence, **the value of σ cannot be less than** (-1). Thus, **the value of σ must lie between** (-1) **and** $(+1/2)$.

Poisson devised an hypothesis of the molecular structure of elastic substances, according to which σ must be equal to 0·25 for all elastic solids. The following table gives values of the elastic constants for a number of substances, and it will be observed that σ is in some cases greater, and in other cases less, than 0·25. Those who uphold Poisson's theory maintain that the discrepancies in the value of σ are due either to crystalline structure or to want of homogeneity in the material tested. The most conclusive evidence on this point is obtained from the results of experiments on "quartz fibres." These fibres are made by rapidly lengthening a small bead of fused quartz; as fused quartz does not contract with cooling, and as the fibres are devoid of any trace of crystalline structure, while chemically the fibres are composed of pure SiO_2, the conditions demanded by Poisson's theory are complied with. According to Threlfall and Boys, for quartz fibres—

Young's modulus $E = 5{\cdot}1785 \times 10^{11}$ dynes per sq. cm.
Bulk modulus $k = 1{\cdot}435 \times 10^{11}$,, ,, ,,
Simple rigidity $n = 2{\cdot}8815 \times 10^{11}$,, ,, ,,

Then from (3), (p. 233)—

$$1 + \sigma = \frac{E}{2n} = \frac{5{\cdot}1785}{5{\cdot}7630} = 0{\cdot}898.$$

$$\therefore \quad \sigma = -0{\cdot}102.$$

We must therefore conclude that there is no experimental evidence in favour of Poisson's hypothesis.

TABLE OF ELASTIC CONSTANTS.

	n	k	E	σ
Aluminium . .	$2{\cdot}38$—$3{\cdot}36 \times 10^{11}$	—	$7{\cdot}4 \times 10^{11}$	0·13
Brass	3·44—4·03 ,,	$10{\cdot}2$—$10{\cdot}85 \times 10^{11}$	9·48—10·75 ,,	0.226—0.469
Copper	3·5 —4·5 ,,	17·0 ,,	10·3 —12·8 ,,	0·25 —0·35
Glass	1·2 —2·4 ,,	3·4— 4·2 ,,	5·4 — 7·8 ,,	0·20 —0·26
Phosphor Bronze	3·6 ,,	—	9·8 ,,	0·36
Platinum . . .	6·6 —7·4 ,,	—	15·0 —17·0 ,,	0·16
Silver	2·5 —2·6 ,,	—	7·0 — 7·5 ,,	0·37
Steel	7·7 —9·8 ,,	14·7—19·0 ,,	18·0 —29·0 ,,	0·25 —0·33

Torsion of a right circular cylinder.—A right circular cylinder may be built up from circular discs of card placed face to face, with their centres in a straight line perpendicular to all the discs. If, now, the discs are displaced so that their centres remain in the same straight line as before, while each disc is twisted through a small constant angle relatively to the disc immediately beneath it, then the cylinder has been subjected to a uniform torsion. The relative twist of any two discs is proportional to the perpendicular distance between them; and the twist per unit length of the cylinder is equal to the relative twist of its ends divided by its length.

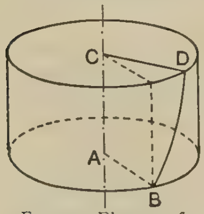

FIG. 94.—Element of twisted cylinder.

Let us now suppose that a solid elastic cylinder is subjected to a uniform torsion. Let Fig. 94 represent a short element of the cylinder, cut off by two planes perpendicular to the axis; then two radial straight lines AB and CD, which originally lay in the vertical plane CAB passing through the axis CA of the cylinder, are twisted relatively to each other through an angle θ (say). The twist per unit length of the cylinder is equal to $(\theta \div CA)$. Each of the plane faces remains undistorted, but the material between these faces has suffered a shearing strain, and the value of the shear is directly proportional to the distance from the axis.

Let Fig. 95 represent a ring, cut from a short element of a solid cylinder by two cylindrical surfaces coaxial with the cylinder. Let two planes passing through the axis be drawn before the cylinder is twisted; if the angle between these planes is very small, they will cut off from the ring a portion which approximates to a rectangular parallelepiped. When the cylinder is twisted, the parallelepiped is sheared as shown in Fig. 95. Let the radii of the two cylindrical surfaces be nearly equal, their mean value being denoted by **r**; then if the distance between the upper and lower flat faces of the ring is d, and the twist per unit length of the cylinder is ϕ, a radius in the upper face is twisted through an angle ϕd relatively to the lower face, and therefore the shearing strain of the parallelepiped is equal to—

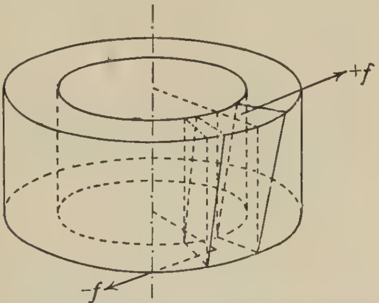

FIG. 95.—Ring cut from an element of a twisted cylinder.

$$(\mathbf{r}\phi d) \div d = \mathbf{r}\phi.$$

In order to produce this shear, the upper and lower faces of the parallelepiped must be subjected to tangential forces $(+f)$, $(-f)$, acting in the direction of the arrows (Fig. 95). The slant faces of the parallelepiped must also be subjected to tangential forces (p. 228), but these are applied by the contiguous portions of the ring, and therefore need not be considered in connection with the equilibrium of the ring as a whole. Let either of the flat faces of the ring have an area equal to a; and let the ring be divided into equal N parallelepipeds, each of which is similar to the parallelepiped represented in Fig 95; then the upper face of each parallelepiped will have an area a/N, the tangential stress applied to it will be equal to $f \div (a/\mathrm{N}) = \mathrm{N}f/a$, and the rigidity n of the material will be given by the equation—

$$n = \frac{stress}{strain} = \frac{\frac{\mathrm{N}f}{a}}{\mathbf{r}\phi}$$

$$\therefore \quad f = \frac{1}{\mathrm{N}} \cdot na\mathbf{r}\phi.$$

Now, if we select similar parallelepipeds situated at opposite sides of the ring, the forces acting on their upper faces will be equal in magnitude but opposite in directions; hence, the forces acting on the upper faces of all the parallelepipeds into which the ring has been divided will not tend to produce any linear motion of the ring. On the other hand, all the forces will exert torques acting in a uniform direction about the axis, the value of each torque being equal to—

$$f\mathbf{r} = \frac{1}{\mathrm{N}} \cdot na\mathbf{r}^2\phi;$$

and since there are N similar parallelepipeds comprised in the ring, the total torque due to the forces applied to the upper surface of the ring is equal to—

$$\mathrm{N}f\mathbf{r} = na\mathbf{r}^2\phi.$$

If the radii of the concentric cylindrical surfaces of the ring are r_a and r_b, where $r_b > r_a$, the area a is given by the equation—

$$a = \pi(r_b^2 - r_a^2);$$

and, when r_b and r_a are very nearly equal—

$$\mathbf{r}^2 = \frac{r_b^2 + r_a^2}{2} \text{ (compare p. 48).}$$

Hence, the torque about the axis, due to the forces applied to the upper face of the ring, is equal to—

$$n\phi\pi(r_b^2 - r_a^2)\frac{r_b^2 + r_a^2}{2} = \frac{n\phi\pi}{2}(r_b^4 - r_a^4) \quad . \quad . \quad . \quad (1)$$

An equal but oppositely directed torque must be applied to the lower face of the ring.

Now, let the short element of the cylinder be divided into concentric rings by cylindrical surfaces of radii $r_0, r_1, r_2, r_3 \ldots . r_n$. The torque applied to the upper face of any one of these rings can be found by substituting in equation (1) above; and since all the torques act in the same direction about the axis, the total torque applied to the upper face of the element is equal to—

$$\frac{n\phi\pi}{2}\{(r_1^4-r_0^4)+(r_2^4-r_1^4)+\ldots\ldots+(r_{n-1}^4-r_{n-2}^4)+(r_n^4-r_{n-1}^4)\}$$

$$=\frac{n\phi\pi}{2}(r_n^4-r_0^4).$$

If the cylinder is solid, $r_0=0$, while $r_n=r$, the external radius of the cylinder; then the torque τ that must be applied to the upper face of the cylinder is given by the equation—

$$\tau=\frac{n\phi\pi r^4}{2}\quad\ldots\ldots\ldots\quad(2)$$

An equal torque, acting about the axis in an opposite direction, must be applied to the lower face of the element of the cylinder.

The whole of the cylinder may be considered to be divided into short elements similar to that which has been examined, and the torque applied to the upper face of any element is equal in magnitude, but oppositely directed, to that applied to the lower face of the element immediately above it; hence the torques, applied to opposite sides of a section which separates two elements, are equal and opposite, and therefore produce equilibrium. The only external forces which need be applied are comprised in the equal and opposite torques which must act respectively on the upper and lower faces of the cylinder, and the numerical value of each of these torques is given by equation (2) above.

If θ is the twist of one end of the cylinder relatively to the other end, and l is the length of the cylinder, then $\phi=\theta/l$, and

$$\tau=\frac{n\pi\theta r^4}{2l};$$

hence, the torque per unit twist, τ_1, is given by the equation—

$$\tau_1=\frac{\tau}{\theta}=\frac{n\pi r^4}{2l}\quad\ldots\ldots\ldots\quad(3)$$

It is easily proved that if the cylinder is hollow, its external and internal radii being R and r respectively, then—

$$\tau_1=\frac{n\pi(R^4-r^4)}{2l}.$$

EXPT. 20.—By a dynamical method, determine the torque per unit twist of a wire, and thence calculate the value of the rigidity of the material of which the wire is composed.

Determine the value of the torque per unit twist in the manner explained on p. 112, and substitute the value obtained in equation (3) above. The greatest possible care should be exercised in determining r, the radius of the wire, as the percentage error in determining this quantity is quadrupled in evaluating τ_1. For, if the true radius is r, and an error $+\delta$ is made in measuring it, the incorrect value is equal to $r+\delta$, and the percentage error is equal to $100\,\delta/r$. Let τ_1 be the true torque per unit twist expressed in terms of n, r, and l; and let Δ be the error introduced owing to the error δ made in measuring the radius; then $\tau_1+\Delta$ is the incorrect value of the torque per unit twist, and the percentage error is equal to $100\,\Delta/\tau_1$. The value of τ_1 is given by (3) above, and—

$$\tau_1+\Delta=\frac{n\pi\theta(r+\delta)^4}{2l}=\frac{n\pi\theta r^4}{2l}+\frac{n\pi\theta\,.\,4r^3\delta}{2l}+\ldots$$

where terms involving the square and higher powers of δ are neglected. Then—

$$\Delta=\frac{n\pi\theta\,.\,4r^3\delta}{2l},$$

and $$\frac{100\,\Delta}{\tau_1}=100\frac{n\pi\theta\,.\,4r^3\delta}{2l}\div\frac{n\pi\theta r^4}{2l}=100\,.\,\frac{4\delta}{r}.$$

A number of equidistant points on the wire should be chosen, and two measurements of the diameter, at right angles to each other, should be made at each point. If the diameters vary but little, their arithmetical mean may be used as the diameter of the wire.

EXPT. 21.—By a statical method, determine the torque per unit twist of a wire, and thence calculate the value of the rigidity.

The best apparatus for the performance of this experiment is due to Dr. Barton.[1] The upper end of the wire is clamped to the crosspiece (Fig. 96). A heavy cylinder, with its axis vertical, is attached to the lower end of the wire, and a needle, fixed axially to the lower surface of the cylinder, fits freely in a small hole in the stand. Two parallel threads leave opposite sides of the cylinder tangentially, pass over two small frictionless pulleys, and are attached to the ends of a horizontal bar, to the middle of which a scale-pan is fixed. A mass m placed on the scale-pan produces a tension of $mg/2$ dynes in each thread, and if d is the diameter of the cylinder which the threads leave tangentially, the torque applied to the wire is equal to $mgd/2$ dyne-cm. The twist of the lower end of the wire is observed by means of a horizontal pointer which moves over a scale graduated in degrees. The twist measured in degrees must be converted into radians by multiplying by $\pi/180$. Let the twist be equal to a degrees; then the torque per unit twist—

---

[1] *A Text-Book of Sound*, by Dr. Edwin H. Barton (Macmillan), p. 130.

$$= \frac{\frac{mgd}{2}}{\frac{a\pi}{180}} = \frac{90gd}{\pi} \cdot \frac{m}{a}.$$

Observations of a should be made, first as weights are added to, and then as they are removed from the scale-pan; on squared paper plot the mean value of a for each value of m, and determine the best mean value of m/a.

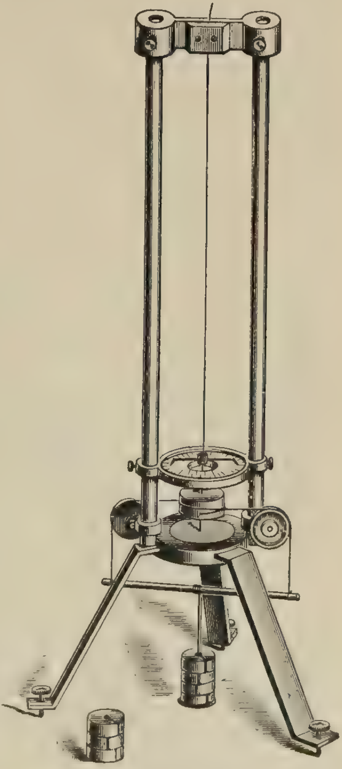

Fig. 96.—Apparatus for twisting a wire statically.

Using a given wire, the value of n determined dynamically is generally somewhat larger than the value determined by a statical method. This is due partly to the fact that a wire, under the action of a torque, does not at once acquire its final twist. The twist increases under the application of a constant torque; and therefore, during a brief application of the torque (for example, during an oscillation of a heavy body attached to the wire) the torque per unit twist has a larger value than when the torque is applied for a long time.

Most wires, when twisted, acquire a "permanent set," that is, they do not untwist entirely on the removal of the torque that produced the twist. This appears to be due to want of homogeneity and to crystalline structure (p. 224). Prof. Boys found that fibres of fused quartz, when twisted, do not acquire a permanent set; these fibres are perfectly homogeneous, consisting of pure SiO_2, and they possess no trace of crystalline structure.

Most wires are made by drawing the metal through a small circular aperture; as a result, the outer layers of the wire are more close-grained, and therefore possess a higher rigidity than the inner core. For this reason the value of *n*, determined from experiments on a fine wire, is greater than that determined from experiments on a thick wire of the same material.

The rigidity of a given material almost universally decreases as the temperature increases. For instance, the rigidity of fused quartz decreases by 0·013 per cent. for each centigrade degree rise of temperature.

Determination of *n* from experiments with a flat spiral spring.—A spiral spring can be made by winding a metallic wire round a circular cylinder, in such a manner that the wire cuts all planes which are perpendicular to the axis of the cylinder at a constant angle. If each turn of the wire is practically parallel to a plane perpendicular to the axis of the cylinder, the spiral is said to be flat.

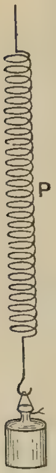

FIG. 97.—Linear oscillations of a body supported by a flat spiral spring.

Fig. 97 represents a flat spiral spring. At each end of the spiral, the wire is bent twice at right-angles, so that it extends radially to the centre of the cylinder, and thence along the axis of the cylinder. Let the upper end of the wire be clamped, while a force *f* acts along the axis of the cylinder on the lower end of the wire.

Let an imaginary plane, passing through the axis of the cylinder, cut the wire at any point P; then in order that the part of the wire below P may be in equilibrium, the vertical downward force at the lower end of the wire must be balanced by a vertical upward force at P. Since action and re-action are equal and opposite, two opposite forces equal to *f*, must act tangentially on opposite sides of the section of the wire at P, and these forces are equivalent to a shearing-stress equal to f/a, if *a* is the sectional area of the wire. A similar stress acts tangentially on all sections made by planes passing through the axis of the cylinder. The shear θ produced is $(f/a) \div n$ (p. 228), and if l is the total length of the spiral part of the wire, the lower end of the wire descends through a distance $\theta l = (lf/an)$.

The force f, acting along the axis of the cylinder, exerts a torque fR about the point P, where R is the radius of the cylinder. This torque produces a twist in the wire, and since the torque is constant for all points on the wire, the twist must be uniform; if ϕ is the twist per unit length of the wire, then (p. 237)—

$$fR = \frac{n\pi r^4}{2}\phi,$$

where r is the radius of the wire.

$$\therefore\ \phi = \frac{2fR}{n\pi r^4}.$$

The total twist imparted to the whole wire of length l is equal to—

$$l\phi = \frac{2fRl}{n\pi r^4}.$$

The radial portion of the wire, extending from the lower end of the spiral to the axis, twists through the angle $l\phi$, and therefore the end which lies on the axis of the cylinder descends through a distance—

$$Rl\phi = \frac{2fR^2l}{n\pi r^4}.$$

Hence—

$$\frac{\textit{Extension due to uniform shear}}{\textit{Extension due to twist}} = \frac{\dfrac{lf}{\pi r^2 n}}{\dfrac{2fR^2l}{n\pi r^4}} = \frac{r^2}{2R^2}.$$

Hence, when the radius r of the wire is small, in comparison with the radius R of the cylinder on which the spiral lies, the extension due to the uniform shear may be neglected, and the depression d of the lower end of the wire, due to a vertical force f, is given by the equation—

$$d = \frac{2fR^2l}{n\pi r^4}.$$

If f_1 denotes the force that produces unit depression—

$$f_1 = \frac{f}{d} = \frac{n\pi r^4}{2R^2l}$$

Expt. 22.—Obtain the value of f_1 by determining the period of the vertical linear oscillations of a mass M attached to the lower end of the wire, (p. 93) and thence calculate the value of n.

The increase of inertia due to the wire may be eliminated by the following procedure. Let a mass M_1 oscillate in a period T_1; then if m is the inertia due to the wire—

$$T_1 = 2\pi\sqrt{\frac{M_1+m}{f_1}},$$

$$\therefore\ T_1^2 = \frac{4\pi^2}{f}\cdot(M_1+m).$$

Let a mass M_2, of two or three times the value of M_1, oscillate in a period T_2; then—

$$T_2^2 = \frac{4\pi^2}{f_1}(M_2+m),$$

$$\therefore\ T_2^2 - T_1^2 = \frac{4\pi^2}{f_1}(M_2-M_1),$$

$$\text{and } f_1 = \frac{4\pi^2(M_2-M_1)}{T_2^2-T_1^2}.$$

M_2 and M_1 must be chosen so that the spring is never strained so much as to cause the wire to make an appreciable angle with the horizontal.

Uniform bending of a beam.—Let a beam AB (Fig. 98) be supported with its ends projecting beyond two knife-edges, G and E, which lie in a horizontal plane; and let equal forces F act vertically downwards at points near the ends of the beam, and equidistant from the knife-edges. Let an imaginary plane be drawn perpendicular to the beam at any point P, and consider the conditions of equilibrium of that part of the beam lying to the right of P. If the weight of the beam be neglected, the only forces acting on this portion are, the force F acting downwards at B, and the equal force F acting upwards at the knife-edge E. These forces tend to produce no linear motion of the portion PB of the beam. If the force at B is at a perpendicular distance $(D+\delta)$ from P, while it is at a perpendicular distance D from the knife-edge E, then the force F at B exerts a torque $F(D+\delta)$ about the point P, and the force $(-F)$ at E exerts a torque $(-F\delta)$ about P, so that the resultant torque is equal to—

$$F(D+\delta) - F\delta = FD.$$

This torque is independent of the position of P, and thus **the torque due to the action of the external forces is the same for all sections between the knife-edges, and its value is FD.**

For the portion PB of the beam to be in equilibrium, it must be acted on by a torque (–FD) exerted by the portion PA to the left of P. This torque is exerted across the cross-section at P, and is called into play by the bending of the beam.

A short element of the beam, immediately to the left of P, (Fig. 98) is represented by QP in Fig. 99. This element, before it is bent, is bounded by parallel planes drawn through Q and P perpendicular to the length of the element (Fig. 99, *a*). When the element is bent, each longitudinal fibre is bent into a circular arc (Fig. 99, *b*) with its centre of curvature on a line, perpendicular to the plane of the paper, drawn through some point C; this line will be called the **axis of bending**. The cross-sections through Q and P are now no longer parallel, but are inclined to each other at an angle QCP; therefore the various longitudinal fibres are now no longer equal in length, since they are at various distances from the axis of bending. Fibres at a certain distance R from the axis of bending are neither shortened nor lengthened by the bending; the surface in which these fibres lie will be called the **neutral surface**, and the line in which this surface cuts the plane of the paper will be called the **neutral axis** of the beam. Fibres above the neutral surface are lengthened, while those below the

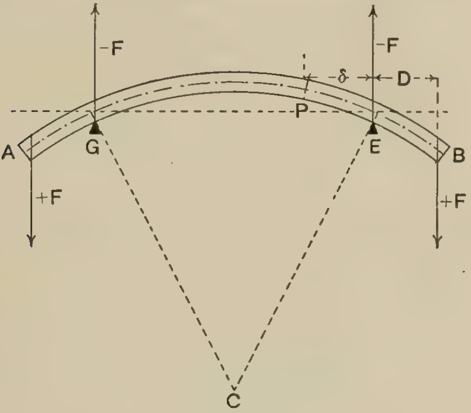

FIG. 98.—Beam supported on two knife-edges, and loaded at its ends.

neutral surface are shortened. Let S be the length of a longitudinal fibre lying in the neutral surface; then all longitudinal fibres in the element must have had this length before the element was bent. Let a fibre at a distance x above the neutral surface, and therefore at a distance $(R+x)$ from the axis of bending, have a length $(S+s)$ when the element is bent; then, since all fibres subtend the same angle QCP at C, it follows that—

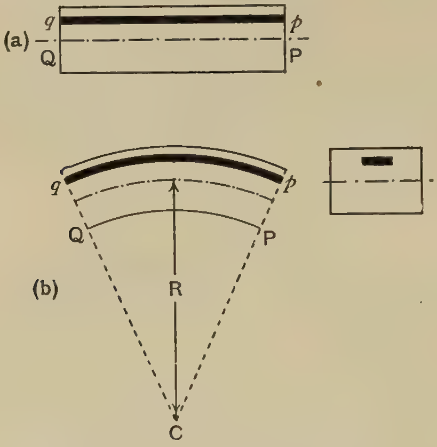

FIG. 99.—Element of beam, before and after bending.

$$\frac{S+s}{R+x}=\frac{S}{R}.$$

$$\therefore\ (S+s)R=(R+x)S, \text{ and } \therefore\ sR=xS;$$

thus

$$\frac{s}{S}=\frac{x}{R}.$$

Now, s/S is the elongational strain of the fibre; hence, if we may assume that the sides of the fibre are free from stress, the strain s/S must be due to a longitudinal tensile stress f_1, where $(s/S)=(f_1/E)$, and E is Young's modulus for the material of the beam.

$$\therefore f_1=\frac{Es}{S}=\frac{Ex}{R}.$$

Let qp (Fig. 100) represent the fibre bent into an arc of a circle of radius $(R+x)$; let the sectional area of the fibre be a, so that the tensile stress f_1 corresponds to forces, each equal to f_1a, applied to its ends, in directions perpendicular to the radii pC and qC respectively. Since these forces do not act in a straight line, the fibre cannot be in equilibrium under their action alone; the upper and lower surfaces of the fibre must be acted upon by forces which produce equilibrium with the resultant of the stretching forces applied to its ends. Through p and q draw straight lines perpendicular to pC and qC respectively, and let these lines meet in O. Make OF and OG each equal to f_1a, and complete the parallelogram OFMG; the diagonal OM is equal to the resultant of the stretching forces applied to the ends of the fibre. The angle OFM is equal to the angle qCp; and if the fibre is very short, the circular arc into which it is bent is approximately equal in length to the straight line joining q to p. The isosceles triangles OFM and qCp are similar, so that—

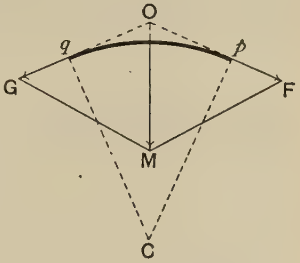

FIG. 100.—Longitudinal fibre of bent beam.

$$\frac{OM}{OF} = \frac{qp}{qC},$$

$$\therefore \frac{OM}{f_1a} = \frac{S+s}{R+x} = \frac{S}{R},$$

$$\therefore OM = f_1a\frac{S}{R}.$$

Now, aS is equal to the volume of the fibre before it is bent. Thus, in order that the fibre may be bent into the circular arc qp, the tensile stresses on its ends being equal to f_1, the substance of the fibre must be acted upon by a force, directed away from the axis of bending C, and equal to f_1/R per unit volume of the fibre. When R is large (that is, when the bending is not great) the value of (f_1aS/R) will be small in comparison with f_1a, and in these circumstances we may neglect the forces which act on the upper and lower surfaces of the fibre, and consider only the stretching forces f_1a applied to its ends.

Position of the neutral surface.—The stretching force, f, applied to each end of a fibre of area a, situated at a distance x above the neutral surface, is given by the equation—

$$f = f_1 a = \frac{\mathrm{E}as}{\mathrm{S}} = \frac{\mathrm{E}}{\mathrm{R}} \cdot ax.$$

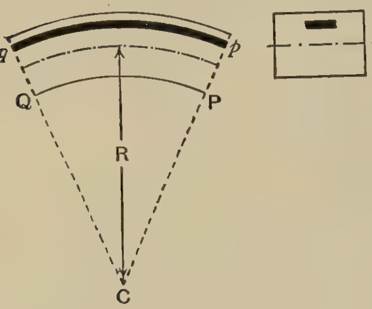

FIG. 101.—Element of bent beam.

When x is positive (that is, when the fibre is on the side of the neutral surface remote from the axis of bending) the force tends to stretch the fibre; when x is negative, the force tends to compress the fibre. All forces applied to the fibres must be normal to the cross-section at P (Fig. 101); no resultant force, tending to produce linear motion, is transmitted across the section at P, since a torque alone acts on the portion PB (Fig. 98) of the beam; thus it follows that the algebraical sum of all the forces acting across the section at P must be equal to zero. Hence—

$$\Sigma f = \frac{\mathrm{E}}{\mathrm{R}} \Sigma ax = 0,$$

$$\therefore \quad \Sigma ax = 0.$$

Hence, if the area of the cross section of each fibre in the beam is multiplied by its distance from the neutral surface, and the sum of the products is obtained, this sum must be equal to zero. This is the condition that the centre of gravity of the cross-section shall lie in the neutral surface (p. 41). Let a plane perpendicular to the axis of bending be called the plane of bending. Then **the neutral surface is perpendicular to the plane of bending, and it passes through the centre of gravity of the cross section of the beam.**

The bending torque.—The torque due to the external forces acting on the portion PB (Fig. 98) of the beam is equal to FD The forces acting across the cross-section at P must reduce to a numerically equal torque. Now the force f, which stretches a fibre at a distance x above the neutral surface, will exert a torque x about the line in which the cross section is cut by the neutral surface. The total torque about this line, due to all the forces which stretch the fibres above the neutral surface, and shorten the fibres below that surface, is equal to Σfx, and this must be equal to the torque FD due to the external forces. Now—

$$f = f_1 a = \frac{\mathrm{E}as}{\mathrm{S}} = \frac{\mathrm{E}ax}{\mathrm{R}};$$

so that

$$\mathrm{FD} = \Sigma fx = \frac{\mathrm{E}}{\mathrm{R}} \Sigma ax^2.$$

The quantity Σax^2 is found by multiplying the cross-sectional area of each fibre by the square of its distance from the line in which the neutral surface cuts the cross-section of the beam, and then obtaining the sum of the quantities so obtained; the quantity is called **the moment of inertia of the cross section,** (compare p. 47), and will be denoted by K. Then—

$$\mathrm{FD} = \frac{\mathrm{E}}{\mathrm{R}} \mathrm{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The moment of inertia K of the cross-section does not involve any measurement of mass; if we obtained a uniform thin lamina of unit mass per unit area, then an element of area a, at a distance x from an axis through the centre of gravity of the lamina, would possess a mass a, and the moment of inertia of the lamina about an axis through its centre of gravity would be equal to Σax^2. Hence, we can determine the moment of inertia of any cross-section from the moment of inertia of a uniform lamina of the same shape, by supposing that the lamina has unit mass per unit area. Thus, the moment of inertia of a circular cross-section, of radius r, about a diameter, is equal to $\pi r^2 \,.\, r^2/4 = \pi r^4/4$ (compare p. 53). The moment of inertia of a rectangular cross-section, of breadth b and depth d, about an axis through its centre of gravity and parallel to its breadth, is equal to—

$$bd \,.\, \frac{d^2}{12} = \frac{bd^3}{12} \text{ (compare p. 50).}$$

If the beam represented in Fig. 98 is uniform in cross-section, the value of K will be constant for all cross-sections; and since the bending torque FD is constant for all cross-sections, it follows from (1) above that R is constant for all points between the knife-edges; that is, the neutral axis is bent into an arc of a circle. In general, **if a uniform beam is bent by torques applied near its ends, then the neutral axis is bent into an arc of a circle.**

Engineers use a strip of steel (Fig 102), bent by torques applied to its ends, to draw circular arcs which are too large to be drawn by the aid of compasses.

Expt. 23.—Determine the value of E for a beam, by bending it by torques applied near its ends.

Support the beam on two knife-edges which lie in a horizontal plane, with one quarter of the beam overhanging each knife-edge: in this position the beam is very little bent by its own weight, each half being balanced on a knife-edge. Hang scale-pans on the beam near its ends, at equal distances from the knife-edges. As the scale-pans are loaded, the middle point of the beam rises: measure the elevation of the middle point of the beam for each load. Let L be the distance between the knife-edges; then the radius of curvature R of the neutral axis is related to the vertical displacement h of the middle point of the beam, by the equation—

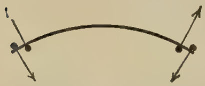

Fig. 102.—Strip of steel bent into an arc of a circle by means of pins.

$$h(2R - h) = \left(\frac{L}{2}\right)^2.$$

Therefore, when h is so small that h^2 may be neglected—

$$R = \frac{L^2}{8h}.$$

If M is the mass in either scale-pan, $F = Mg$;

$$\therefore \quad MgD = EK\left(\frac{8h}{L^2}\right),$$

and
$$E = \frac{L^2 gD}{8K} \cdot \frac{M}{h}.$$

Vary the load M, and observe the corresponding values of h. Plot M against h, and draw a straight line so as to pass, as fairly as possible, between all the points, and thence determine the average value of M/h.

Determination of n, k, σ, and E by Searle's method.—If Young's modulus E, and the simple rigidity n, have been determined for a given substance, then the bulk modulus k and Poisson's ratio σ can be calculated by the aid of the equations given on pp. 232 and 233. If the substance to be experimented with is in the form of a wire, E can be determined in the manner explained on p. 221, and n in the manner explained on p. 238. The method of determining E referred to, requires the use of a wire several metres long, and as the wire is seldom or never quite homogeneous throughout its length, some uncertainty exists as to the value of E for the short length used in determining n; therefore, great accuracy cannot be expected in the calculated values of k and σ. Mr. Searle has devised a simple method by means of which the value of E can be determined for the same short length of wire that is used in determining the value of n.

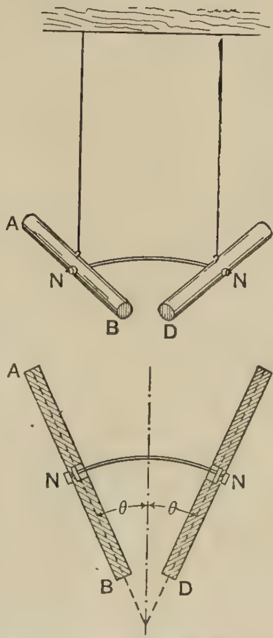

FIG. 103.—Searle's apparatus for determining n and E. (Perspective view above, sectional plan below.)

A piece of wire, about 30 cm. long and 1 mm. in diameter, is used. Each end of this wire is soldered into a hole drilled axially in a cheese-headed screw; the screws are slipped into holes bored in the middles of two brass rods, AB and CD (Fig. 103) and are clamped by means of small nuts N, N. The wires must enter the screw-heads exactly on the axes of the rods AB and CD. The rods may be about 30 cm. long and 1·5 cm. in diameter; they are suspended by means of thin silk fibres (Fig. 103) so that the rods and the wire which connects them lie in the same horizontal plane. The fibres must be vertical, so that they can exercise no appreciable control over the oscillations of the rods in a horizontal plane. In their positions of equilibrium, the rods will be parallel to each other, and the wire will extend perpendicularly from one to the other. When the rods are twisted equally in opposite directions about the silk fibres as axes, the wire is bent and exerts a torque on each rod, tending to bring it back to its position of equilibrium. If τ_1 is the torque per unit twist exerted on a rod, while I is the moment of inertia of the rod, then the period of an oscillation T of the rod is given by the equation—

$$T = 2\pi\sqrt{\frac{I}{\tau_1}},$$

In starting an experiment, a loop of thin cotton may be slipped over the ends B and D of the rods, so that the rods are held in the positions shown in Fig. 103; on burning the cotton, the rods are released and commence to oscillate regularly.

If the angular oscillations are small in amplitude, the wire will never be very much bent, and in this case the distance between the ends of the wire will never differ perceptibly from the length of the wire. Consequently, the distance between the lower ends of the silk fibres remains practically constant, and the silk fibres remain vertical during the oscillations of the rods. Any linear forces exerted on the wire would be due to the horizontal components of the tensions of the silk fibres; since the fibres remain vertical, it follows that no linear force is exerted on the wire. Thus, the wire is bent by torques applied at its ends, and therefore the bending torque is constant for all points of the wire (p. 242), and at any instant the wire is bent into an arc of a circle. The torque τ exerted on either end of the wire is given by the equation—

$$\tau = \frac{E}{R} K \text{ (p. 247)},$$

where $K = \pi r^4/4$ (p. 247), r being the radius of the cross-section of the wire.

Let l be the length of the wire; at any instant, let the rods be twisted in opposite directions through angles each equal to θ; then the axes of the rods form the extreme radii of the circular arc into which the wire is bent (Fig. 103), and the angle between these axes is equal to 2θ, so that $2\theta = l/R$. Therefore—

$$\tau = EK\frac{2\theta}{l},$$

and τ_1 the torque per unit twist of either rod, is given by the equation—

$$\tau_1 = \frac{\tau}{\theta} = \frac{2EK}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

τ_1 is determined in terms of T and I; then E is calculated from the values of τ_1, K, and l.

The value of n can be determined by clamping one rod in a horizontal position, so that the other rod is suspended by the wire. If T′ is the period of oscillation of the suspended rod when it oscillates in a horizontal plane, and τ_1' is the torque per unit twist due to the torsion of the wire, then—

$$T' = 2\pi \sqrt{\frac{I}{\tau_1'}},$$

where I has the same value as in the previous experiment, and τ_1', from p. 237, is given by the equation—

$$\tau_1' = \frac{n\pi r^4}{2l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now $$1 + \sigma = \frac{E}{2n} \text{ (p. 233)},$$

and from equations (1) and (2), remembering that $K = \pi r^4/4$ in (1), we have—

$$\frac{E}{2n} = \frac{\tau_1}{2\tau_1'} = \tfrac{1}{2}\left(\frac{T'}{T}\right)^2.$$

Hence, σ is determined independently of the value of r, so that an important source of error is eliminated.

Determination of E from experiments with a flat spiral spring.—If a wire is straight when unstrained, it will be bent into an arc of a circle of radius R by torques, each equal to τ, applied to its ends, where—

$$\tau = \frac{1}{R} EK \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The meaning of the factor 1/R must now be considered. Let normals be drawn to the curve into which the wire is bent, at points A, B (Fig. 104), unit distance (measured along the wire) apart. These normals meet at the centre of curvature C, and AC = BC = R. The angle ϕ at which these normals meet is equal to (AB/BC) = (1/R); and ϕ is also equal to the angle between the tangents to the curve at the points A and B. Hence, ϕ, **the bend per unit length of the wire, is equal to 1/R; and the torque τ is related to the bend ϕ per unit length, according to the equation—**

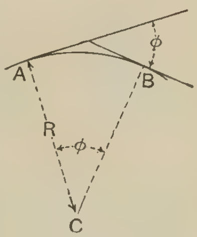

FIG. 104.—Portion of bent wire.

$$\tau = EK\phi.$$

It will now be shown that a similar relation exists when the wire is not originally straight.

Let us suppose that a wire, when unstrained, is curved in any manner, provided that it lies in a plane. When this wire is further bent in this plane, there will be a neutral surface in which the fibres are neither lengthened nor shortened. Let the radius of curvature of the neutral surface be R_0 before, and R after the bending. Before the bending, let planes be drawn through the centre of curvature so as to cut the neutral surface perpendicularly at two points at a small distance S apart; if we confine our attention to the element of the wire cut off by these planes, the original length of a fibre at a distance x from the neutral surface, and on the side remote from the centre of curvature, has the value—

$$S \cdot \frac{R_0 + x}{R_0} = S\left(1 + \frac{x}{R_0}\right),$$

and after the bending its length is—

$$S \cdot \frac{R + x}{R} = S\left(1 + \frac{x}{R}\right),$$

so that the elongation is $Sx\left(\frac{1}{R} - \frac{1}{R_0}\right)$, and the strain is equal to—

$$\frac{Sx\left(\frac{1}{R} - \frac{1}{R_0}\right)}{S\left(1 + \frac{x}{R_0}\right)}.$$

When the maximum value of x is small in comparison with R_0, we may neglect (x/R_0) in the denominator in comparison with unity. In this case the strain is equal to—

$$x\left(\frac{1}{R} - \frac{1}{R_0}\right).$$

If tangents are drawn to the neutral axis at points unit distance apart, then the original inclination ϕ_1 of these tangents was $1/R_0$, and their final inclination $\phi_1 + \phi$ is $1/R$; therefore—

$$\frac{1}{R} - \frac{1}{R_0} = (\phi_1 + \phi) - \phi_1 = \phi.$$

Therefore the strain of the fibre is equal to $x\phi$. It is obvious that ϕ denotes the additional bend per unit length imparted to the wire.

If the area of the fibre is a, the force f acting on its end is given by the equation—

$$f = \mathrm{E}\phi a x.$$

If the wire is bent by two opposite torques, each equal to τ, applied to its extremities, then it can be proved, in the manner used on p. 242, that a pure bending torque τ acts at each point of the wire, and therefore (compare p. 246).

$$\Sigma f = \mathrm{E}\phi \,.\, \Sigma a x = 0.$$

Hence the neutral surface passes through the centre of gravity of the cross-section of the wire. Also—

$$\tau = \Sigma f x = \mathrm{E}\phi \,.\, \Sigma a x^2 = \mathrm{EK}\phi,$$

where K is the moment of inertia of the cross-section of the wire, about its intersection with the neutral surface.

Since τ is constant for all points of the wire, the additional bend, ϕ, imparted to unit length of the wire by the torques, is constant; therefore if the total length of the wire is l, and the tangents at its extremities suffer a relative angular displacement θ owing to the bending, it follows that $\phi = \theta/l$. To obtain a clear idea of θ, let it be supposed that one end of the wire is clamped in such a manner that the tangent at this end is fixed; then the bending turns the tangent at the other end through an angle θ, and any two tangents drawn at points unit distance apart suffer a relative angular displacement $\phi = \theta/l$. Even if the original curvature of the wire varied from point to point, the additional bend per unit length produced by torques applied to its ends is constant for all points of the wire.

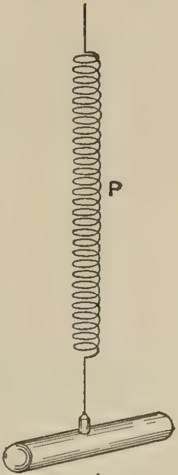

FIG. 105. Angular oscillations of a body supported by a flat spiral spring.

Now let a wire be formed into a flat spiral spring in the manner explained on p. 240, and let the upper axial extremity of the wire be clamped so that the spiral hangs vertically below it, and let an inertia bar be fixed to the lower axial extremity of the wire (Fig. 105). If the bar is twisted through an angle θ about the axis of the spiral, the tangent to the lower end of the spiral is turned through an angle θ, and the bend per unit length (measured along the spiral) is θ/l, where l is the length of the spiral portion of the wire. In order to produce this bend per unit length, a torque τ must be applied to the bar where—

$$\tau = \frac{\theta}{l}\mathrm{EK};$$

therefore the torque per unit twist, τ_1, is given by the equation

$$\tau_1 = \frac{\tau}{\theta} = \frac{\mathrm{EK}}{l}.$$

Then, if I is the moment of inertia of the bar, its period of angular oscillation T is given by the equation—

$$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I}}{\tau_1}}.$$

EXPT. 24.—Determine the value of E for a wire formed into a flat spiral, by observing the period of angular oscillation of an inertia bar fixed to the lower end of the spiral.

The value of E obtained by this method is not very accurate, since the wire becomes flattened to some extent during the winding of the spiral. However, a fairly accurate value can be obtained if a wire of about 1 mm. diameter is wound on a rod of 4 or 5 cm. diameter. If experiments 22 and 24 are performed with the same spiral, the values of σ and k can be calculated from the values of n and E obtained.

Pendulums cannot be used in portable time-keepers; consequently, a watch or a chronometer is controlled by a small fly-wheel which oscillates about its axis under the action of a steel spring, having the form of a spiral lying entirely in a plane perpendicular to the axis of the fly-wheel. The ends of the spring are fixed in such a manner that the rotation of the fly-wheel produces merely a bending torque. The period of oscillation T of the fly-wheel is given by the equation—

$$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I}}{\tau_1}} = 2\pi\sqrt{\frac{\mathrm{I}l}{\mathrm{EK}}},$$

where l is the length of the spring, and K is the moment of inertia of the cross-section of the spring about its intersection with the neutral surface. The period is regulated by slightly altering the point at which the stationary end of the spring is clamped.

When the temperature rises, the elasticity of the spring diminishes, and at the same time the balance wheel expands and its moment of inertia increases. The value of E for steel decreases at the rate of 0·02 per cent. for each degree centigrade rise of temperature. The coefficient of linear expansion of brass is equal to 0·000019, and therefore the radius of the balance wheel increases by 0·0019 per cent. per degree centigrade rise of temperature. The moment of inertia of the balance wheel is proportional to the square of the radius, and thus increases at the rate of 0·0038 per cent. per degree. Hence, the most important change produced by a rise of temperature is due to the diminution in E, which will cause an increase of about 0·02 per cent. per degree in the value of $1/E$. Since the period of oscillation is proportional to $\sqrt{1/E}$, it follows that T increases by 0·01 per cent. for each degree rise of temperature. This means, that a rise of one degree centigrade will cause a watch with a steel spring and an uncompensated balance wheel, to lose one second in every 10,000 seconds, or 8·6 seconds per day.

Until a few years ago, it was the custom to provide expensive watches and chronometers with a compensated balance wheel,[1] that is, a wheel made so that, as the temperature rises, its moment of inertia I decreases at the same rate as the elasticity E. The modern practice is to use a simple brass balance wheel, and a spring made of nickel steel (a variety of "invar"), for which E increases, with a rise of temperature, at the same rate as the moment of inertia of the wheel. The compensation secured by this means, at a comparatively small cost, is far more perfect than that obtained by the aid of the expensive compensated balance wheel previously used.

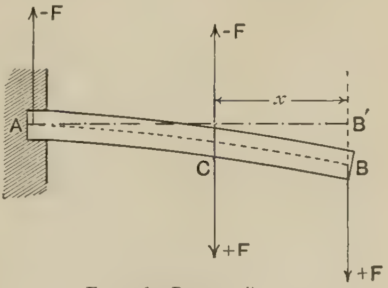

FIG. 106.—Bent cantilever.

Bending of a cantilever.—A cantilever is a beam of which one end is fixed in a horizontal position, while the other end is free. When the free end is loaded, it is deflected vertically downwards, and the beam is bent; but the fixed end remains horizontal.

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[1] See the author's *Heat for Advanced Students* (Macmillan), p. 58.

Let AB (Fig. 106) represent a cantilever, the end A being fixed in a horizontal position, while a force F acts vertically downwards near the end B. Let us consider the conditions of equilibrium of that portion of the beam between the end B and a transverse section at C. If the bending of the beam is small, the section at C will be sensibly vertical; and since the portion CB of the beam is acted upon by the external downward force F, an equal force must act upwards on CB at C. Since action and reaction are equal and opposite, a force F acts vertically downwards on the portion AC at C. Hence, two equal but opposite forces act tangentially to the section C. The magnitudes of these equal but opposite forces are independent of the position of the section C; hence, every transverse section is acted upon by a shearing stress equal to F/a, where a denotes the transverse sectional area of the beam.

Let the external force F act at a perpendicular distance l from the section at which the beam leaves the clamp that holds the end A in a horizontal position. Then, if n is the rigidity of the material of which the beam is composed, and δ is the shearing displacement of the section to which the external force F is applied, the shearing stress F/a produces a shearing strain δ/l (p. 226), and—

$$n = \frac{F/a}{\delta/l},$$

$$\therefore \ \delta = \frac{Fl}{na}.$$

It will be proved that δ is negligibly small in comparison with the downward deflection of the end B of the beam due to bending.

Let the external force F act at a perpendicular distance x from the section C; then the portion of the beam to the right of C is acted upon by a torque equal to Fx, and this must be balanced by an equal torque transmitted across the section C, due to the stretching and compression of the fibres of the beam. It can be proved, by using reasoning identical with that explained on p. 246, that the neutral surface will pass through the centre of gravity of the cross-section; and if K is the moment of inertia of the cross-section about its intersection by the neutral surface, then—

$$Fx = \frac{EK}{R} = EK\phi,$$

where E is Young's modulus for the material of the beam, R is the radius of curvature of the neutral axis at C, and ϕ is the bend per unit length of the neutral axis at C. Hence, the bend per unit length of the beam is not constant, but varies directly as x; thus, the bend is equal to zero at the position where the external force F acts (that is, where $x=0$), and has a maximum value Fl/EK at the point where the beam leaves the clamp. Hence, in this case the beam is not bent into an arc of a circle (compare p. 247).

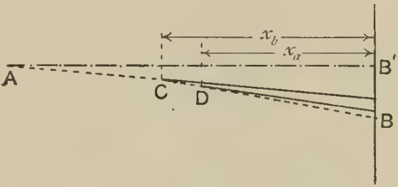

FIG. 107.—The neutral axis of a bent cantilever.

Let AB (Fig. 107) represent the neutral axis of the bent beam, and let the horizontal straight line AB′ represent the neutral axis of the same beam before it was bent. If the bending is small, the points B and B′ will lie in a vertical straight line, and B′B will be the vertical deflection of the point at which the beam is loaded. Let B′B be denoted by Δ.

Let C and D be two adjacent points on the neutral axis, at distances x_b and x_a respectively from the line B′B. Let ϕ denote the bend per unit length at a point midway between C and D; then the angle α between the tangent at C and that at D is given by the equation—

$$\alpha = \phi(x_b - x_a) = \frac{F\mathbf{x}}{EK}(x_b - x_a),$$

where $\mathbf{x}$ denotes the mean distance of the points C and D from B′B. The distance between the points where these tangents cut the vertical line B′B is practically an arc of a circle of radius $\mathbf{x}$, subtending an angle α at a point midway between C and D; hence this distance is equal to—

$$\mathbf{x}\alpha = \frac{F}{EK}\mathbf{x}^2(x_b - x_a) = \frac{F}{EK}\frac{(x_b^2 + x_b x_a + x_a^2)}{3}(x_b - x_a)$$

$$= \frac{F}{EK}\frac{x_b^3 - x_a^3}{3}.$$

The student is referred to p. 48 for the reason why the value of $\mathbf{x}^2$ is written down as $(x_b^2 + x_b x_a + x_a^2)/3$.

Now, let the neutral axis be divided into infinitely short elements by points at distances x_0, x_1, x_2, x_3, ... x_n from B′; and at each of these points, let a tangent be drawn to the neutral axis. Then Δ, or B′B, is equal to the sum of the distances between the points in which consecutive tangents cut the line B′B, and therefore—

$$\Delta = \frac{F}{3EK}\{(x_1^3 - x_0^3) + (x_2^3 - x_1^3) + (x_3^3 - x_2^3) + \ldots (x_n^3 - x_{n-1}^3)\}$$
$$= \frac{F}{3EK}(x_n^3 - x_0^3).$$

Remembering that $x_n = l$, while $x_0 = 0$, we have—

$$\Delta = \frac{Fl^3}{3EK}. \quad . \quad . \quad (1)$$

Values of K for beams of rectangular and circular cross-sections are given on p. 247.

For a beam of rectangular cross-section, of breadth b and depth d, the value of K is equal to $bd^3/12$. For such a beam, mounted as a cantilever—

$$\Delta = \frac{4Fl^3}{Ebd^3}.$$

The vertical deflection δ of the cantilever due to shearing (p. 256) is given by the equation—

$$\delta = \frac{Fl}{nbd},$$
$$\therefore \frac{\Delta}{\delta} = \frac{4l^2}{d^2} \cdot \frac{n}{E}.$$

Now, $(1 + \sigma)/E = 1/2n$ (p. 233). Hence, $n/E = 1/\{2(1 + \sigma)\}$, and in general σ does not differ much from 1/4. Thus, n/E does not differ much, in general, from 2/5. When the length l is large in comparison with the depth d of the cantilever, the ratio of Δ to δ will be very large, as it is proportional to $(l/d)^2$; hence, in these circumstances, the deflection of the cantilever is due almost exclusively to bending, and the effect of shearing may be neglected.

The angle θ, which the tangent at B (Fig. 107) makes with the horizontal straight line AB′, is equal to the sum of the angles between consecutive tangents to the neutral axis. The angle α between tangents at distances x_b and x_a from B′ is given by the equation—

$$\alpha = \frac{Fx}{EK}(x_b - x_a) = \frac{F}{EK}\frac{(x_b + x_a)}{2}(x_b - x_a)$$

$$= \frac{F}{2EK}(x_b^2 - x_a^2), \text{ (compare p. 48),}$$

$$\therefore \theta = \frac{F}{2EK}\{(x_1^2 - x_o^2) + (x_2^2 - x_1)^2 + \ldots (x_n^2 - x_{n-1}^2)\}$$

$$= \frac{F}{2EK}(x_n^2 - x_o^2) = \frac{Fl^2}{2EK}. \quad . \ . \ . \ (2)$$

θ is the inclination of the beam to the horizontal, at the point where the force F acts.

Beam supported at its ends, and loaded at its middle point.—Let a beam be supported on two knife-edges A and B (Fig. 108) in a horizontal plane, and let a force F act vertically downwards at its middle point. Each knife-edge exerts a vertical upward force F/2 on the beam. The point C, at which the downward force F acts, will be the point where the downward deflection is greatest; that is, the beam slopes downwards on either side of C, and therefore in the neighbourhood of C the beam is horizontal. Hence, if we divide the beam into two equal portions by a transverse section through C, each half is equivalent to a cantilever with a force F/2 acting upwards at its free end. If the distance between the knife-edges is equal to L, then the length *l* of the equivalent cantilever is equal to to L/2. Hence, the deflection Δ at the middle point of the beam is found by substituting F/2 in place of F, and L/2 in place of *l*, in the formula (1) for the deflection of the cantilever (p. 258). Thus—

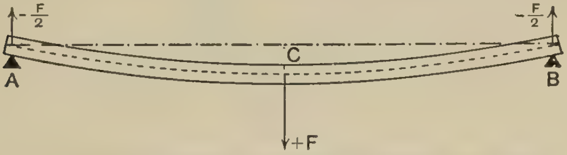

FIG. 108.—Beam supported near its ends on knife-edges, and loaded at its middle point.

$$\Delta = \frac{(F/2)(L/2)^3}{3EK} = \frac{FL^3}{48EK},$$

$$\therefore \ E = \frac{FL^3}{48\Delta K}. \quad . \ . \ . \ (3)$$

For beams of a given material, E is constant. If the cross-section is also constant, K is constant; hence the value of FL^3/Δ also is constant. The meaning of this relation is, that Δ/F, **the deflection per unit load, is proportional to the cube of the length** L.

For a beam of rectangular section, of breadth b and depth d, the value of K is $bd^3/12$. For such a beam—

$$E = \frac{FL^3}{4bd^3\Delta}.$$

Hence, for beams of a given material and length, but of different rectangular sectional areas, the value of $(F/bd^3\Delta)$ is constant; hence Δ/F, **the deflection per unit load, is inversely proportional to** bd^3.

Let θ denote the angle of slope of the beam at a point over either of the knife-edges; then from equation (2), p. 259,

$$\theta = \frac{\left(\frac{F}{2}\right)\left(\frac{L}{2}\right)^2}{2EK} = \frac{FL^2}{16EK},$$

$$\therefore\ E = \frac{FL^2}{16\theta K}\ .\ .\ .\ .\ (4)$$

The value of θ must be expressed in circular measure.

Expt. 25—Determine the value of Young's modulus from experiments on the flexure of a beam, supported on knife-edges near to its ends and loaded at its middle point.

Apparatus suitable for the performance of this experiment is represented in Fig. 109. Two cast iron tripods, of the shape represented, have knife-edges ground on their tops; the beam is supported on these knife-edges, and is loaded by means of a scale pan hung from a knife-edge which rests on the middle of the beam. A small scale, graduated in half-millimetres, is counterweighted so that it is balanced in a vertical position on two points which rest on the middle of the beam. The tripods carry a wooden shelf which supports a low-power microscope, by means of which the scale can be viewed; the microscope is provided with cross-wires in the eye-piece, and the scale reading, coincident with the horizontal cross-wire, is observed for each load placed on the scale pan. Plot the loads against the scale-readings, and thence determine the mean value of (F/Δ) in equation (3). Thence calculate the value of E.

The apparatus, represented in Fig. 109, can be modified so as to permit of the performance of Expt. 23, p. 248.

To determine the value of E in terms of the angle of slope above either knife-edge, a small square of looking glass is clamped to the beam above one knife-edge, the plane of the looking glass being perpendicular to the length of the beam. A vertical scale is reflected in the looking glass, and its image is viewed by the aid of a telescope; the angle of slope, θ, is determined in the manner explained in the Author's *Light for Students* (Macmillan), p. 26. Plot the loads against the values obtained for θ, and thence obtain the average value of (F/θ) in equation (4) above.

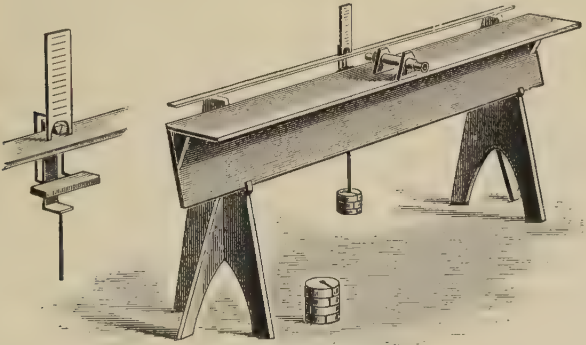

FIG. 109.—Apparatus for experimenting on the bending of a beam.

QUESTIONS ON CHAPTER VII.

1. The coefficient n of shear elasticity of a substance is equal to 3×10^{11} c.g.s. units, and Young's modulus E for the same substance is equal to 7×10^{11} c.g.s. units. Calculate the value of the bulk modulus k, and Poisson's ratio σ, for this substance.

2. If the coefficient n of shear elasticity of steel is equal to 8×10^{11} c.g.s. units, and the bulk modulus k of the same substance is equal to 15×10^{11} c.g.s. units, calculate the value of these moduli in poundals per sq. ft., and in pounds per sq. in.

3. A substance is subjected to tensile strains equal to a, b, and c, in the directions of the rectangular axes of x, y and z respectively.

Prove that the resultant strain is equivalent to that due to the superposition of a uniform dilatational strain equal to $(a+b+c)$, together with two shears, one equal to $\{(2/3)(a+b+c)-2b\}$ in the plane of (x, y) and the other equal to $\{(2/3)(a+b+c)-2c\}$ in the plane of (x, z).

4. A substance is subjected to tensile strains equal to 3×10^{-4}, 2×10^{-4}, and 5×10^{-4}, in the directions of the axes of x, y, and z respectively. Resolve the resultant strain into a uniform dilatational strain, together with two shears, one in the plane of (x, y), and the other in the plane of (x, z). Calculate the stresses that will produce the resultant strain, if—

$$k=15 \times 10^{11} \text{ dyne/(cm.)}^2, \text{ and } n=8 \times 10^{11} \text{ dyne/(cm.)}^2.$$

5. A substance is subjected to a uniform tensile stress f_1; calculate the elongational strain produced, if the substance cannot expand or contract laterally, and the bulk modulus and simple rigidity of the substance are equal to k and n respectively.

6. The coefficient of linear expansion of steel is equal to $1 \cdot 2 \times 10^{-5}$, per degree C., and the value of Young's modulus for the same substance is equal to 2×10^{12} c.g.s. units. Calculate the value of the linear compressive stress that must be applied to the ends of a steel cylinder, in order to prevent it from expanding longitudinally when the temperature is raised through 20°C.

7. An inertia bar is suspended by means of a steel wire, and the period T of its angular oscillations is determined. Calculate the period of oscillation of another inertia bar, made of the same substance and similar in shape, but of twice the linear dimensions of the first bar, if it be suspended by means of a steel wire of half the linear dimensions of the first wire.

8. The coefficient of linear expansion of a substance is equal to 2×10^{-5}, and the coefficient of shear elasticity of the same substance decreases at the rate of 0·1 per cent. for each centigrade degree rise of temperature. An inertia bar made from this substance is suspended by means of a wire of the same substance, and it is found that N oscillations are performed in ten minutes when the temperature is 5°C. How many oscillations will be performed in ten minutes when the temperature is 20°C.?

9. A solid rod has a circular cross-section of radius R; and a tube of the same substance has an annular cross-section of external radius R and internal radius r. The rod and tube are supported on the same pair of knife-edges, and are equally loaded midway between the points of support. Calculate the ratio of the sags produced in the two cases.

10. A watch gains time at the rate of 2 minutes in 24 hours. In what ratio must the hair spring be lengthened, in order that correct time may be kept?

CHAPTER VIII

ELASTICITY (*continued*)

Deformation of cross-section, due to bending.—In the investigation carried out on pp. 242–247, it was assumed tacitly that the cross-section of a beam remains unaltered when the beam is bent. This assumption is admissible when the bending is slight, but a simple experiment shows that it is inadmissible when the bending is considerable.

EXPT. 26.—Obtain a piece of rubber, about two inches long and one inch by half an inch in section. Bend the rubber lengthways into an arc of a circle of about 1·5 inches radius; it will be seen that the section no longer remains rectangular. The shape assumed by the rubber is represented in Fig. 110. Not only the longitudinal fibres, but also the fibres parallel to the width of the rubber, are bent into circular arcs. If the longitudinal fibres are bent so that their concavity is below, then the transverse fibres become bent so that their concavity is above. The curvature of the longitudinal fibres tends to break the substance, hence it may be called the *clastic*[1] curvature. The name *anticlastic curvature* is given to the transverse bending of the substance; this bending takes place in a plane at right angles to that of the clastic curvature.

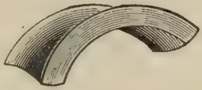

FIG. 110.—Bent piece of rubber.

Let ABCD (Fig. 111) represent the cross-section of a beam; and let GH represent the cross-section of its neutral surface. When the beam is bent about an axis, parallel to GH and on the same side of GH as DC, the fibres above GH are lengthened longitudinally and contracted laterally (p. 223), while the fibres below GH are shortened longitudinally and expanded laterally.

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[1] Greek *klastos*, broken.

Hence, if we consider a narrow rectangle, EF, of which the longer sides are parallel to the depth of the beam, it becomes evident that this will assume the form E′F′ when the beam is bent.

A longitudinal fibre, at a distance x above GH, will suffer an extensional strain equal to x/R, where R is the radius of the elastic curvature (p. 244); and the lateral strain will be equal to $\sigma x/R$, where σ denotes Poisson's ratio. Hence, the lateral shrinkage is proportional to the distance of the fibre above GH, and the lateral expansion is proportional to the distance of the fibres below GH. If we divide the cross-section of the unbent beam into rectangles similar to EF, each of these will assume the form E′F′ when the beam is bent, and the complete cross-section of the beam will assume the form A′B′C′D′ (Fig. 111).

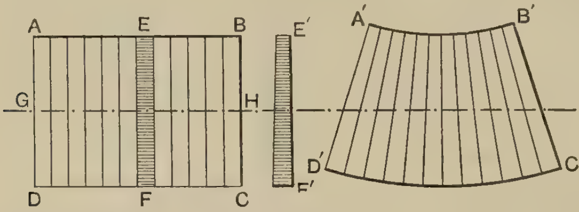

FIG. 111.—Cross-section of a beam, before and after the beam has been bent longitudinally.

To determine the radius of the anticlastic curvature, we may proceed as follows. Let b be the breadth, and d the depth of the beam, and let r be the radius of the anticlastic curvature of the neutral surface. The surface AB (Fig. 111) is at a distance $d/2$ above the neutral surface, and therefore the length of the arc A′B′ is equal to—

$$b\left(1 - \frac{\sigma}{R} \cdot \frac{d}{2}\right).$$

Assuming that A′B′ is an arc of a circle, the radius of the circle will be $\left(r - \frac{d}{2}\right)$, and the angle subtended at the centre of the circle by the shortened transverse fibre A′B′, is equal to that subtended by the unshortened transverse fibre of length b in the neutral surface. Thus—

$$\frac{b}{r} = \frac{b\left(1 - \frac{\sigma d}{2R}\right)}{r - \frac{d}{2}},$$

$$\therefore r - \frac{d}{2} = r - \frac{\sigma r}{R}\cdot\frac{d}{2},$$

so that

$$\frac{\sigma r}{R} = 1.$$

Hence, $\sigma = R/r$; that is, **Poisson's ratio σ is given by the ratio of the clastic, to the anticlastic radius of curvature.**

EXPT. 27.—Determine the clastic and anticlastic radii of curvature of a uniformly bent beam, and thence calculate the value of σ.

A steel bar about 50 cm. long, 2 cm. wide, and 0·3 cm. thick may be used. It is supported on knife-edges, with its ends overhanging as described on p. 248. Scale-pans are hung from the ends of the bar, and the radius of clastic curvature is determined in the manner explained on p. 248. The radius of anticlastic curvature is measured by the aid of special appliances. At opposite sides of the middle cross-section of the bar, two straight steel rods, each about 30 cm. long and 2 mm. in diameter, are fixed (Fig. 112): these rods may be clamped in barrel binding screws which are soldered to the sides of the bar. The upper end of one rod carries a small bracket, in which a roller of about 2 mm. diameter can rotate between centres: a light aluminium pointer is fixed to the roller, and moves over a semicircular scale divided into degrees. A fine copper wire, fixed to the top of the other rod, is

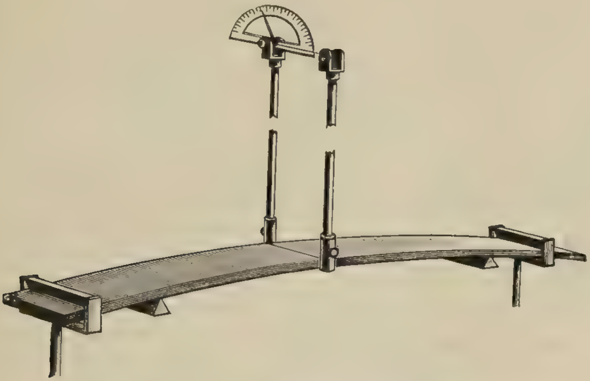

FIG. 112.—Apparatus for measuring the anticlastic curvature due to the longitudinal bending of a bar.

wound round the roller and fixed to it. Thus, any relative angular motion of the rods causes the pointer to rotate. Let each rod have a length D, while the diameter of the roller is δ. If the tops of the rods move toward each other, so that the angle between the rods increases by ψ, then an additional length ψD of wire is wound round the roller, and the roller rotates through an angle

$$(\psi D) \div (\delta/2) = 2\psi D/\delta \text{ radians.}$$

By observing the angular rotation of the pointer attached to the roller, the value of ψ can be obtained. Now, the rods obviously point toward the centre of curvature; and if b is the breadth of the bar, and r is the radius of its anticlastic curvature, then $b/r = \psi$. Hence r can be determined.

The loads on the ends of the bar should be increased step by step, the deflection of the pointer being observed for each load. At first the deflection increases proportionately with the load; but when the elastic curvature becomes considerable, the deflection increases at a smaller rate. In determining the value of $\sigma = R/r$, only those observations should be used in which the deflection of the pointer is proportional to the load.

Bending of a blade.—A blade may be defined as a long strip of metal of rectangular section, very wide in comparison with its thickness. If the cross-section of a bent blade were deformed in the same way as that of a beam (p. 264), it would assume a shape similar to ABC, Fig. 113, *a*. Now, the cross-section of the neutral surface must be a straight line DE parallel to the axis of bending. A longitudinal fibre above the neutral surface is extended, and the forces acting on its ends produce a resultant directed toward the axis of bending (p. 245); a longitudinal fibre below the neutral surface is shortened, and the forces acting on its ends produce a resultant directed away from the axis of bending. Thus, the cross-section would be acted upon by forces such as those represented by the arrows in Fig. 113, *a*, and these forces obviously tend to straighten the cross-section; that is, they produce a lateral tension in fibres above the neutral surface, and a lateral pressure in fibres below the neutral surface.

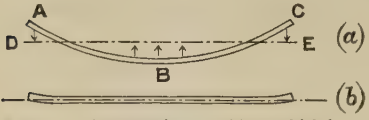

FIG. 113.—Cross-section of a blade which is bent longitudinally.

Experiment shows that when a blade is bent, its cross-section is practically rectangular, except for a very slight upward curvature near the edges (Fig. 113, *b*).

Let Fig. 114 represent two mutually perpendicular sections of part of a bent blade, very much magnified. Let AB be a cross-section plane passing through the axis of bending, while BC is a longitudinal section by a plane perpendicular to the axis of bending. Consider a longitudinal fibre PQ, at a distance x above the surface in which there is no longitudinal extension (this surface is represented by dotted lines). Let f_1 be the longitudinal tensile stress to which this fibre is subjected; then this stress produces a longitudinal extensional strain f_1/E, and a lateral compressional strain $\sigma f_1/E$ (p. 223). Let the forces which straighten the cross-section produce a lateral tensile stress F_1 in the fibre PQ; then this stress will produce a lateral elongational strain F_1/E, and a longitudinal compressional strain $\sigma F_1/E$. Hence, the total longitudinal elongation per unit length of the fibre is equal to $\{(f_1/E - (\sigma F_1/E)\}$, and the total lateral elongation per unit breadth of the fibre is equal to $\{(F_1/E) - (\sigma f_1/E)\}$. Now, the longitudinal fibre is bent into an arc of a circle of radius $(R + x)$ (p. 244), and therefore the longitudinal strain is equal to x/R. Thus—

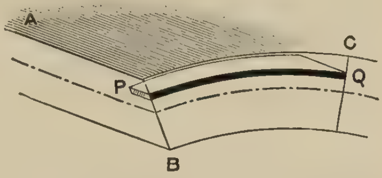

FIG. 114.—Longitudinal and transverse sections of a bent blade.

$$\frac{f_1}{E} - \frac{\sigma F_1}{E} = \frac{x}{R},$$

$$\therefore\ f_1 - \sigma F_1 = \frac{Ex}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The cross-section of the blade remains practically rectangular, but possibly the width of the blade may be altered by the forces which straighten it laterally; let the blade suffer a lateral elongation equal to δ per unit width. Then—

$$\frac{F_1}{E} - \frac{\sigma f_1}{E} = \delta,$$

and $$F_1 - \sigma f_1 = E\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Solving (1) and (2) for f_1, we obtain

$$(1 - \sigma^2) f_1 = \frac{Ex}{R} + E\sigma\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equation (3) shows that, **although the fibres, for which $x=0$, are not stretched, yet each is subjected to a longitudinal tensile stress equal to $E\sigma\delta/(1-\sigma^2)$**; this stress merely neutralises the longitudinal compressional strain produced by the lateral straightening of the blade. Since the cross-section of the blade remains rectangular, δ must have a constant value for all the fibres; hence, if we determine δ for any one fibre, we obtain its value for all the others. It will now be shown that, when the longitudinal bending is not very great, δ is negligibly small.

If the blade is bent by torques applied to its ends, the forces acting across the section AB (Fig. 114) must reduce to a pure torque (compare p. 247); therefore the longitudinal fibres which lie in the surface passing through the centre of gravity of the cross-section are subject to zero stress (compare p. 246, and note that in that case the unstretched fibres were subjected to zero stress). The centre of gravity of the cross-section is midway between the upper and lower surfaces of the blade; hence **longitudinal fibres lying midway between the upper and lower surfaces of the blade are subject to zero longitudinal stress; that is, for these fibres $f_1=0$.** Fibres lying *nearly* midway between the upper and lower surfaces of the blade will be subjected to negligibly small longitudinal stresses.

Now consider the longitudinal section BC; since no forces act on the free side surfaces of the blade, the forces acting across the section BC must reduce to a pure torque, and therefore **a fibre which passes through the centre of gravity of the section BC will be subjected to zero lateral stress; that is, for such a fibre $F_1=0$.** The centre of gravity of the section BC will lie nearly, but not exactly, midway between the upper and lower surfaces of the blade; hence, the fibre which is subjected to zero lateral stress will be subjected to a longitudinal stress which is negligibly small. Hence, in equation (2) above, when $F_1=0$, the value of f_1 is negligibly small, and therefore δ is negligibly small.

Neglecting δ in equation (3) above, we find that—

$$f_1=\frac{E}{1-\sigma^2}\cdot\frac{x}{R}.$$

Let f be the longitudinal force acting on the end of the fibre of area a; then $f=f_1a$, and the torque τ acting across the whole cross-section of the blade is given by the equation—

$$\tau=\Sigma fx=\frac{E}{(1-\sigma^2)R}\Sigma ax^2=\frac{E}{(1-\sigma^2)R}\cdot K, \quad . \quad . \quad (4)$$

where $K = bd^3/12$: the breadth of the blade being b and its thickness d (compare p. 247). It is obvious that τ is equal to the bending torque applied at each end of the blade.

EXPT. 28.—Determine the value of $E/(1-\sigma^2)$ for a blade.

Support the blade on two knife-edges, with one quarter of its length overhanging at each end. Hang light aluminium scale-pans, of known mass, over the ends of the blade by means of threads. Determine the value of R for a known value of τ in the manner explained on p. 248.

An experiment on the uniform bending of a blade affords insufficient data for the determination of E. But from (4) we have—

$$\frac{E}{(1-\sigma^2)} = \frac{E}{(1+\sigma)(1-\sigma)} = \frac{R\tau}{K} = G \text{ (say)}, \quad . \quad . \quad . \quad (5)$$

And from p. 233, $E/(1+\sigma) = 2n$, where n is the simple rigidity of the substance. Then—

$$\frac{2n}{1-\sigma} = G,$$

and if n can be determined, the value of σ can be calculated and substituted in (5). The value of n can be found by suspending an inertia bar from the blade and determining its period of oscillation T ; then, if I is the moment of inertia of the bar, and τ_1 is the torque that will produce unit twist of the blade, we have—

$$T = 2\pi\sqrt{\frac{I}{\tau_1}}.$$

The relation between τ_1 and n for a blade must now be investigated. The method used differs considerably from that which was employed in connection with the torsion of a cylindrical wire. In the first place it is necessary to investigate some geometrical properties of the surface called the helicoid.

The geometry of the helicoid.—Let a straight line (which will be called a *generating line*) move so that its middle point always lies on another perpendicular straight line (called the *axis*). If the generating line moves without rotating about the axis, a plane rectangular area will be generated. If the generating line moves so that it rotates about the axis, through an angle proportional to the distance traversed by its middle point, a surface called a **helicoid** will be generated. The general characteristics of a helicoid can be understood by reference to Fig. 115. The angle through which the generating line rotates while its middle point moves unit distance along the axis, will be called the *twist per unit length* of the helicoid; this will be denoted by ϕ. While the middle point of the generating line moves a distance l along the axis, the generating line twists through an angle $l\phi$. Let AOB (Fig. 116) represent part of the axis of a helicoid generated by a line which rotates in a clockwise direction (viewed in the direction from B to A) as its middle point moves from B to A. Let OD be one position of the generating line. Describe a plane containing DO and OA, and in this plane draw DF parallel to OA. From M, a point on the axis just above O, draw MF parallel to OD; then MFDO is a rectangle, and MF = OD. Join F and O by a straight line, and let the angle FOD = ψ. From O draw OC perpendicular to both OD and OA; then since OD and OA lie in the helicoid, OC is normal to the helicoid at the point O. Through FO and OC, describe a plane, cutting the helicoid in the curve OE. A little consideration will show that **the curve OE is convex towards the observer.**

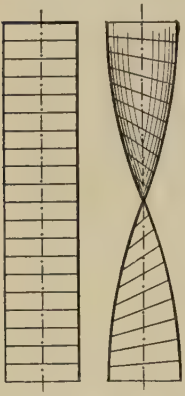

FIG. 115.—The helicoid.

Let G be any point on the straight line OF, and with O as centre and OG as radius, describe the arc GH in the plane FOC. If the twist of the helicoid is small, the arc GH will approximate to a straight line perpendicular to the plane FDO. Thus, GH is the perpendicular distance of the point H from the plane FDO. Let OG = r; then H is a point on a generating line at a distance $r \sin \psi$ above OD, and this generating line has rotated through an angle $\phi \, . \, r \sin \psi$ with respect to OD. Also, H is a point at a distance $r \cos \psi$ from the axis; therefore the rotation of the generating line on which H lies has carried it through a distance $(r \cos \psi) \times (\phi r \sin \psi) = \phi r^2 \sin \psi \cos \psi$ from the plane FDO. Thus—

$$GH = \phi r^2 \sin \psi \cos \psi;$$

that is, HG $\propto r^2$; so that, by doubling the value of r, we quadruple the value of HG. In other words, as we pass along OE, we recede more and more quickly from the plane FDO.

If the intersecting plane passing through OC were chosen so as to cut the helicoid in the curve OP, it can be proved easily that OP would be **concave towards the observer.**

When the curvature of OHE is small, an arc of the curve will be approximately equal to its chord. It is obvious that the straight line joining O to H is equal to OG, since HG is an arc of a circle with O as centre; hence, when the twist per unit length of the helicoid is small, the length of the arc of OH is approximately equal to the straight line OG. Thus, if the plane area MFDO were twisted so as to form the portion MEDO of the helicoid, any line drawn in MFDO would remain unchanged in length. Hence we may conclude that **when an infinitely thin rectangular strip is twisted into a helicoid of small twist per unit length, no elongations or shortenings are produced.**

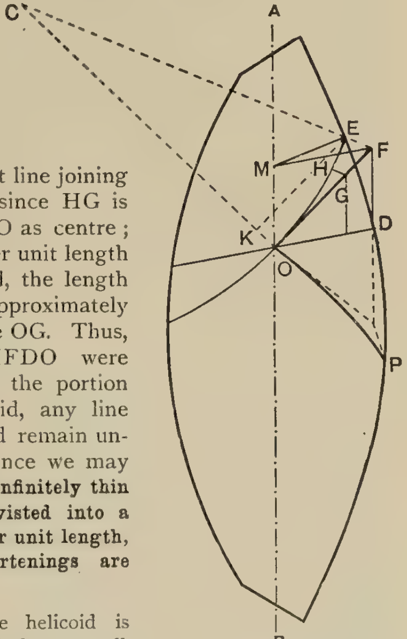

Fig. 116.—The geometry of the helicoid.

When the twist of the helicoid is small, the arc OHE will be a small element of a curve, and it will approximate to an arc of a circle. Let C be the centre of the circular arc OHE: join E to C, and from E draw EK perpendicular to OC. Then, since FO and EK are parallel and equal, OK is equal to the straight line joining F to E, and—

$$OK = FE = \phi \,.\, \sin\psi \,.\, \cos\psi \,.\, (OF)^2\,;$$

also

$$OK\{(2OC) - OK\} = EK^2\,;$$

and when OK is small, its value may be neglected in comparison with 2OC. Thus, if OC = R

$$R = \frac{(EK)^2}{2OK} = \frac{(OF)^2}{2OK} = \frac{1}{2\phi \,.\, \sin\psi \cos\psi} = \frac{1}{\phi \,.\, \sin 2\psi}.$$

$$\therefore \; \frac{1}{R} = \phi \,.\, \sin 2\psi.$$

Let $1/R$, **the reciprocal of the radius of curvature, be called the curvature.** Notice that when $\psi=0$, $\sin 2\psi=0$, and $1/R=0$; the section of the helicoid then coincides with the straight line OD, for which $R=\infty$. As ψ increases, $\sin 2\psi$ increases, until it reaches the value unity for $\psi=45^\circ$. Thus, **when the section is inclined at an angle of 45° to the generating line OD, the curvature** $1/R=\phi$. As ψ increases from 45° to 90°, $\sin 2\psi$ decreases from unity to zero, and $1/R$ diminishes from ϕ to zero. **For all sections with a value of ψ between 0° and 90°, R is positive; that is the centre of curvature is behind the helicoidal strip; and the section is convex towards the observer. For values of ψ between 0° and (-90°), R is negative; the centre of curvature is in front of the strip, and the section is concave towards the observer.** The maximum numerical values of $1/R$ correspond to $\psi=(+45^\circ)$ and $\psi=(-45^\circ)$; for the first of these values $1/R=\phi$, and for the second $1/R=-\phi$.

Model of the helicoid.—Let a long strip of paper, about 2 or 3 inches wide, be folded at right angles to its length, at regular intervals; and let a piece of cotton be threaded through the middle points of all the folds. Let the folds be opened out until the two sides of each fold are inclined at an angle of about 90°, and then let the cotton be pulled taut, its ends being fixed to two arms of a retort stand. If the lower edge of the paper is now fixed, and the upper edge is twisted about the cotton as axis, the paper takes the form of a helicoid, the folds corresponding to the generating lines (Fig. 117). Two pieces of cotton, wound over the surface so as to make angles of $(+45^\circ)$ and (-45°) with the generating lines, render evident the curvature of the corresponding sections of the helicoid (Fig. 117).

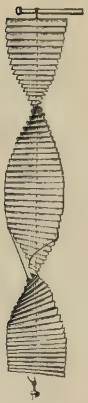

FIG. 117. Paper model of the helicoid.

Torsion of a blade or strip.—If an infinitely thin strip is twisted into an helicoid, no stretchings or compressions are produced, and therefore there are no strains. If a strip of finite thickness is twisted, it is obvious that its neutral surface will assume the form of a helicoid, and the fibres which do not lie in the neutral surface will be strained. Let Fig. 118 represent part of a strip, of which the upper end is to be twisted in an anti-clockwise sense, viewed from above, while the lower end remains stationary Let ABCD be a narrow rectangular portion of the strip the sides AB and CD being inclined at an angle of 45° to the generating lines of the helicoid. Then the portion ABCD will assume the form represented in Fig. 119. The neutral surface, indicated by dotted lines, will lie midway between the front and back surface of the strip, and a layer of fibres in front of the neutral surface (above the dotted section in Fig. 119) will be stretched in the direction AB, and compressed in the direction AD. It can be proved easily that a layer of fibres at a distance x above the dotted section in Fig. 119 suffers a longitudinal extensional strain equal to x/R (compare p. 244) due to the bending in the direction AB, together with a lateral compressional strain equal to x/R due to the bending in the direction AD. Hence, the layer must be subjected to a tensile stress, say f_1, in the direction AB, together with a numerically equal compressive stress in the direction AD. Now, the longitudinal tensile stress f_1 produces an extensional strain equal to f_1/E, together with a lateral compressive strain equal to $\sigma f_1/E$ (p. 223). The lateral compressive stress f_1 produces a lateral compressive strain f_1/E, together with a longitudinal extensional strain equal to $\sigma f_1/E$.

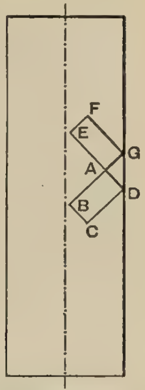

FIG. 118. — Strains produced by twisting a strip into the form of a helicoid.

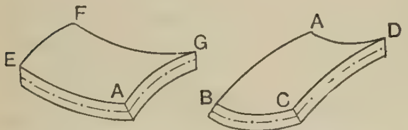

FIG. 119.—Elements of the strip represented in Fig 118, showing the effects of twisting.

Hence, the resultant extensional strain, x/R, is given by the equation—

$$\frac{x}{R} = \frac{f_1}{E} + \frac{\sigma f_1}{E} = \frac{1+\sigma}{E} \cdot f_1 = \frac{f_1}{2n} \text{ (p. 233)}$$

Thus, referring to Fig. 118, a fibre parallel to AB, and at a distance x in front of the neutral surfaces, is subjected to a tensile stress f_1, given by the equation—

$$f_1 = \frac{2nx}{R} = 2nx\phi,$$

since $1/R = \phi$, where ϕ is the twist per unit length of the strip.

AEFG (Fig. 118) represents a narrow rectangular portion of the strip, the sides EA and FG being inclined at an angle of (–45°) to the generating lines of the helicoid. The twisting of the strip will cause this portion to assume the shape AEFG (Fig. 119) and it is obvious that **a fibre parallel to** AE, **and at a distance x in front of the neutral surface** (above the dotted section in Fig. 119) **will suffer a compressive stress f_1 equal to $2nx\phi$.**

For a fibre at a distance $(-x)$ behind the neutral surface, the change in the sign of x converts a tensile into a compressive stress, or *vice versâ*.

Now consider the conditions of equilibrium of a prism ABCDE (Fig. 120), bounded by a rectangular portion ABCD of the edge of the strip, together with two planes EAB and EDC, inclined at 45° to ABCD. Let a layer, parallel to AED and passing through the line pq, be at a distance x from the neutral surface; then the mutually perpendicular edges of this layer are subjected to the tensile and compressive stresses f_1 indicated by the arrows, and these stresses are equivalent (p. 230) to a tangential stress f_1 acting on the slant edge pq. Thus, the slant edge pq is subjected to a tangential stress $f_1 = 2nx\phi$, which acts downwards when x is positive (that is, when the layer is in front of the neutral surface) and upwards when x is negative. The tangential stresses acting on the slant edges of two similar layers, equidistant from, and on opposite sides of the neutral surface, will reduce to a torque tending to twist the surface ABCD in an anti-clockwise direction. Conse-

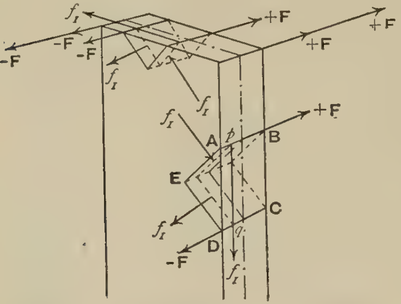

FIG. 120.—Stresses called into play by twisting a strip.

quently, the tangential stresses acting on the slant edges of all the layers into which the prism ABCDE can be divided, will reduce to a torque tending to twist the surface in an anti-clockwise direction. This resultant torque can be balanced by a torque tending to twist the surface ABCD in a clockwise direction, and this torque can be produced by two equal forces (+F), (−F), acting in the directions of the arrows along the edges AB and CD of the prism.

Let the length AD of the slant face of the prism be denoted by h, and let the thickness of the layer pq be denoted by δ, while the thickness AB of the strip is equal to d. Then the tangential stress f_1, acting on the slant edge pq of the layer, gives rise to a tangential force $f_1 h\delta$; and the torque, exerted by this force about a point in the neutral surface, is equal to $f_1 h\delta x$. Then, to determine the value of F, we have the equation—

$$Fh = \Sigma f_1 h\delta x = 2n\phi h\Sigma(\delta x^2).$$

By reference to p. 49, it will be seen[1] that—

$$\Sigma(\delta x^2) = d(d^2/12) = d^3/12.$$

Hence, $$F = 2n\phi . \frac{d^3}{12} = \frac{n\phi d^3}{6}.$$

If we divide the whole of the side face of the strip into rectangular elements similar to ABCD, then equal and opposite forces will act along the line which divides each pair of elements; and the only forces left unbalanced will be a force +F, acting from front to back at the top right-hand corner of the strip (Fig. 120), and an equal but opposite force acting along the bottom right-hand corner. It can be proved, without difficulty, that the top face of the strip is acted on by forces which reduce to a force +F acting from front to back at the top right-hand corner of the strip, and an equal but opposite force acting from back to front at the top left-hand corner. The forces acting on the remaining faces may be dealt with in a similar manner. Thus, a force +2F must act from front to back at the top right-hand corner of the strip, and an equal but opposite force must act at the top left-hand corner. If b is the breadth of the strip, these forces reduce to a torque $\tau = 2Fb$, tending to twist the top of the strip in an anti-clockwise direction, viewed from above.

---

[1] $\Sigma(\delta x^2)$ is obviously equal to the moment of inertia of a thin rod, of unit mass per unit length, that is, of total mass d, where d is the length of the rod.

An equal but opposite torque must act on the bottom of the strip. If l is the length of the strip, while θ is the total twist of one end relative to the other end of the strip, then the twist per unit length, $\phi = \theta/l$. Hence—

$$\tau = 2\text{F}b = \frac{nbd^3}{3} \cdot \frac{\theta}{l},$$

and τ_1, the torque per unit twist, is given by the equation—

$$\tau_1 = \frac{\tau}{\theta} = \frac{nbd^3}{3l}.$$

Let τ_1' denote the torque per unit twist of a wire of radius r, and of length and sectional area equal to those of the strip, so that $\pi r^2 = bd$. Then (p. 237)—

$$\frac{\tau_1'}{\tau_1} = \frac{n\pi r^4}{2l} \div \frac{nbd^3}{3l} = \frac{3\pi r^4}{2bd^3} = \frac{3(\pi r^2)^2}{\pi} \div \frac{2(bd)^2 d}{b} = \frac{3b}{2\pi d}.$$

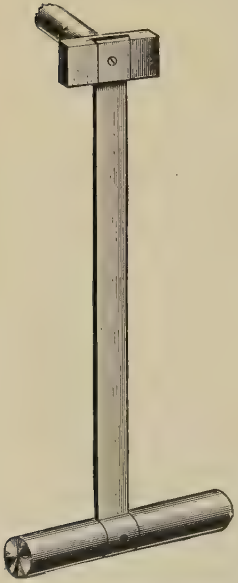

FIG. 121.—Arrangement for determining the coefficient of shear elasticity of a blade.

When b is much greater than d, the torque τ_1', which will produce unit twist in the wire, is very much greater than the torque τ_1 which will produce unit twist in the strip. Hence, where a delicate torsional suspension is required, as for instance in suspended coil galvanometers, a strip is used in preference to a wire; the strength of the suspension depends on the area of the strip, and the deflection produced by a given torque will be many times greater than if a wire of the same strength were used.

EXPT. 29.—Determine the value of n for a blade.

The steel blade used in experiment 28 (p. 269) may be used. An inertia bar is fixed to its lower end in the manner indicated in Fig. 121; the screw passes through a hole in the blade. The upper end of the blade is clamped to an arm of a firm retort stand, by means of a screw that passes through another hole in the blade. Great care should be exercised in measuring the thickness d of the blade.

Determination of the bulk modulus k of a solid.—When E and n have been determined for a rod, wire, or blade, the value of k can be calculated from the formulæ given on p. 233. But if the value of k is required for a short solid cylinder of large diameter, neither E nor n can be determined by the methods previously described; hence, if the value of k is required for such a cylinder, it must be determined by an independent method. This can be accomplished by measuring the decrease in length produced when the cylinder is subjected to a uniform external pressure. In this case, the decrease of volume per unit volume is three times the decrease of length per unit length (p. 218), if the substance is homogeneous and isotropic. Thus, if L is the length of the cylinder, and l is the decrease in length produced by an uniform compressive stress f_1, the value of k can be determined from the equation—

$$k = \frac{f_1}{3(l/L)}.$$

The value of l can be determined by enclosing the cylinder in a strong glass vessel which contains water, and observing two marks on the cylinder through fixed microscopes provided with micrometer eye-pieces, before and after the water is subjected to a high pressure.

If the value of k is required for the substance of which a hollow cylindrical vessel is composed, the following method may be used. A graduated glass tube G, of fine bore, is fixed in one end of the cylinder, and the whole is filled with water up to a point near the top of the glass tube. The upper end of the glass tube is open to the atmosphere, so that the water is at atmospheric pressure. The cylinder is supported in the manner indicated in Fig. 122, and its lower end is loaded; the distance through which the water sinks in the tube indicates the increase in the internal volume of the vessel produced by the tensile stresses due to the load.

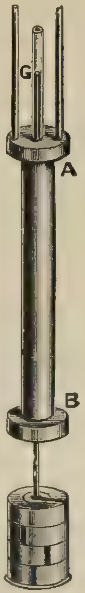

FIG. 122. Apparatus for the determination of the bulk modulus of the metal of which a hollow cylinder is composed.

The relation between the increase in the internal volume of the hollow cylinder, the tensile stress, and k, can be found by comparing the strain of the hollow cylinder with that of a solid cylinder of the same material and of the same dimensions. In imagination, the hollow cylinder may be converted into the solid one by filling it with a solid core of the same material as that of the walls; if the solid cylinder is subjected to the same stress (force per unit area) as that applied to the walls, any alteration of volume produced in the core will be identical with the alteration in the internal volume of the hollow vessel; for the core continues to fill the hollow vessel, and there is no lateral stress on the stretched fibres of either the core or the hollow vessel. Now, from the reasoning explained on pp. 231–232, it follows that a longitudinal tensile stress f_1 produces a dilatational strain equal to $f_1/3k$, together with two shears which entail no alteration in volume. Hence, if V denotes the internal volume of the unstrained cylindrical vessel, and v denotes the increase in the internal volume produced by the longitudinal tensile stress f_1, we obtain the equation—

$$\frac{v}{V} = \frac{f_1}{3k},$$

from which k can be calculated in terms of v, V, and f_1.

Bulk modulus of a liquid.—In determining the bulk modulus of a liquid, it is necessary to observe the decrease in the volume of the liquid, produced by a known compressive stress. If we imagine the liquid to be contained in a vessel of the shape of a thermometer, then the compressive stress will not only diminish the volume of the liquid, but it will also increase the internal volume of the containing vessel. This increase in the volume of the containing vessel cannot be determined with accuracy; hence, Regnault enclosed the vessel in a larger one so that equal compressive stresses could be applied outside and inside the vessel. Fig. 123 is a diagrammatic representation of

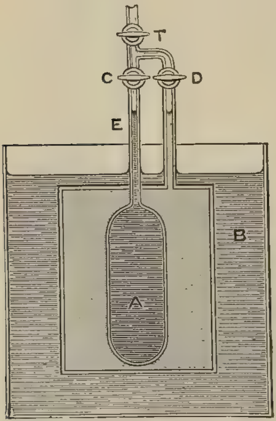

FIG. 123.—Diagrammatic representation of Regnault's apparatus for determining the bulk modulus of a liquid.

Regnault's apparatus; the liquid was contained in a vessel A, which was itself enclosed by a larger vessel. The tube T was connected to a reservoir of compressed air, and by opening the stop-cock C the liquid in A was connected with the reservoir, and by opening the stop-cock D the interior of the containing vessel was connected with the reservoir; when both C and D were opened, the external pressure acting on the vessel A was equal to the pressure to which the liquid in A was subjected. The apparent compression of the liquid was determined by observing the motion of the surface of the liquid in the narrow glass tube E. At first sight it would appear that, if the walls of the vessel A were very thin, no alteration would be produced in the internal volume of A by subjecting both the contained liquid and the outside of the vessel to the same pressure. It must be remembered, however, that the uniform compressive stress to which the walls of the vessel A are subjected, will cause each particle of the walls to shrink in volume, and therefore the volume enclosed by the walls will be diminished. Similarly, when a vessel containing a liquid is heated, the expansion of the material of which the vessel is composed causes the internal volume of the vessel to increase; and the increase in the internal volume of the vessel is independent of the thickness of the walls of the vessel.[1]

The alteration in the internal volume of the vessel, produced by a uniform compressive stress f_1 applied inside and outside, can be determined by comparison with a solid vessel of the same material and dimensions, subjected to a uniform external stress f_1. In imagination, the hollow vessel can be converted into the solid one by filling it with a solid core of the same material as the walls of the vessel; since the compressive stress will be uniform throughout the solid, every particle will contract by an amount proportional to the stress f_1. The core continues to fill the shell, and the strains in the shell are due merely to the stress f_1; hence the strains in the shell are the same, whether the stresses on its internal surface are supplied by the contained liquid or by the solid core, and therefore the diminution in the internal volume of the vessel is exactly equal to the diminution in the volume of the core. Thus, if V is the internal volume of the unstrained vessel, and v' is the diminution of volume produced by the stress f_1 applied both inside and outside, then $v'/V = f_1/k$, where k is the bulk modulus of the substance of which the vessel is composed.

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[1] See the Author's *Heat for Advanced Students* (Macmillan), pp. 65–66.

Let v be the apparent compression of the liquid, as determined from the motion of the liquid in the tube E; and let v'' be the true compression of the liquid. Then $v = v'' - v'$. Thus—

$$\frac{v}{V} = \frac{v''}{V} - \frac{v'}{V}.$$

Now, $v''/V = f_1/K$, if K denotes the bulk modulus of the liquid. Therefore—

$$\frac{v}{V} = f_1\left(\frac{1}{K} - \frac{1}{k}\right)$$

The value of K can be determined most accurately by subjecting the vessel to a tensile stress, as explained in the preceding section.

Let v_1 be the apparent compression of the liquid when the internal stress is equal to f_1 and the outside stress is zero; and let v_2 be the apparent expansion of the liquid when the internal stress is zero and the external stress is f_1. Then, if v is the apparent compression of the liquid when the internal and external stresses are both equal to f_1, it follows that—

$$v = v_1 - v_2.$$

This follows from the law that **a given stress produces the same strain, whether the substance is originally strained or unstrained;** provided that the strains do not surpass the elastic limits. For instance, the extension of a wire, within the elastic limits, is proportional to the load; a given increase of load produces the same strain, whatever may have been the original load. Now, the internal stress f_1 compresses the liquid and expands the containing vessel, with the result that the volume of the liquid apparently diminishes by v_1; on applying the external stress f_1 the compression of the liquid is unaffected, but the volume of the containing vessel is diminished by v_2, just as if the internal stress were equal to zero.

Regnault measured v_1 and v_2 as well as v, and from these values calculated the value of k, the bulk modulus of the containing vessel, on the assumption that Poisson's ratio is equal to 1/4 (p. 233). Since this assumption is inadmissible, the more accurate method of determining k, which is described in the text, has been used in more recent experiments.

The **compressibility** of a liquid is defined as the compressive strain (diminution of volume per unit volume) produced by a definite compressive stress. According to Regnault the **compressibility of water** per atmosphere[1] of pressure is equal to $4 \cdot 8 \times 10^{-5}$. More recent experiments show that at 0° C, the compressibility of water per atmosphere lies between the limits $5 \cdot 03 \times 10^{-5}$ and $5 \cdot 12 \times 10^{-5}$. The compressibility of water diminishes as the temperature rises; according to Pagliani and Vincentini, a minimum value of $3 \cdot 89 \times 10^{-5}$ is reached between 60° and 70° C., while at 99° C the compressibility is equal to $4 \cdot 09 \times 10^{-5}$.

The mean value of the **compressibility of mercury** is about $3 \cdot 8 \times 10^{-6}$. The compressibility of mercury, and that of most other liquids (water excepted), increases with the temperature.

Tensile strength of liquids.—Ordinary observation shows that an extremely small force suffices to separate a liquid into parts, and apparently leads to the conclusion that there is no perceptible cohesion between the particles of a liquid. It may be remarked, however, that the separation of a liquid into parts always takes the form of an inflowing of the surface, and never results from an internal rupture; the fact that such an inflowing is produced readily does not disprove cohesion between the particles of a liquid, any more than the strength of a strip of paper under a uniform tensile stress is disproved by observing the ease with which the paper can be torn when it has been snipped at the edge.

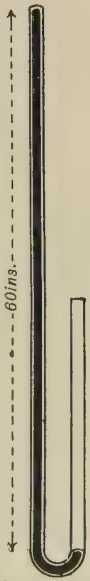

Fig. 124.—Osborne Reynolds's experiment, to show the tensile strength of mercury.

Prof. Osborne Reynolds observed that in certain circumstances the mercury in a barometer tube 60 inches long adhered to the top of the tube, and remained as an unbroken column, even when the mercury at the lower end of the tube was subjected to a scarcely perceptible pressure. A tube similar to that of an ordinary syphon barometer, but about 60 inches in length (Fig. 124), was cleaned, and its inner surface was wetted by sulphuric acid or by water freed from air. The tube, in a horizontal position, was then filled with mercury ; on raising the closed end until the tube was vertical, the mercury column broke, and its surface stood at a height of 30 inches ; but after lowering the tube into a horizontal position, and taking great care to remove some small air bubbles which made their appearance, it was found possible at last to raise the closed end until the tube was vertical, without rupturing the mercury column. The short limb of the tube was then connected to an air pump, and exhausted ; the mercury still remained as an unbroken column 60 inches long.

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[1] A standard atmosphere is practically equal to 10^6 dynes per square centimetre.

Now, there was no question of the mercury being upheld by sticking to the *side* walls of the tube ; hence, the film of water or acid at the top of the tube must have adhered to the glass above it, and supported the column of mercury hanging below it. Whenever a rupture of the mercury column occurred, a small bubble of air was always detected ; this bubble started the rupture, just as a scratch of a diamond will start a crack in a strained piece of glass. Hence it follows that the mercury, as well as the film of acid or water, withstood a tensile stress equal to two atmospheres. From this we conclude that the molecules of a liquid cling together, and a considerable force is needed to pull them asunder ; the difficulty of removing every trace of air from a liquid prevents us from determining the absolute stress which is necessary to rupture it, but mercury has been observed to sustain a tension of 72·5 lbs. per square inch, while sulphuric acid has been observed to sustain a tension [illegible] high as 173 lbs. per square inch.

Prof. Worthington has found that when water is subjected to a dilatational stress, the increase in its volume is equal to the decrease in volume that would be produced by an equal compressive stress. Mr. H. H. Dixon,[1] has used an experimental method, due originally to Berthelot, to determine the minor limit of the tensile stress of water in terms of its elasticity under tensile stress. A strong capillary tube, closed at one end and drawn out at the other, was filled with water at 28° C. ; the tube was cooled to 18° C., so as to allow a small bubble of air to enter it, and the open end was then closed. On raising the temperature of the tube the air gradually dissolved in the water, which finally filled the tube completely. On re-cooling the tube to 18° C., the water still filled it completely, and therefore it must have suffered a dilatational strain, measured by the ratio of the volume of the bubble of air to the volume of the water. The bulk modulus of elasticity of water being known, the dilatational stress to which the water was subjected was calculated. It was found to be equal to 150 atmospheres, or 2,250 lb. per sq. inch.

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[1] *The Tensile Strength of Water*, by H. H. Dixon ; *Proc. Roy. Soc. of Dublin* 12, 7, pp. 60–65, April, 1909.

It will be explained, in an ensuing chapter, that the phenomena of surface tension lead us to conclude that the tensile strength of liquids is really very great, amounting to many thousands of atmospheres; and the experiments described above show that there is nothing intrinsically absurd in this conclusion, although the phrase "weak as water" has been accepted as implying the irreducible minimum of strength.

QUESTIONS ON CHAPTER VIII

1. A boxwood metre-scale is placed in a vertical position, with its lower end clamped in a vice, and with a heavy body fixed to its upper end (compare Fig. 65, p. 163). The scale is bent, by pulling the heavy body to one side, and is then released: determine the period of oscillation of the heavy body.

2. The heavy body, referred to in question 1, is symmetrical in shape, and its moment of inertia about a vertical axis can be calculated. The body is twisted through a small angle about its vertical axis of symmetry, and is then released; determine the period of its torsional oscillation about its position of equilibrium.

3. Explain how the values of n, k, σ, and E for boxwood can be obtained by the aid of the results deduced in answer to questions 1 and 2.

4. It is found that a torque equal to 1,000 dyne-centimetres is required to twist one end of a cylindrical wire through a radian, relatively to its other end. A strip of the same material is equal in length and sectional area to the wire, and the breadth of the strip is ten times as large as its thickness; what torque will be needed to twist one end of the strip through a radian, relatively to its other end?

5. A uniform metal rod, so long that its length may be considered to be infinitely great, is supported in such a manner that it can move freely in a direction parallel to its length, while lateral motion is rendered impossible. A pressure equal to f_1 dynes per sq. cm. is applied suddenly to one of the flat ends of the rod; determine the

velocity with which the resulting compressional strain travels along the rod.

6. A uniform cylindrical metal rod, so long that its length may be considered to be infinitely great, is supported in such a manner that it can rotate freely, while any other kind of motion is rendered impossible. A torque is applied suddenly to one end of the rod, tending to twist it about its axis of symmetry: determine the velocity with which the resulting torsional strain travels along the rod.

7. Equal and opposite compressive stresses are applied to the ends of a uniform straight rod of length l. Prove that the effect produced will be as follows:—

(*a*) If the force F applied to either end of the rod is less than $(\pi/l)^2\mathrm{EK}$, the rod will be compressed longitudinally while remaining straight.

(*b*) If $\mathrm{F}=(\pi/l)^2\mathrm{EK}$, the rod will be in equilibrium either when it is straight, or when it is bent into a loop of the wave curve

$$y=a \sin (2\pi x/2l).$$

(*c*) If $\mathrm{F}>(\pi/l)^2\mathrm{EK}$, the rod will continue to bend more and more until it breaks.

(The radius of curvature R of a wave curve is given by the equation—

$$\frac{1}{\mathrm{R}}=\left(\frac{2\pi}{\lambda}\right)^2 y,$$

where λ denotes the wave length (see p. 336).

8. An elastic rod of uniform sectional area, and so long that its length may be considered to be infinitely great, is supported in such a manner that it can bend freely, any other kind of motion being rendered impossible. A part of the rod is bent into the shape of a wave curve (p. 85) and is then released; determine the velocity with which the resulting flexural waves travel along the rod.

9. A rod of finite length can move freely in a direction parallel to its length; determine the pressure in the rod, at a distance x from its front end, when the rod moves with an acceleration α, produced by a force applied to its rear end.

10. A long flexible cord, of mass m_1 per unit length, is subjected to a tension equal to f; the cord is struck transversely, and the deformation produced by the blow is seen to resolve itself into two transverse disturbances which travel along the cord in opposite directions. Determine the speed with which either transverse disturbance travels along the cord.

CHAPTER IX

SURFACE TENSION

Introduction.—When a liquid partly fills a vessel, the free surface of the liquid is smooth and definite in shape: this definite free surface is the characteristic by which a liquid is recognised. A drop of liquid, clinging to the underside of a plate of glass, also has a definite free surface. The fact that the drop can cling to the underside of the glass, proves that the liquid adheres to the glass; or, expressing this conclusion in terms of the molecular theory of matter, we may say that those molecules of the liquid which are contiguous to the glass, are attracted by the molecules of the glass. To explain the definite shape of the drop, some other property of the liquid must be assumed; for the only molecules of the liquid which are attracted to any appreciable extent by the glass, are those which are very close to it, and therefore most of the molecules comprised in the drop must be in equilibrium under the action of their mutual attractions. Now, whatever forces may be exerted on a molecule in the interior of a liquid, they must be such that the liquid can flow; for, the shape assumed by a liquid is that of the vessel in which it is contained. Thus, the definite shape of the drop leads us to conclude that its surface must act as an elastic containing vessel. If the amount of liquid contained in the drop is gradually increased, a stage is reached at which the elastic vessel is no longer strong enough to support its contents, and a portion of the liquid falls away. It must be noticed that this falling away of a portion of the liquid occurs, not by internal rupture of the drop, but by the formation of a constriction in its surface; and this constriction increases until part of the drop becomes detached. Fig. 125 represents several stages in the formation and breaking away of a drop.

EXPT. 30.—To examine the formation and breaking away of a drop of liquid.

At temperatures above 80° C, liquid aniline is less dense than water, while at low temperatures, aniline is denser than water. Further, aniline does not mix with water, so that if the two liquids are contained in a beaker, they are separated by a definite free surface. These facts have been utilised by Mr. C. A. Darling,[1] in the arrangement of a beautiful experiment, which exhibits the various stages in the formation and breaking away of a drop of a liquid.

Procure a beaker about 9 inches in height and about 4·5 inches in diameter ; fill it with distilled water to a height of about 7 inches, and add about 80 c.c. of aniline. Place the beaker on a sand-bath, and allow the temperature to rise to about 80° C. The hot aniline ascends to the surface of the water, becomes cooled by contact with the air, and collects in the form of a pendant drop, an inch or more in diameter. As the drop increases in size, it develops a neck which becomes thinner and thinner ; at a certain stage, two constrictions develop in the neck, and finally the neck becomes severed at two places. Thus, when the large drop breaks away, it is followed by a droplet, called **Plateau's spherule**, in honour of its discoverer. As the large drops descends, it exhibits an oscillatory change of shape due to the tension of its surface layer. When the aniline reaches the bottom of the beaker, it once more becomes hot, and again ascends to the surface ; so that another drop presently forms and breaks away. The process of formation and breaking away is so slow, that each stage can be observed at leisure.

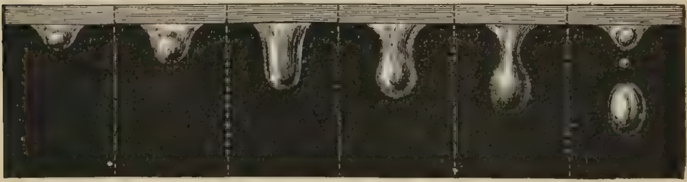

FIG. 125.—Stages in the formation and breaking away of a drop of water.

Thus we are led to conclude that the surface of a liquid acts somewhat like a stretched elastic membrane. If a straight line be drawn on the surface of such a membrane, the portions of the membrane on opposite sides of the line are pulled away from each other, and rupture is prevented only by the cohesion between particles on opposite sides of the line. Let a straight line, one centimetre in length, be drawn on the surface of a liquid ; then the portions of the surface on opposite sides of this line are pulled away from each other with a force which is defined as the **surface tension** of the liquid. Hence, surface tension is a force per unit length, and its dimensions are—

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[1] *Nature*, March 10, 1910, p. 37.

$$\frac{ML}{T^2} \div L = \frac{M}{T^2}.$$

Let the surface of a liquid be bounded by a rectangular framework, comprising two parallel rails rigidly connected by a cross piece, and a second cross piece which can slide along the rails while remaining parallel to its original direction. Let the length of the cross piece be l ; then, if the liquid adheres to the cross piece, this will be acted upon by a force equal to Sl, where S is the surface tension of the liquid. The force Sl tends to diminish the area of the surface. While the temperature of the liquid remains constant, let the cross piece be displaced through a distance d in the direction which increases the area of the surface ; then, the work done by the agent which moves the cross piece is equal to Sld, and the increase in the area of the liquid is equal to ld. Thus, **the surface tension of a liquid may be defined as the work done in enlarging the surface of the liquid by one square centimetre under isothermal conditions.**

The definition of surface tension, as the energy per unit area of the surface of a liquid, is inadmissible, although very widely accepted The objection to this definition may be made clear by a comparison When a perfect gas expands isothermally, the energy of the gas (that is, the kinetic energy of its molecules) remains constant, the external work performed during the expansion being derived solely from the heat which enters the gas.[1] Now, a transmission of heat occurs when the surface of a liquid is stretched isothermally, and therefore the energy of the new surface is not equal merely to the mechanical work done in forming it ; allowance must be made for the energy which enters or leaves the surface in the form of heat. Of course, no change of temperature is produced by the heat which enters or leaves the surface during an isothermal process. If heat enters the surface during an isothermal increase of area, then the energy of the newly formed surface is equal to the work done in overcoming the surface tension, *plus* the mechanical equivalent of the heat absorbed.

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[1] See the Author's *Heat for Advanced Students* (Macmillan), p. 298.

EXPT. 31.—To measure the surface tension of water.

Make a rectangular framework of platinum wire, similar to that represented in Fig. 126; the dimensions of the rectangle may be about 3 cm. by 1·5 cm. Hold the framework in a Bunsen flame until it becomes red hot; this will remove every trace of grease from the wire. Hang the framework so that it is partly immersed in water, and support it from one arm of a balance; the upper side of the rectangle should be two or three millimetres above the surface of the water. Add weights to the scale pan on the opposite side of the balance until the weight of the partly immersed framework is counterbalanced. Now depress the framework until its upper side sinks below the surface of the water; on raising it, a film is formed, extending from the upper side of the framework to the surface of the water. This film has two surfaces, each of which exerts a tension of S dynes per cm.; thus if the breadth of the framework is b, the downward force exerted by the film is equal to $(2 \times Sb)$, and if this force can be counterbalanced by adding a small mass m to the scale pan on the opposite side of the balance, then—

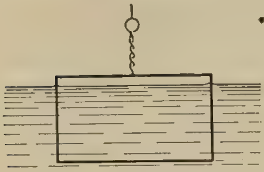

FIG. 126.—Method of measuring the surface tension of a liquid.

$$2Sb = mg.$$

It is impossible to obtain a film of pure water when the upper side of the rectangle is more than about 3 mm. above the surface of the water. Adding a little soap to the water renders it possible to obtain a much longer film, but the surface tension of the soap solution is less than that of the water.

Repeat the above experiment, using ether, paraffin oil, and benzene in place of water.

EXPT. 32.—To measure the surface tension of mercury.

Procure a piece of thin sheet copper, cut to the shape represented in Fig. 127; the lower edge is sharpened, and the two projecting horns are made as thin as possible. Amalgamate the lower part of the copper (left white in Fig. 127), by dipping it in a solution of mercuric nitrate. Wash the amalgamated surface, dry it in a spirit flame, and pour pure mercury over it, until no trace of scum remains. Then hang the piece of copper from one arm of a balance, so that the horns just dip into mercury contained in a suitable vessel. Counterbalance the piece of copper by means of weights added to the scale pan on the other side of the balance.

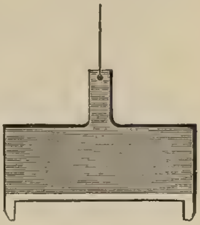

FIG. 127. — Method of measuring the surface tension of mercury.

Now, depress the copper so that the amalgamated surface is immersed in the mercury, and form a film of mercury by raising the copper quickly but steadily. It is best to use an ordinary balance which weighs to a centigram, since in this type of balance the beam is raised during weighing. Add weights to the scale pan on the other side of the balance, until the mercury film just breaks when the beam is raised so far that the pointer of the balance is vertical; the additional weight, expressed in dynes, is equal to the sum of the tensions of the two surfaces of the mercury film. After a little practice, an accurate value of the surface tension of mercury can be obtained by this method; this value is equal to 450 dynes per cm.

Elementary molecular theory of surface tension.—The attraction exerted by one molecule on another may be assumed to be negligibly small when the distance separating the molecules exceeds a certain small value. Thus, we may imagine a spherical surface to be described about each molecule, its radius being equal to the greatest distance at which an appreciable attraction can be exerted on another molecule; this sphere is called the **sphere of molecular attraction.** The molecules of a substance in the gaseous state are widely separated, and one molecule remains within the sphere of attraction of another only during the short time immediately preceding and following an impact. In the liquid state, the molecules of a substance are packed relatively closely; that is, the sphere of attraction of a molecule embraces a large number of neighbouring molecules, each of which is attracted by a finite force. Since action and reaction are equal and opposite, it follows that a molecule in the interior of a liquid is attracted by its immediate neighbours; but, since the attractions are exerted in various directions, their average resultant is equal to zero, and there is no tendency to move the molecule in any particular direction. Thus a molecule can move through the body of a liquid unopposed, except by impacts with its neighbours; and therefore a molecule neither loses nor gains energy while it moves from one position to another within the body of a liquid.

A molecule near the surface of a liquid is under different conditions. Any molecules in the gaseous state, in the space above the liquid, are beyond the sphere of attraction of a molecule in the surface; hence, a molecule in the surface is attracted only by the molecules lying around it in the surface, and those immediately beneath it in the interior of the liquid. Hence, in order that a molecule may arrive at the surface from the interior of a liquid, it must have overcome a resultant attraction tending to pull it back into the interior. From this we may infer that, when a molecule approaches the surface from the interior of a liquid, work is done by it at the expense of its kinetic energy. But the temperature of the liquid depends on the kinetic energy of its molecules, and since the surface is at the same temperature as the interior of the liquid, it follows that a molecule approaching the surface must gain kinetic energy from its neighbours, and these must gain energy in the form of heat from some external source. When the surface of a liquid is increased in area, a number of molecules which previously were in the interior of the liquid will form the new surface layer; and in order that these molecules may possess the same average kinetic energy as those in the interior of the liquid, heat must have been absorbed from some external source. Thus **when the surface of a liquid is enlarged isothermally, heat is absorbed.** If the surface is enlarged adiabatically (that is, under such conditions that heat neither enters nor leaves the liquid), then the liquid will be cooled.

The molecules, lying side by side in the surface of a liquid, cling one to another, and thus endow the surface with properties somewhat resembling those of a stretched elastic membrane; hence the existence of surface tension. When a molecule is in the surface, its potential energy is greater than when it was in the interior of the liquid; for work must be done in bringing the molecule to the surface, against the resultant force which pulls it toward the interior of the liquid. Any system of bodies arranges itself in such a manner that the potential energy of the system is a minimum (p. 44); since the total surface energy is diminished by diminishing the area of the surface, it follows that the surface tends to become as small as possible.

A very slight film of grease produces a great diminution in the surface tension of water. This is due, probably, to the fact that the water molecules exert a very small attraction on the grease molecules; therefore when the water molecules in the surface are separated, one from another, by interspersed grease molecules, it is impossible for them to cling together with the same force as when the grease molecules are absent.

EXPT. 33.—Wash a glass vessel under the tap for some considerable time, and then fill it with tap water. Be careful not to touch the water or the inside of the vessel with the fingers. Scatter some lycopodium powder over the water surface, and then touch the surface with a finger that has been slightly greased by contact with the hair of the head. The surface tension of the water is diminished where the surface is touched, with the result that the uncontaminated surface, near to the sides of the vessel, contracts, carrying the lycopodium powder with it. As a result, a large, nearly circular patch of surface is left free from lycopodium in the middle of the vessel. The lycopodium powder merely serves the purpose of making visible the motion of the surface layer of water.

EXPT. 34.—Scatter lycopodium powder over a clean water surface, and then touch the surface with (1) a crystal of sodium carbonate, (2) a crystal of sodium chloride, (3) a crystal of sodium hypochlorite, (4) a stick of caustic soda, and (5) a crystal of sodium sulphate. The motion of the lycopodium powder indicates whether the aqueous solutions of these substances possess surface tensions greater or less than that of water. Scrape each crystal so as to remove any trace of grease or other impurity from the part which is brought into contact with the water surface.

In order that a molecule may escape from the surface of a liquid, it must approach the surface from the interior with a velocity sufficient to carry it beyond the range of molecular attraction of the surface layer. Hence, when a liquid evaporates, it is only the more quickly moving molecules that escape, while those moving more slowly remain behind; consequently, a liquid is cooled by evaporation. In order that a gram of liquid shall evaporate, a quantity of heat equal to the latent heat of vaporisation must be supplied; this quantity of heat is thermally equivalent to the work done in carrying the molecules, comprised in a gram of the liquid, from the interior to the surface, and thence into the space above the liquid, beyond the range of molecular attraction of the surface molecules. Thus, there is an intimate connection between the latent heat of vaporisation and the surface tension of a liquid.

The latent heat of vaporisation of a liquid decreases as the temperature of vaporisation rises.[1] The connection between latent heat and surface tension leads us to conclude that the surface tension of a liquid will decrease as the temperature of the liquid rises. At the critical temperature there is no difference between the liquid and its vapour; hence, at this temperature the latent heat of vaporisation, and the surface tension of the liquid, must both be equal to zero.

EXPT. 35.—To demonstrate the decrease in the surface tension of a liquid, produced by a rise in temperature.

Fill a glass vessel with clean water (p. 291), and scatter lycopodium powder over the surface of the latter. Hold a piece of hot metal (such as a heated soldering bit) at a small distance above the surface of the water; the lycopodium moves away from the point at which the surface is most heated, thus showing that the surface layer of the water is pulled away from that point (compare experiment 33, p. 291).

Relation between surface tension and surface energy.—When a liquid surface is enlarged isothermally, heat is absorbed; and when the surface is enlarged adiabatically, the temperature of the surface falls. The quantity of heat absorbed at constant temperature, per unit increase of area, can be determined by an application of thermodynamic principles.[2]

At a constant temperature, the value of the surface tension is independent of the area of the surface of the liquid. Hence, if surface tension is plotted against surface area, the isothermal relation between the two can be represented by a horizontal straight line parallel to the axis along which areas are measured. Let AB (Fig. 128) represent the isothermal relation between the surface tension S and the surface area, at a temperature T measured on the absolute dynamical scale. At a slightly lower temperature $(T - t)$, the surface tension will be greater; let its value be $(S + s)$, and let the isothermal relation between surface tension and surface area now be represented by the horizontal straight line CD. Let EF and GH represent portions of adiabatics, that is, curves representing the relation between surface tension and surface area when no heat is allowed to enter or leave the surface layer of the liquid. When the surface layer is enlarged adiabatically, the temperature falls, and therefore the surface tension increases. The straight lines AB and CD are very close together, since t is small and therefore s is small; hence, the portions of the adiabatics, cut off by AB and CD, are very small, and may be treated as straight lines. Since EF and GH are very near to each other, they will be approximately parallel; hence EFHG is a parallelogram. Let the side GE of the parallelogram, which represents an isothermal enlargement of surface, be equal to a; then, since the perpendicular distance between AB and CD is equal to s, the area of EFGH is equal to sa.

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[1] See the Author's *Heat for Advanced Students*, p. 155. (Macmillan.)

[2] For a discussion of the thermodynamic principles used in this section, see the Author's *Heat for Advanced Students*, Ch. XVI., pp. 333–365. (Macmillan.)

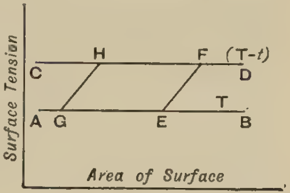

FIG. 128.—Surface tension cycle.

Let it be supposed that the surface is originally at a temperature T, its area being that corresponding to the point E (Fig. 128). Let the surface be enlarged adiabatically until its temperature falls to $(T-t)$; its properties now correspond to the point F. Next, let the surface contract isothermally until it attains the area corresponding to the point H; and during this process let a quantity Q_2 of heat be given out. As the surface contracts, it performs external work. Next, let the surface contract adiabatically until its temperature rises to T; the point now reached is G. Finally, let the surface be enlarged isothermally, until the point E is reached; and let a quantity Q_1 of heat be absorbed, work being done by the agent which stretches the surface. The surface has been brought back to its original condition, and therefore its energy has the same value as at first. A quantity of heat Q_1 has been absorbed at a temperature T, and a quantity Q_2 has been given out at a temperature $(T-t)$, so that (Q_1-Q_2) units of heat have disappeared.

During the isothermal enlargement of the surface by a sq. cm. at the temperature T, the work done on the surface by the external agent was Sa, since the surface tension is equal to the work done in enlarging the surface by unity at a constant temperature (p. 287). During the isothermal contraction of a sq. cm. at $(T-t)$, the external work done by the surface was

$(S+s)a$. The external work done during the adiabatic contraction must be equal to the internal work done during the adiabatic enlargement of the surface; for the two alterations of area are equal, and the surface tension has the same average value in both cases. Hence in completing the cycle, a net amount of external work equal to—

$$(S + s)a - Sa = sa$$

has been performed. It has been proved already that sa is the area of the cycle EFGH.

It is obvious that the cycle is reversible; that is, the energy transformation which occurs in traversing any part of the cycle in one direction, is reversed if the same part of the cycle is traversed in an opposite direction. Thus, the cycle is similar to that of a reversible heat engine. The most important difference between the surface tension cycle and a gas cycle is, that a net amount of external work is performed when the surface tension cycle is traversed in an anti-clockwise sense, while a net amount of work must be performed by an external agent when a gas cycle is traversed in an anti-clockwise sense. This difference is due to the fact that a gas can do external work only by expanding, while a surface can do external work only by contracting.

By the first law of thermodynamics, the heat $Q_1 - Q_2$ which has disappeared is equivalent to the nett work, sa, performed during the cycle. Therefore—

$$Q_1 - Q_2 = \frac{sa}{J},$$

when J is the mechanical equivalent of unit quantity of heat.

As a consequence of the second law of thermodynamics, the efficiency of the cycle is the same as that of a reversible heat engine working between the temperatures T and $(T - t)$; thus—

$$\frac{Q_1 - Q_2}{Q_1} = \frac{T - (T - t)}{T};$$

$$\therefore \quad \frac{sa}{JQ_1} = \frac{t}{T};$$

$$\therefore \quad \frac{Q_1}{a} = \frac{s}{t} \cdot \frac{T}{J}.$$

Q_1/a is the value of the heat absorbed by each square centimetre added to the surface at the absolute temperature T. s/t is the increase in the surface tension due to a fall of temperature of 1° C; this quantity is called the **temperature coefficient of the surface tension.** The following table gives the surface tension S_o for a number of liquids at 0° C, together with the temperature coefficient β. If S_t denotes the surface tension of a liquid at t° C., then—

$$S_t = S_o - \beta t.$$

Liquid	S_o	β
Alcohol (C_2H_6O)	25·3	0·087
Benzene (C_6H_6)	30·6	0·132
Ether ($C_4H_{10}O$)	19·3	0·115
Mercury	441·3	0·379
Water	75·8	0·152

At 0° C. the surface tension of water is equal to 75·8 dynes per centimetre, and each degree rise of temperature entails a decrease of 0·152 dynes per cm. Hence for water $s/t = 0·152$. Thus Q_1/a, the heat absorbed by each additional square centimetre added to a water surface, is equal to $\frac{273 \times 0·152}{4·2 \times 10^7}$ gram-calories, and this is equivalent to $273 \times 0·152 = 41·5$ ergs. **While the surface of water at 0° C is being increased by one square centimetre, 75·8 ergs of work are performed by the agent which stretches the surface, and 41·5 ergs enter the surface in the form of heat.** Hence, at 0° C the **surface energy** of water is equal to 117·3 ergs per sq. cm. It is clear that the surface energy of a liquid is much greater than its surface tension (see p. 287).

EXPT. 36.—To determine the surface tension, and its temperature coefficient, for water and other liquids.

Obtain a piece of thermometer tubing about 6 inches long, its internal diameter being half a millimetre or less. Select a piece of tubing which has a uniform bore; the uniformity of the bore can be tested by introducing a short thread of clean mercury into the tube, and measuring its length in different parts of the tube. Clean the inside of the selected tube with nitric acid, and then run plenty of tap water through it. Distilled water should not be used, as it is generally contaminated with grease derived from the red lead used in making joints, or the oil used in lubricating taps, of the distilling apparatus. The salts found in tap water produce no appreciable effect on the surface tension of the water. The piece of thermometer tubing is passed through corks which close the ends of a wider tube (Fig. 129); a paper scale, graduated in millimetres, is placed within the outer tube and behind the thermometer tube. The upper end of the thermometer tube is then passed through a cork which fits the mouth of a wide boiling tube. Well wash the inside of the boiling tube with running tap water (p. 291), leave some tap water in it, and fit the thermometer tube in place in the manner indicated in Fig. 129. A small glass tube passes through the cork which closes the mouth of the boiling tube; by pinching a piece of rubber tube connected to this glass tube, the water may be caused to rise and wet the inside of the thermometer tube; this must be done before each observation is made.

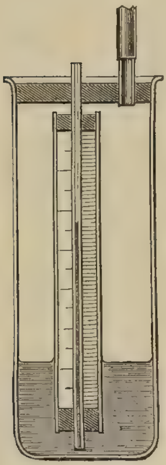

FIG. 129.—Apparatus for the determination of the surface tension, and its temperature coefficient, for water and other liquids.

Place the boiling tube in a large beaker containing hot water, or better still, in a thermostat. Observe the height h to which the water rises in the thermometer tube, measured from the surface of the water in the boiling tube; repeat the observation at different temperatures.

The water wets the inside of the thermometer tube; hence, where its surface comes in contact with the inside of the tube, it forms part of a cylinder of radius r, where r is the internal radius of the tube. The circumference of this cylinder is equal to $2\pi r$, and a force S acts vertically upwards across each centimetre length of this circumference; thus the resultant upward force due to the surface tension is equal to $2\pi r\mathrm{S}$.

The mass of the liquid column supported by the surface is equal to $\pi r^2 \times h \times \rho$, where ρ is the density of the liquid (for water $\rho = 1$, nearly); hence the downward force exerted by gravity on the column is equal to $\pi r^2 h\rho g$, and since this force is in equilibrium with the upward force due to the surface tension, we have—

$$2\pi r\mathrm{S} = \pi r^2 h\rho g;$$

$$\therefore \quad \mathrm{S} = \frac{hr\rho g}{2}.$$

The internal diameter of the thermometer tube should be measured by the aid of a travelling microscope.

Obtain the value of the surface tension of water for a number of different temperatures between 0° and 100° C, plot these values against the corresponding temperatures, and determine the temperature coefficient; that is, the diminution in the value of the surface tension for each degree rise of temperature.

Perform a similar set of observations for benzene.

Contamination of the surface of a liquid.

EXPT. 37.—Scatter some small parings of camphor over a water surface; if the surface is uncontaminated, the fragments of camphor rotate and move from place to place in a manner that suggests a wild unceasing dance. On touching the surface with a finger that has been slightly greased by contact with the hair of the head, the motion of the camphor particles suddenly ceases.

Camphor slowly dissolves in water, and in doing so produces a diminution in the surface tension. The camphor dissolves more quickly at salient points than elsewhere; thus, the surface tension of the water is diminished greatly near to salient points of a piece of camphor, and the water pulls the camphor away from the point at which the surface tension has been diminished most. The effect is thus similar to that which would be produced if a salient point of the camphor repelled the water in its neighbourhood. A thin film of grease diminishes the surface tension of the water to such an extent that the solution of the camphor can produce no further diminution.

Lord Rayleigh has measured the thickness of an oil film which will just stop the motion of camphor particles on water; he allowed a weighed drop of oil to spread over a surface of known area, and found that the motion of camphor is stopped when the oil film has a thickness of two one-millionths of a millimetre (2 $\mu.\mu.$). In these circumstances, the surface tension of water is 28 per cent. smaller than if its surface were uncontaminated. Lord Rayleigh also found that a layer of oil, one one-millionth of a millimetre (1 $\mu.\mu.$) or less in thickness, produces practically no effect on the surface tension of water. As a layer of oil increases in thickness from 2 $\mu.\mu.$ upwards the surface tension diminishes, but the change is much more gradual than that which occurs when the thickness of the film is increased from 1 $\mu.\mu.$ to 2 $\mu.\mu.$

Plateau found that, when a magnetised needle is suspended so that it lies on the surface of water, its oscillations subside much more rapidly than when the needle is wholly immersed in the water. Marangoni explained this, by assuming that the surface of the water is contaminated slightly with grease; as the needle moves over the surface, the grease is concentrated in front of it, and a comparatively pure water surface is left behind it. Thus, there will be a backward pull due to the greater surface tension of the pure water, and this pull tends to bring the needle to rest.

In a storm at sea, the waves may be calmed to a great extent by pouring oil on the surface of the water. When the wind acts on a portion of the oily surface, it urges the oil forward and leaves a comparatively pure water surface behind; hence, there is a backward pull on the surface set in motion by the wind, and this pull tends to stop the motion. Even when waves have been formed, they will die down as they pass over that part of the surface which has been greased; for the wave motion entails stretching of the surface layer of the water, and at any instant the stretching will be greater at some parts of the wave than at others; a pull will be exerted on the less stretched by the more stretched portions of the surface, and hence the energy of the wave will be used up in producing motion in the surface layer of the water.

The surface tension of a liquid depends, to a considerable extent, on the gas in contact with its surface. Thus, according to Ramsay and Shields, the surface tension of water in contact only with its own vapour is equal to 73·21 dynes per cm. at 0° C., and to 70·20 dynes per cm. at 20° C. The surface tension of water in contact with air is equal to 75·8 dynes per cm. at 0° C. Stockle has found that the surface tension of mercury, in contact only with its own vapour, is equal to 435·6 dynes per cm. at 15° C. If the surface of mercury is exposed to air, the value of the surface tension is increased, but the increase becomes less as the time of exposure is extended.

Some phenomena due to surface tension.—When mercury is spilt on a table it forms beads which, when small, are practically spherical, while the larger beads are flattened more or less. Mercury does not "wet" the table; that is, its molecules do not attract the molecules of the wood appreciably. Therefore, the shape assumed by a bead of mercury is due, partly to the gravitational attraction exerted on each of its molecules, and partly to the elastic properties of its surface. When the bead is small, the elastic surface assumes a practically spherical shape; for a sphere has a smaller surface than any other solid of equal volume. If water is spilt on a table which has been dusted with lycopodium powder, it forms beads somewhat similar in shape to those formed by mercury. The surface of the water can be seen to be coated with a layer of lycopodium powder, which prevents the water molecules from coming within the range of molecular attraction of the molecules of the wood. Similar beads of water are formed by the first few drops of rain which fall on a dusty road.

When a camel-hair brush is dipped into water, the hairs remain apart while the brush is immersed, but they cling together when the brush is removed from the water. Some water remains in the spaces between the hairs of the brush, and its free surface tends to contract, and so pulls the hairs together.

If a sieve, made from wire gauze of about twenty meshes to the inch, is dipped into melted paraffin-wax, and is then shaken so that the holes are left clear, the wires when cold will be coated with solid paraffin-wax. Water does not wet paraffin-wax; if a piece of paper is placed on the bottom of the sieve, water can be poured in, and afterwards the paper can be removed: the water does not flow through, because its surface stretches across each hole, and is strong enough to support the water above it.

Those who have had experience of camping, may have remarked that, during a shower of rain, a tent leaks at any point which is touched internally by the finger. If the inside of the canvas is examined, it will be seen that there are numberless small water surfaces projecting into the tent; the water does not wet the threads of the canvas, and thus the water surfaces remain isolated until they are connected by a touch of the finger, and then a large drop is formed, with the result that the surface tension is no longer strong enough to support the water that collects.

"Rain-proof" fabrics are made in the following manner. The fabric, say a cloth for overcoats, is dipped into a very dilute solution of resin in petroleum. During the drying of the cloth, the petroleum evaporates ; afterwards, the cloth is pressed between hot metal rollers, with the result that the resin is melted ; on cooling, a thin film of resin is left on each fibre of the wool. Now, water does not wet resin ; thus rain-drops falling on the coat assume the form of small beads, which roll off without penetrating the cloth. A basin made of this cloth may be filled with water, and none will pass through, for the reason explained above in connection with the paraffin-coated sieve.

It has been found that yellow fever is disseminated by mosquitoes. The mosquito larvæ live in stagnant water ; each possesses a tubular tail-like appendage by means of which it breathes. At the free end of this appendage there are three hairs, which are crossed above the opening when the larva is swimming about, thus preventing water from entering the respiratory ducts. When the larva wishes to breathe, it thrusts the end of its tubular appendage through the surface of the water, bends back the hair-like processes, and hangs suspended from the surface of the water with the mouth of the tube open to the air. Thus, the existence of the mosquito larva, and therefore the existence of the mosquito itself, depends on the strength of the water surface. A little oil poured on every pond or tract of stagnant water in a neighbourhood, serves to exterminate the mosquito larvæ, and so to prevent the dissemination of yellow fever.

EXPT. 38.—Coat an ordinary sewing needle with a trace of vaseline, place it on a square of blotting paper, and float this on water. The paper becomes wet and sinks, while the needle remains floating. It is supported by the elastic surface of the water.

Spreading of a liquid over the surface of a solid or another liquid.—When a drop of liquid is placed on the horizontal surface of a solid, the form it ultimately assumes will depend on a number of circumstances. The more the drop spreads, the lower will its centre of gravity descend, and the less will be its gravitational energy. But as the drop spreads, its surface increases in area, and therefore work must be done in enlarging the surface. To determine the condition of equilibrium of the drop, we must remember that **the decrease in gravitational energy due to an infinitesimal additional spreading, must be equal to the work done in producing the corresponding enlargement of the surface.** For, if the decrease of gravitational energy is greater than the work done in enlarging the surface, the liquid must gain a certain amount of kinetic energy (compare p. 38) and therefore the spreading will continue; on the other hand, if the gravitational energy lost is less than the work performed in enlarging the surface, work must be done by some external agent, and therefore the drop could not be in equilibrium without the application of some external force.

In the first instance, let it be supposed that no attraction is exerted between the molecules of the solid and those of the liquid. In this case, the molecules of the liquid which are in contact with the solid are in the same condition as if the solid were absent; that is, the surface dividing the liquid from the solid has the same tension as the free surface of the liquid. The surface of the liquid must be parallel to that of the solid where the two come into contact; for, if the liquid and solid surfaces were to meet at an angle, the liquid surface would exert a pull normal to the solid surface, and this pull could not be sustained in the absence of attraction between the molecules of the liquid and those of the solid. If the drop is small, it will assume an approximately spherical form; in this case the gravitational energy is very small, and equilibrium will be attained when the surface has the smallest possible area. If the drop is large, it will assume a form somewhat like that of a tea-cake (Fig. 130).

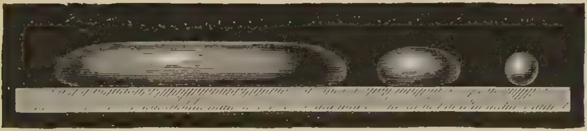

FIG. 130.—Drops of liquid resting on a surface which they do not wet.

Next, let it be supposed that a finite attraction is exerted by the molecules of the solid on those of the liquid. The free surface of the liquid may now meet that of the solid at a finite angle θ. Let S_2 denote the surface tension of the liquid ; then S_2 ergs of work must be performed in increasing the free surface of the liquid by one square centimetre. With regard to the free surface of the solid, there is no difficulty in seeing that this may be the seat of energy ; for a surface molecule is pulled toward the interior of the solid by a finite force, and therefore the surface of the solid is strained ; let it be assumed that S_1 ergs of work could be performed by each square centimetre of the solid surface, during the relief of this strain. When the liquid comes into contact with the solid, the surface of the solid will be partly relieved of its strain by the attraction of the liquid molecules ; let the remaining work which could be obtained from each square centimetre of the solid surface be denoted by S'_1. Let S'_2 ergs of work be obtainable from the molecules comprised in one square centimetre of the liquid surface where it is in contact with the solid. Then the total work obtainable from a square centimetre of the solid-liquid interface is equal to $S'_1+S'_2$; let this be denoted by S_{12}.

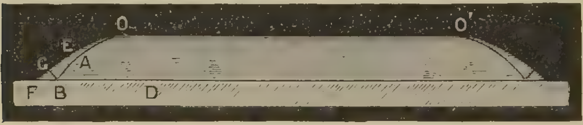

FIG. 131.—Drop of liquid resting on a surface which it wets.

Now let it be supposed that a drop of liquid rests on a solid, the free surface AB (Fig. 131) of the liquid meeting the solid-liquid interface at an angle $ABD=\theta$. If the liquid spreads infinitesimally, so that its new surface acquires the section OEF, the angle EFB will be but slightly less than the angle ABD. Let any small surface element, of area a, be chosen on the surface OAB ; if normals be drawn to the element around its boundary, these normals will cut the surface OEF in a curve which bounds a small surface element ; and it may be supposed that, during the spreading, the surface element of OAB moves normally outwards until it forms the corresponding surface element of OEF. Since the surface OAB is convex outwards, the element of OEF will be slightly larger in area than the corresponding element of OAB, and work must be done in increasing the area of the surface element. If the surface element of OAB is at a distance h below the flat upper surface OO′ of the liquid, the pressure tending to push the element outwards is equal to $g\rho h$, where ρ is the density of the liquid (p. 34); and the force tending to push the element outwards is equal to $g\rho h a$. Since the liquid is in equilibrium, the elasticity of its surface must pull the element back with a force equal to $g\rho h a$; and if the element moves through a distance δ during the incipient spreading of the drop, the work done against the elastic surface force must be equal to $g\rho h a\delta$, and this is the value of the work done in enlarging the surface element. But $a\delta$ is the volume of the liquid which must have flowed into the small space swept out by the surface element, and $g\rho a\delta h$ is the gravitational energy that would be lost by a volume $a\delta$ of liquid, if it were removed from the surface OO′ and taken to a distance h below that surface (compare p. 34). Thus, **the work done in enlarging each surface element of OAB, as it moves normally outwards to form an element of the new surface OEF, is just equal to the gravitational energy lost by the liquid that flows into the space swept out by the element.** From B draw BG perpendicular to GF; then the work done in converting the original boundary surface OAB, into the part OEG of the final boundary surface, is just equal to the gravitational energy lost by the liquid that flows into the space BAOEG.

During the spreading, a strip of the solid, of width FB and area a (say), is covered up; the work done in enlarging the solid-liquid interface by a, is aS_{12}; and the mechanical energy gained by diminishing the area of the solid by a, is aS_1. Also, since $GF = FB \cos\theta$, it follows that the free surface of the liquid is enlarged by $a \cos\theta$, and therefore the work done during this enlargement is $a \cos\theta \times S_2$. Thus—

$a(S_2 \cos\theta + S_{12} - S_1) =$ *gravitational energy lost by the liquid which flows into the space* BGF.

The gravitational energy lost is equal to the work done by gravity if the space BGF were filled with liquid removed from the surface OO′. The area of the cross section BGF is equal to $(1/2)BG \times GF = \frac{1}{2}(FB \sin\theta)(FB \cos\theta)$, and $a = FB \times l$, where l is the peripheral length of the base of the drop. Thus the space which has the section BGF, and extends all round the drop, has a volume equal to $(1/2)(a^2/l)\sin\theta\cos\theta$. In filling his space with the liquid, the gravitational energy lost is equal to $(1/2)g\rho h(a^2/l)\sin\theta\cos\theta$. Thus—

$$a(S_2\cos\theta + S_{12} - S_1) = \tfrac{1}{2}g\rho h(a^2/l)\sin\theta\cos\theta.$$

$$\therefore\quad S_2\cos\theta + S_{12} - S_1 = \tfrac{1}{2}g\rho h(a/l)\sin\theta\cos\theta.$$

The term on the right hand side of the last equation can be made as small as we please, by making a sufficiently small; while the left-hand side of the equation is independent of the value given to a. Thus, when the spreading is infinitesimal, and a is therefore infinitely small—

$$S_2\cos\theta + S_{12} - S_1 = 0$$

$$\therefore\quad \cos\theta = \frac{S_1 - S_{12}}{S_2} \quad \dots\dots \quad (1)$$

$$= \frac{S_1 - (S'_1 + S'_2)}{S_2}.$$

If there is no attraction between the molecules of the liquid and those of the solid, $S'_1 = S_1$, and $S'_2 = S_2$; in this case:—

$$\cos\theta = \frac{S_1 - (S_1 + S_2)}{S_2} = -1,$$

and $\theta = 180$. Hence the form of the drop is that represented in Fig. 130.

In the case of mercury in contact with glass, experiment shows that θ is equal to 140°, which indicates that $S_1 - S_{12}$ has a negative value numerically less than S_2. In this case—

$$(S'_1 + S'_2 - S_1) < S_2;$$

$$\therefore\quad (S_1 - S'_1 + S_2 - S'_2) > 0.$$

Hence, when a free mercury surface is brought into contact with glass, a certain amount of mechanical energy is liberated. Consequently, there is some attraction between the mercury and the glass molecules; but the surface tension of pure mercury is very great (450 dynes per cm.); and therefore the attraction between two mercury molecules is very great in comparison with the attraction between a mercury and a glass molecule.

In the case of water and most other liquids in contact with glass, θ is less than $\pi/2$, and therefore—

$$(S_1 - S'_1 - S'_2) < S_2;$$

$$\therefore\quad (S_1 - S'_1) < (S_2 + S'_2).$$

Thus, when a water surface is brought into contact with glass, the mechanical energy given up during the relief of the strain in the glass surface, is less than twice the mean of the values S_2 and S'_2.

If $(S_1 - S_{12})/S_2$ is greater than $(+1)$ or less than (-1), there is no angle for which equation (1) is satisfied. In this case the liquid cannot form a drop on the surface of the solid, but spreads completely over the latter.

When a drop of one liquid is placed on the surface of another liquid, similar conditions must be complied with. In this case, S_2 denotes the surface tension of the drop, S_1 denotes the surface tension of the liquid on which the drop is placed, and S_{12} denotes the surface tension of the interface between the two liquids. For the drop to be in equilibrium, there must be a definite angle of contact between the two liquids; that is, $(S_1 - S_{12})/S_2$ must be less than $(+1)$ and greater than (-1).

$$\therefore\ (S_1 - S_{12}) < S_2, \text{ and } \therefore\ (S_{12} + S_2) > S_1.$$

Also—

$$(S_1 - S_{12}) > (-S_2), \text{ and } \therefore\ (S_1 + S_2) > S_{12}.$$

Thus **the sum of any two of the quantities S_1, S_2, and S_{12}, must be greater than the third quantity.** Hence, it must be possible to construct a triangle of which the sides are equal to S_1, S_2, and S_{12}. This triangle is called **Neumann's triangle**; if it cannot be constructed (that is, if one of the three quantities S_1, S_2, and S_{12}, is greater than the sum of the other two), then it is impossible for a drop of one of the liquids to be formed on the surface of the other.

It has been found that for no two pure liquids can Neumann's triangle be constructed; hence, a drop of a pure liquid spreads completely over the surface of any other pure liquid on which it may be placed. For instance, a drop of pure water, when placed on the surface of pure mercury, spreads over the entire surface. If, however, the mercury surface is greasy, a drop of water can collect on it; for the grease diminishes the surface tension of the water, and so renders possible the construction of Neumann's triangle. Similarly, a small trace of oil, when placed on pure water, spreads over the entire surface; but if more oil is placed on the surface, it may collect in the form of a thin disc, owing to the fact that the first trace of oil has diminished the tension of the water so much that Neumann's triangle may be constructed. This explains the formation of discs of grease on the surface of broth.

Expt. 39.—To determine the surface tension of water in contact with oil.

Adjust the framework of platinum wire represented in Fig. 126, so that it dips into water in the manner explained in connection with expt. 31, p. 288; then cover the surface of the water with oil, to such a height that the oil extends up to the supporting filament. On immersing the rectangular framework in the water, and then carefully withdrawing it, a film of water is formed with its surfaces in contact with the oil. The method of measuring the surface tension of the oil-water interface is similar to that explained in connection with expt. 31.

Expt. 40.—To determine the angle of contact of mercury with glass.

Clean and dry the inside of a small spherical flask of about 4 or 5 cm. diameter. Pour clean mercury into the flask until it is nearly full, and then close the mouth of the flask with a rubber stopper through which a thick piece of glass rod has been pushed. Invert the flask (Fig. 132), and adjust the glass rod so that the surface of the mercury is plane, up to the line in which it meets the glass: this adjustment can be made with great accuracy by observing the image of a sheet of printed paper reflected at grazing incidence in the surface; any distortion or "fuzziness" in the image indicates that the surface is not plane. Measure the diameter d of the circle of contact of the mercury and glass by the aid of a pair of dividers and a centimetre scale; then, if θ is the angle which the free mercury surface makes with the mercury-glass interface—

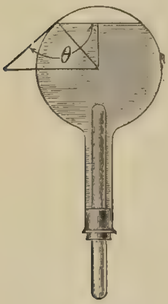

Fig. 132.—Angle of contact of mercury with glass.

$$\frac{d}{2} = r\cos\left(\theta - \frac{\pi}{2}\right),$$

where r is the radius of the spherical flask.

Expt. 41.—To determine the angle of contact between pure water and clean glass.

Fix a cork, by means of marine glue, to one side of a piece of plate glass about two inches square. Make a mop, by pushing part of a small bundle of cotton wool into one end of a piece of glass tube. Hold the plate glass by the cork, and scrub the free surface of the glass with soap applied to the mop. Well wash the surface with running tap water; then, by means of a fresh mop, scrub the surface with strong nitric acid.

If the surface is once more washed with running tap water, and is then held so that its plane is vertical, the water will be seen to form a uniform film over the surface; this film becomes thinner and thinner, until at last the colours of thin films appear. The waer shows no tendency to collect into drops; when such drops are formed, either the water or the glass surface is greasy. This experiment shows that $(S_1 - S_{12})/S_2$ is either equal to, or greater than, unity.

Relation between the surface tension and the dimensions of a drop of mercury supported on a solid.—Let a drop of mercury be supported on a clean surface, say that of a piece of plate glass. Let the drop be so large that the central part of its upper surface is practically plane: this can be tested by observing the image of a piece of printed paper reflected in the surface. Let an imaginary vertical plane divide the drop into halves at the section AB (Fig. 133); and let a slice be cut from one half of the drop by two parallel vertical planes, perpendicular to the plane through AB, and at a distance δ apart; this slice is shown isolated in the lower part of Fig. 133. Next, let a horizontal plane HKL be drawn through the point H at which the surface is vertical.

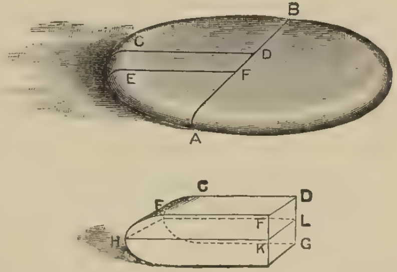

FIG. 133.—Large drop of mercury supported on a horizontal plane surface. (A slice cut from the drop is shown below.)

The upper portion CDLKH of the slice must be in equilibrium under the action of the forces acting on it. Let $FK = h$; then the liquid to the right of the vertical planes FL exerts a hydrostatic pressure, varying from zero at F to $g\rho h$ at K. Hence, the average pressure exerted on the plane FL is $g\rho h/2$; and since the area of the plane is $h\delta$, the total force acting on it, from right to left, is equal to $g\rho h^2\delta/2$. The surface to the right of the section FD exerts a force S, acting from left to right; and since no other forces act parallel to HK on the portion CDLKH of the slice—

$$S\delta = \frac{g\rho h^2\delta}{2};$$

$$\therefore\ S = \frac{g\rho h^2}{2}.$$

This equation suffices to determine the surface tension S, in terms of the distance h and the density of the mercury.

The angle of contact between the mercury and glass can be determined from the following considerations. Let the total height GD of the drop be denoted by H. Then the force, due to hydrostatic pressure, acting from right to left on the plane FG, is equal to $g\rho H^2\delta/2$. If the free surface of the mercury makes an angle α with the free surface of the glass where the two meet, it is clear that $\theta = \pi - \alpha$, where θ is the angle of contact of the mercury and glass (p. 301). Now, the free surface of the mercury exerts a horizontal force $S\delta \cos \alpha$ on the surface of the glass, and an equal but opposite force is exerted by the glass on the mercury surface; therefore the surface forces, acting from left to right on the slice CDGH, are together equal to $S\delta(1 + \cos \alpha)$, and

$$S\delta(1 + \cos \alpha) = \frac{g\rho H^2 \delta}{2};$$

$$\therefore \qquad \cos \alpha = \frac{g\rho H^2}{2S} - 1.$$

EXPT. 42.—Determine the surface tension, and the angle of contact of mercury with glass, from observations made on a drop of mercury supported on a glass plate.

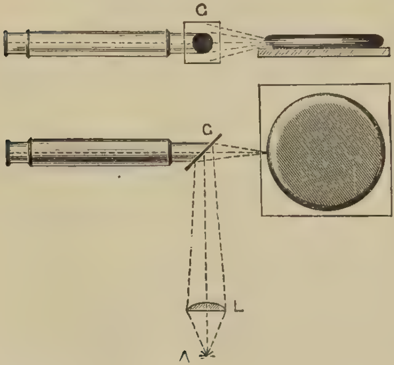

FIG. 134.—Determination of the surface tension of a drop of mercury. (Side view of experimental arrangement above, plan below.)

A piece of plate glass must be supported on a small table provided with levelling screws; when its surface is level, a drop of mercury about two inches in diameter may be formed on it. The thickness H of the drop may be measured by the aid of a spherometer. The value of h must be determined by the aid of a microscope, mounted so that its axis remains horizontal while it is raised or lowered by means of a screw provided with a graduated head. The chief difficulty is to determine the exact height at which the surface of the drop is vertical; the usual method is to observe the profile of the drop, but by this means different observers of equal skill often obtain discrepant values of h. The following method can be used with advantage. A cap, which fits over the nozzle of the microscope, is cut away at an angle of 45°, and to this cap a piece of thin plate glass G (Fig. 134) is cemented. The axis of the microscope is directed toward the middle of the drop. A source of light A (such as a four-volt Osram lamp) is placed in the horizontal plane which passes through the axis of the microscope, and a lens L is used to focus the rays from A so that, after being reflected at an angle of 45° from G, they form an image of A on the side of the drop. On looking through the microscope, a thin horizontal luminous line is seen; this is formed by the rays which are reflected from that part of the surface of the drop which is vertical, and on adjusting the microscope so that the luminous line is focussed on the horizontal cross wire in the eye-piece, we may be certain that the axis of the microscope is directed toward the vertical part of the surface of the drop. The microscope may then be raised, and at the same time advanced toward the drop, until the image of a few particles of lycopodium, scattered on the horizontal part of the upper surface of the drop, are focussed on the horizontal cross wire.

If a drop of mercury is formed on a piece of plate glass which has been dusted with lycopodium, in the manner described in connection with expt. 43, the mercury does not come into contact with the glass, and therefore $\alpha = 0$. Let the thickness H_1 of the drop be measured by the aid of a spherometer, then—

$$2S = \frac{g\rho H_1^2}{2}.$$

Afterwards the thickness H of a drop formed on clean glass can be measured; and the value of α can be determined from the equation—

$$S(1 + \cos\alpha) = \frac{g\rho H^2}{2}.$$

EXPT. 43.—To determine the surface tension of a drop of water.

Warm a piece of plate glass, and coat it with paraffin wax; keep the glass warm, and allow the wax to drain off until the layer left on the glass is very thin. Then, before the wax solidifies, dust it with lycopodium powder. Place the glass on a table provided with levelling screws, and form a large drop of water on the surface which has been dusted. Measure H by the aid of a spherometer. In this case $\alpha = 0$ (p. 308), therefore, since $\cos\alpha = 1$, and $\rho = 1$, it follows that

$$S \times 2 = \frac{gH^2}{2};$$

$$\therefore \quad S = \frac{gH^2}{4}.$$

EXPT. 44.—To determine the surface tension, and the angle of contact of a liquid that wets glass.

For this experiment, a cell with parallel plane glass faces at front and back is required; the cell should also be provided with levelling screws. A lens about three inches in diameter, with a concave surface of about ·100 to 200 cm. radius, is supported in the cell with the concave surface downwards; the lens may rest on three screws projecting upwards from a brass plate which lies on the bottom of the cell (Fig. 135). Fill the cell with the liquid to be tested, and blow a large bubble of air underneath the lens; the bubble, which should extend as nearly as possible to the edge of the lens, can be blown by the aid of a piece of bent capillary tube connected to the pressure bottle described on p. 316. If the liquid used is water, the inside of the cell, and the surfaces of the lens, must be cleaned carefully with soap and water, and then well rinsed with running tap water, before the experiment is commenced.

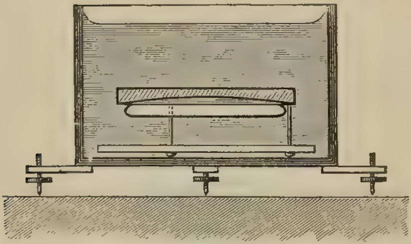

FIG 135.—Bubble of air formed beneath the concave surface of a lens immersed in a liquid.

Let h be the distance between that part of the edge of the bubble where the surface is vertical, and the lower surface of the bubble; this distance can be measured in the manner described above, in connection with the mercury drop. The surface of the bubble comes into contact with the concave surface of the lens along a horizontal circle; let H be the distance from the plane containing this circle to the lower surface of the bubble: H can be measured by the aid of the travelling microscope without difficulty.

Let it be imagined that a slice (similar, if it were inverted, to that represented in Fig. 133) is cut from the bubble. The pressure of the air inside the bubble is uniform, and is equal to that of the liquid in contact with the lower surface of the bubble. Hence, if the lower surface of the bubble is at a distance D below the free surface of the liquid, and if P is the atmospheric pressure, in dynes per sq. cm., then the pressure of the air inside the bubble is equal to $(P + g\rho D)$. The plane surface corresponding to FL (Fig. 133) will be acted upon by a force equal to $(P + g\rho D)h\delta$. The pressure on the external surface of the bubble varies with the depth; on the portion below the horizontal plane that cuts the edge of the bubble where it is vertical, the pressure varies from $(P + g\rho D)$ to $\{P + g\rho(D - h)\}$, and therefore its average value is equal to $\left\{P + g\rho\left(D - \frac{h}{2}\right)\right\}$. Thus the portion of the slice corresponding to CDLKH is acted upon by opposite horizontal forces, respectively equal to $(P + g\rho D)h\delta$ and $\left\{P + g\rho\left(D - \frac{h}{2}\right)\right\}h\delta$, and the resultant of these forces is equal to $g\rho h^2\delta/2$. Then, as before—

$$S\delta = g\rho h^2\delta/2,$$

and
$$S = \frac{g\rho h^2}{2}.$$

The liquid intersects the glass surface at an angle θ, equal to the angle of contact of the liquid with glass. Reasoning similar to that employed on p. 308 shows that the surface of the bubble is subjected to a component force equal to $S \cos \theta$ per cm. in a direction parallel to the glass surface; hence, if the latter is inclined at an angle ϕ to the horizontal along the circle of contact of the liquid, the horizontal force is equal to $S \cos \theta \cos \phi$ per cm. Then it can be proved without difficulty that—

$$S(1 + \cos\theta \, . \, \cos\phi) = \frac{g\rho H^2}{2}.$$

When the radius of curvature of the glass surface is large, ϕ will be small, and $\cos \phi$ will be practically equal to unity. Hence we arrive at the somewhat surprising result, that **the thickness H of a flat bubble of gas, situated below the surface of a liquid, is independent of the depth below the surface.** An increase of pressure would merely compress the bubble laterally until it became approximately spherical, and then a further increase of pressure would diminish its dimension in all directions.

Let d be the diameter of the circle of contact of the liquid with the glass; then $(d/2) = R \sin \phi$, where R is the radius of curvature of the concave glass surface. This equation suffices to determine ϕ.

Conditions of equilibrium of a liquid in a capillary tube.—Let a tube, of small radius r, be placed in a vertical position with its lower end immersed in a liquid of density ρ;

and let the liquid rise in the tube until the lowest point of its curved surface is at a height h above the plane surface of the liquid outside the tube (Fig. 136). The liquid inside the tube is in equilibrium ; hence, no work will be done in raising its surface by an infinitesimal amount (p. 38). Let an imaginary plane, drawn horizontally through the lowest point of the curved surface within the tube, have a volume v of water above it. Then, if the surface within the tube is raised through a very small distance δ, we may suppose this to be done by raising the volume v through a distance δ, and then inserting, between it and the liquid in the lower part of the tube, a volume $\pi r^2\delta$ of the liquid obtained from the level of the plane surface of the liquid outside the tube.

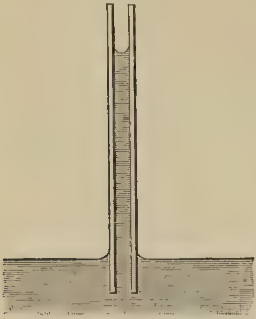

Fig. 136.—Rise of a liquid in a capillary tube.

In raising the volume v through the distance δ, the work done against gravity is equal to $\rho v g\delta$. The volume $\pi r^2\delta$ is raised through a distance h against the force of gravity, so that the work done is equal to $\rho\pi r^2 g\delta h$. In raising the surface through the distance δ, a strip of the internal surface of the glass, of area equal to $2\pi r\delta$, is covered up by the liquid ; and in performing this operation the work done is equal to $2\pi r\delta\,(S_{12} - S_1)$, (p. 303). Hence, for the total work done to be zero,

$$2\pi r\delta(S_{12} - S_1) + \rho g\delta(v + \pi r^2 h) = 0.$$

But $(S_1 - S_{12})/S_2 = \cos\theta$, where S_2 is the surface tension of the liquid, and θ is the angle of contact with glass (p. 304)

$$\therefore\quad -2\pi r S_2 \cos\theta + \rho g(v + \pi r^2 h) = 0.$$

If the liquid wets the inside of the tube, $\theta = 0$. In this case, if the bore of the tube is very small, we may assume that the curved surface is practically a hemisphere of radius r, and therefore v is the volume contained between the walls of a cylinder of length r and radius r, and a hemisphere of radius r. Thus—

$$v = \pi r^2 r - \tfrac{2}{3}\pi r^3 = \frac{\pi r^3}{3}.$$

Thus
$$2\pi r S_2 = \rho g\left(\frac{\pi r^3}{3} + \pi r^2 h\right),$$
$$\therefore\quad S_2 = \rho g\left(\frac{rh}{2} + \frac{r^2}{6}\right).$$

When r is so small that r^2 may be neglected, this result agrees with that obtained by a simple train of reasoning on p. 296.

If the liquid is mercury, then $\theta = 140°$, and $\cos\theta = -0{\cdot}766$. In this case—
$$2\pi r S_2 \times 0{\cdot}766 = -\rho g(v + \pi r^2 h),$$
$$\therefore\quad S_2 = -\frac{\rho g}{1{\cdot}532\pi r}\cdot(\pi r^2 h + v),$$
or, neglecting v—
$$S_2 = -\frac{\rho g r h}{1{\cdot}532}.$$

Now, S_2, the surface tension of the mercury, is essentially positive; and for the right-hand side of the equation just obtained to be positive, h must be negative; that is, the surface of the mercury in the capillary tube must be lower than the surface outside the tube. This phenomenon can be observed without difficulty by immersing the end of a capillary tube in mercury.

If the inside of a capillary tube is coated with paraffin wax, and the lower end is immersed in water, it will be seen that the water surface inside the tube is lower than that outside the tube.

If mercury be poured into a U tube, one limb of which is wide while the other is of capillary bore, it comes to rest in the position represented in Fig. 137. This method does not suffice to determine the surface tension of mercury, unless the angle of contact of mercury and glass has been determined previously. Water behaves in a similar manner when the inside of the tube has been coated with paraffin wax.

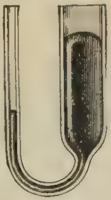
FIG. 137.—Mercury in a capillary tube.

Pressure in the space enclosed by a spherical liquid surface.—If the surface of a liquid is spherical, like that of a small bubble of air in water, or a small bead of mercury, the elastic properties of the surface tend to make it contract; in order that the surface may be in equilibrium, its tendency to contract must be neutralised by an internal pressure, which of itself would produce an expansion. Thus a soap bubble, blown on one end of a tube, will contract and expel the enclosed air if the other end of the tube be left open.

Let the radius of a spherical surface be R, and let the surface be cut into halves by an imaginary plane passing through the centre of the sphere (Fig. 138). Within the surface, let the fluid (gaseous or liquid, as the case may be) exert a pressure of P dynes per square centimetre; then one hemisphere of the fluid exerts a force of $P\pi R^2$ dynes on the other hemisphere; that is, the two hemispheres are urged apart by a force equal to $P\pi R^2$ dynes. Also, they are pulled towards each other by the elastic surface; each half of the surface exerts a force equal to $2\pi RS$ dynes on the other half, since the imaginary plane cuts the surface in a circle of radius R, and a tension of S dynes is exerted across each centimetre length of the circumference of this circle. Thus—

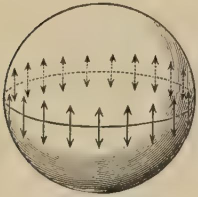

FIG. 138.—Equilibrium of a spherical liquid surface.

$$2\pi RS = P\pi R^2,$$

$$\therefore \quad P = \frac{2S}{R}.$$

Thus, the pressure is inversely proportional to the radius of the spherical surface.

The same result may be obtained by another method. Let a cone of small vertical angle be drawn from the centre of the sphere, and let this cone cut off a small area a from the spherical surface. An outward force, parallel to the axis of the cone, and equal to Pa, acts on this area. Let the spherical surface expand by an infinitesimal amount, so that its radius increases from R to $(R+\delta)$; then the base of the cone increases in area from a to $\{a(R+\delta)^2/R^2\}$, since the area of the base of a cone is proportional to the square of its altitude; thus, the increase of area is equal to—

$$a\left(\frac{R+\delta}{R}\right)^2 - a = a\left\{\left(1+\frac{\delta}{R}\right)^2 - 1\right\} = \frac{2a\delta}{R},$$

when δ is so small that its square may be neglected. Now, the work done in producing this increase of area is equal to $2Sa\delta/R$ (p. 287), and the work done by the force Pa acting through the distance δ is equal to $Pa\delta$. In order that the surface and its contents may be in equilibrium, the work done by the pressure must just suffice to increase the area of the element on which the pressure acts ; therefore—

$$Pa\delta = \frac{2Sa\delta}{R},$$

$$\therefore \; P = \frac{2S}{R}.$$

It should be realised that a soap-bubble possesses two surfaces, each of which exerts a pressure equal 2S/R, so that the pressure within the soap-bubble is equal to 4S/R.

A liquid boils when bubbles of vapour, formed in its interior, can rise through it and escape at its surface. Let R be the radius of a spherical bubble of vapour, at a distance D below the surface ; if p denotes the atmospheric pressure (in dynes per sq. cm.) exerted on the free surface of the liquid, and ρ denotes the density of the liquid, then the pressure P of the vapour within the bubble is given by the equation —

$$P = p + g\rho D + \frac{2S}{R}.$$

If the pressure of the vapour within the bubble were less than this value, the bubble would collapse. Now, by making R small enough, we can make P as great as we please ; hence, for very small bubbles of vapour to be formed within a liquid, the pressure of the vapour must be very great, and therefore the temperature of the liquid must be extremely high—much higher than would be necessary to maintain large bubbles in equilibrium.

If small pieces of capillary tube, sealed at one end, are immersed (while filled with air) in a liquid, the vapour can escape into their interiors, and form fairly large bubbles at their open ends, so that the temperature of the liquid need not rise perceptibly above its boiling point. Any porous substance, such as coke, acts in the same way as the capillary tubes. If a glass of aerated water is allowed to stand until effervescence has ceased, a piece of loaf sugar dropped into it will evoke a plenteous evolution of bubbles. When a liquid has no facilities for producing bubbles of vapour which are fairly large to start with, its temperature may rise far above the boiling point without the occurrence of boiling, and ultimately a large bubble of vapour may be formed, which throws the greater part of the liquid from its containing vessel. This phenomenon is known as **bumping.**

EXPT. 45.—To determine the surface tension of a liquid in terms of the pressure required to form a spherical bubble in its interior.

Let a tube of fine bore be placed in a vertical position with its lower end at a distance D below the surface of a liquid. The liquid rises in the tube, and its surface is approximately spherical (p. 312). If air is forced into the tube, the surface of the liquid is pressed downwards; and as the pressure of the air increases, the surface sinks lower and lower, until finally a hemispherical bubble, of the same radius R as the orifice of the tube, protrudes into the liquid below. The pressure within the hemispherical bubble is greater than the atmospheric pressure by P, where—

$$P = g\rho D + \frac{2S}{R}.$$

At this stage the bubble becomes unstable, for a slight increase in its radius reduces the internal pressure necessary to produce equilibrium, and therefore if the pressure P remains constant the bubble breaks away.

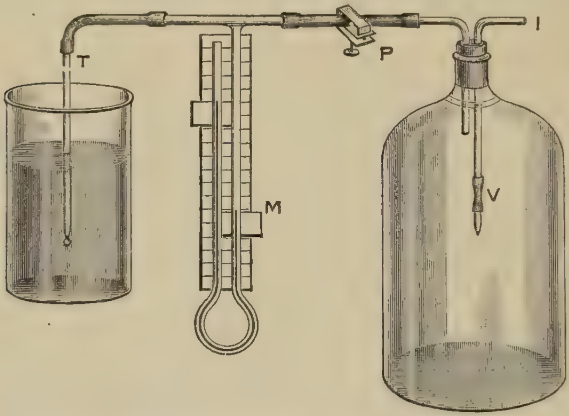

FIG. 139.—Experimental arrangement, for the determination of the surface tension of a liquid by blowing bubbles of air in it.

Draw out a piece of glass connection tubing until its internal radius is about equal to half a millimetre. Cut off the capillary tube close to the wide tube, and wind a single turn of fine wire around the tube at a distance of about 3 or 4 cm. from its capillary orifice. Clamp the tube in a vertical position, with its capillary orifice below the surface of a liquid in a beaker (Fig. 139). Adjust the tube until the wire twisted round it can just be seen on looking tangentially along the underside of the surface of the liquid; this adjustment determines the value of D, the distance of the orifice of the tube below the surface of the liquid. Blow air into the top of the tube from a pressure bottle F (Fig. 139) made by closing the mouth of a Winchester quart bottle with a cork, pierced to receive two glass tubes: one tube allows the air to leave the bottle, and the other (called the inlet tube) allows air to be forced into a bottle by a bicycle pump. The inlet tube is provided with a valve V, made by sealing the end of the tube, blowing a small hole in its side, and covering this with a short length of thin-walled rubber tubing. Between the pressure bottle, and the experimental tube from which the bubbles are blown, is a water manometer which indicates the pressure of the air in the bubble; when the water surfaces in the two limbs of the manometer differ in height by H, the value of P is equal to $g\rho$H. Adjust the flow of air by the aid of a screw clip P, until a bubble is formed every two seconds. The pressure indicated by the manometer rises until the bubble becomes unstable, and then suddenly falls; place pieces of paper with straight edges behind the limbs of the manometer, and adjust them until they mark the positions of the water surfaces just before the bubble bursts. The radius R of the capillary orifice of the tube must be measured by the aid of a travelling microscope.

Determine the value of the surface tension of water at a number of different temperatures below its boiling point. Perform a similar series of experiments with relation to turpentine. The surface tension of a liquid can be measured accurately by this method.

Pressure in the space enclosed by a cylindrical liquid surface.—Let the surface of a liquid form a circular cylinder of length l and radius R, and let the internal pressure, necessary to keep the surface in equilibrium, be P dynes per sq. cm. Divide the cylinder into halves by an imaginary plane through its axis; then the force tending to separate the halves of the cylinder is equal to $P \times l \times 2R$, and equilibrium is produced by the tension $2Sl$ which pulls the halves of the cylindrical surface towards each other.

$$\therefore\ 2Sl = 2Pl R,$$

$$\therefore\ P = \frac{S}{R}.$$

In comparing this result with that obtained for the pressure within a spherical surface, it must be noted that the cylindrical surface is straight along a section parallel to the axis, its curvature being in a section perpendicular to the axis; while in the case of a sphere, the curvature is uniform for all sections drawn through the centre.

A clear idea of the way in which pressure is produced by the curvature of a cylindrical surface may be obtained from the following train of reasoning.

Let AB (Fig. 140) represent a perspective view of a strip of a cylindrical surface cut off by two planes perpendicular to the axis of the cylinder, and let the length AB of the strip be equal to l, its breadth being equal to b. From A and B draw radii intersecting in C, the centre of curvature. The forces acting on the ends A and B of the strip lie in the plane ABC, and are respectively perpendicular to AC and BC; each force is equal to Sb. Through A and B draw lines DE and DF perpendicular to AC and BC, and let these intersect in D. Make DE = DF = Sb; then the resultant of the forces acting on the ends of the strip is equal to DK, the diagonal of the parallelogram EDFK. Also, the angle DFK is equal to the angle ACB, and the triangles DFK and ABC are similar. Thus—

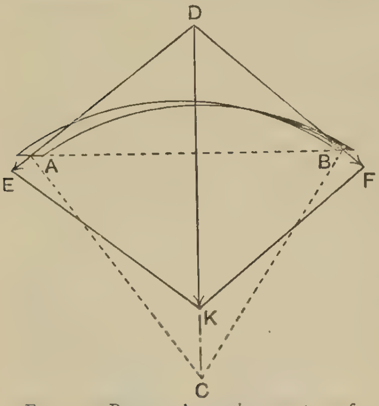

FIG. 140.—Pressure due to the curvature of a cylindrical surface.

$$\frac{\mathrm{DK}}{\mathrm{DF}} = \frac{\mathrm{AB}}{\mathrm{AC}}.$$

Let DK be denoted by f, and let the strip be so short that its length l is practically equal to the chord AB. Thus—

$$\frac{f}{\mathrm{S}b} = \frac{l}{\mathrm{R}}, \text{ and } f = \frac{\mathrm{S}bl}{\mathrm{R}}.$$

Now, the force f is perpendicular to the surface; and since the area of the surface is equal to bl, the force per unit area is equal to S/R.

Hence, **in order that the strip may be in equilibrium, it must be acted upon by a pressure** S/R **directed away from the centre of curvature** C.

EXPT. 46.—To determine the surface tension of a liquid in terms of the mass of a drop that falls from a tube.

If the orifice of a glass tube is coated with paraffin wax, water or an aqueous solution will not wet the tube; each drop, just before it breaks away, hangs suspended in the manner represented in Fig. 141. The exact calculation of the mass of each drop that breaks away has been effected by Lord Rayleigh: it presents considerable difficulty, and so an approximate calculation, based merely on statical consideration, will be given here.

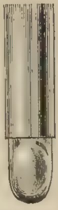

FIG. 141. Drop of water hanging from a glass tube.

Let R be the internal radius of the tube, and let it be assumed that the liquid below the orifice of the tube breaks away when the downward force acting on it exceeds the upward pull due to the tension of the surface. Let the mass that breaks away be denoted by m; then the downward force of gravity on this is equal to mg. Let an imaginary plane be drawn horizontally across the orifice of the tube; then, owing to the cylindrical curvature of the surface of the drop where it leaves the tube, the liquid just below this plane exerts a pressure (S/R) on the liquid above it, and the upward force exerted is equal to $\pi R^2 \times (S/R) = \pi RS$. An equal reaction tends to push the drop downwards, away from the orifice of the tube. The upward force due to the tension of the surface is equal to $2\pi RS$. Hence, for equilibrium to exist—

$$2\pi RS = \pi RS + mg,$$

$$\therefore \pi RS = mg, \text{ and } S = \frac{mg}{\pi R}.$$

In the result obtained by Lord Rayleigh, 3·8 is substituted for π in the formula just obtained, so that—

$$S = \frac{mg}{3{\cdot}8R}.$$

The liquid may be contained in a burette connected by a piece of rubber tube with the tube from which the drops break away. The speed at which the drops are formed can be regulated by a pinch-cock. Collect two hundred drops, formed at the rate of about one per second, and weigh them; thence calculate the value of m. The internal radius R of the tube should be about 0·5 cm.; this can be measured with a travelling microscope. For liquids in which paraffin wax is soluble, a tube coated with that substance must not be used.

QUESTIONS ON CHAPTER IX

1. A U-tube is supported with its limbs vertical, and is partly filled with water. If the internal diameters of the limbs are respectively equal to 1 cm. and 0·1 mm., what will be the difference in the heights at which the water stands in the two limbs?

(Surface tension of water = 70 dyne/cm.)

2. A soap film, 0·001 mm. thick, is at the temperature of melting ice. Calculate the fall in the temperature of the film due to stretching it adiabatically until its area is doubled. Assume that the specific heat of the soap solution, and its density, are each numerically equal to unity; and that the surface tension of the solution decreases at the rate of 0·15 dyne/cm. per degree C. rise of temperature.

3. Calculate the energy per sq. cm. of a mercury surface at 0°C., if the surface tension of mercury at this temperature is equal to 441 dyne/cm., and its value decreases at the rate of 0·379 dyne/cm. per degree C. rise of temperature.

4. Prove that it is impossible for more than three films of any substance to intersect in a single straight line, if the films are in stable equilibrium; also, prove that three films, which intersect in a single straight line, can be in stable equilibrium only when the angle between each pair of films is equal to $2\pi/3$. Discuss the bearing of the results on the conditions of equilibrium of "soap suds" and the froth on beer.

5. A spherical bubble, situated just below the surface of water, has a diameter equal to 0·01 mm. What must be the pressure within the bubble, if the atmospheric pressure on the surface of the water is equal to 10^6 dyne/(cm.)^2?

(Surface tension of water = 70 dyne/cm.)

6. The lower end of a vertical capillary tube is immersed in mercury, and it is observed that the summit of the curved surface of the mercury within the tube is 1 cm. below the plane surface of the mercury outside the tube. Calculate the value of the radius of curvature at the summit of the surface of the mercury within the tube.

(Surface tension of mercury = 450 dyne/cm. Density of mercury = 13·6 gm./(cm.)^3.

7. A spherical soap bubble, 10 cm. in diameter, is filled with air. Calculate the value of the mechanical energy that will be given up, if the bubble is allowed to contract isothermally until its diameter is

infinitesimally small; and thence infer the value of the work done in blowing a spherical soap bubble 10 cm. in diameter.

(The atmospheric pressure outside the bubble may be assumed to be equal to 10^6 dyne/(cm.)^2. The energy given up by the air originally contained in the bubble, as it expands to atmospheric pressure, may be assumed to be equal to the increase in the volume of the air multiplied by the mean of its original and final pressures. The air obeys Boyle's law. The surface tension of the soap solution = 45 dyne/cm.)

8. A soap bubble is formed at one end of a long narrow tube, made from a substance which is an electric insulator. The bubble is placed in the middle of a large room, and is charged electrically to a potential V, measured in c.g.s. electrostatic units. Prove that the bubble will be in stable equilibrium when the tube is open to the atmosphere, if—

$$V = 4\sqrt{(2\pi r S)};$$

where r is the radius of the bubble, and S is the surface tension of the soap solution.

(The electric lines of force terminating on a conducting surface charged with δ c.g.s. electrostatic units per sq. cm., exert an outward pull, normal to the surface, equal to $2\pi\delta^2$ dyne/cm.^{2} The potential of a sphere of radius r, charged with Q c.g.s. electrostatic units, is equal to Q/r.)

9. A beaker is filled with water to a height of 10 cm., and at the bottom of the water lie some pieces of capillary tube, each sealed at one end and filled with air. The internal diameter of the capillary tubes is equal to 0·2 mm. When the water boils, bubbles of steam are formed at the open ends of the capillary tubes, the radius of the bubbles when they break away being equal to that of the tubes. What is the temperature of the water at the bottom of the beaker when boiling is progressing, if the atmospheric pressure on the surface of the water is equal to that of 76 cm. of mercury (equivalent to 10^6 dyne/(cm.)^2)? (The surface tension of the boiling water is equal to 57 dyne/cm., and the vapour pressure of water increases at the rate of 2·7 cm. of mercury per degree rise of temperature, in the neighbourhood of 100° C.)

10. How would you determine whether a water surface is contaminated by contact with paraffin wax, vaseline, tallow, pitch, paraffin oil?

CHAPTER X

SURFACE TENSION (*continued*)

Pressure within a curved liquid surface. Let ABDE (Fig. 142) represent a cylindrical surface element, obtained by bending a flat rectangle, of breadth AB and length BD, about the straight line FG as axis. Let an imaginary plane, perpendicular to FG, be drawn so as to pass through the middle points of the ends AB and ED of the element, and let this plane cut FG in C_1. This plane may be called the *plane of longitudinal bending*: let it cut the element in a circular arc of a radius R_1. If the surface element forms part of a liquid surface of which the tension is S dynes per cm., it follows that the pressure on the concave side of the element must exceed that on the convex side by S/R (p. 318).

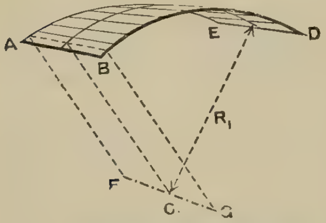

FIG. 142.—Surface element, curved only in one direction.

Now, a curve may be formed by joining an infinite number of short straight lines end to end; similarly, the surface element ABDE might be formed by laying an infinite number of flat transverse strips side by side, as indicated in Fig. 142. Let one of these transverse strips be selected, and let a plane passing through FG be drawn so as to divide the strip into two equal parts; next, let the strip be bent uniformly about an axis perpendicular to this plane and at a distance R_2 from the strip; the plane may be called the *plane of transverse bending*. Finally, let each transverse strip be bent in a precisely similar

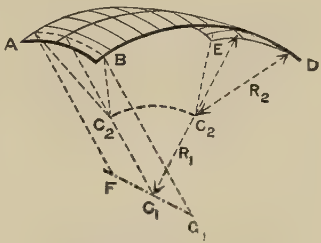

FIG. 143.—Surface element, curved synclastically.

manner; then each strip has been converted into an element of a cylinder of radius R_2, and therefore the pressure on its concave side must exceed that which existed when the strip was flat, by S/R_2. By trimming the various strips, without bending them any further, they can be made to fit together to form a continuous curved surface (Fig. 143). Each plane of transverse bending is obviously perpendicular to the plane of longitudinal bending, and these two mutually perpendicular planes cut the surface in curves which are called **principal sections.** The radii of curvature of the principal sections, (that is, R_1 and R_2) are called the **principal radii of curvature** of the surface. When the concavities of the principal sections are both turned towards the same side of the surface, the curvature of the surface is said to be **synclastic** (compare p. 263). If the surface of a liquid is curved synclastically, the longitudinal bending produces a pressure S/R_1 on the concave side, and the transverse bending produces a pressure S/R_2 on the same side; hence, the pressure on the concave side exceeds that on the convex side by P, where

$$P = S\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

The curvature of a curve is defined as the reciprocal of the radius of curvature of the curve. Thus, the curvatures of the two principal sections are equal to $(1/R_1)$ and $(1/R_2)$, and their sum is defined as the **curvature of the surface.** Hence, **the pressure within a synclastically curved surface exceeds the external pressure by the product of the surface tension and the curvature of the surface.**

In the case of a sphere, the principal radii of curvature are equal, and therefore the internal pressure exceeds the external pressure by $S\{(1/R) + (1/R)\} = 2S/R$, where R is the radius of the sphere (p. 314).

When the concavities of the principal sections are turned towards opposite sides of the surface, the curvature is said to be *anticlastic* (compare p. 263). A surface, of which the curvature is anticlastic, is represented in Fig. 144. In this case, owing to the longitudinal bending of the surface, the pressure beneath it is greater than that above it by S/R_1; and owing to the transverse bending, the pressure above the surface is greater than that beneath it by S/R_2: the total pressure beneath the surface must therefore exceed that above it by $(S/R_1 - S/R_2)$.

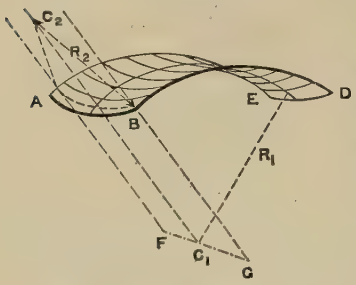

FIG. 144.—Surface element, curved anticlastically.

If it be decided to measure the radii of curvature from the surface to the corresponding axes of bending, then the signs of the radii will be different if the axes of bending are on opposite sides of the surface, and if the sign of R_1 is positive, that of R_2 will be negative. When this convention is agreed to, the pressure in the space enclosed by the surface exceeds the external pressure by P, where—

$$P = S\{(1/R_1) + (1/R_2)\},$$

and this formula applies both to surfaces curved synclastically and to those curved anticlastically.

A **model of a synclastically curved surface** may be made by supporting the corners of a duster in a horizontal plane, and placing some sand in the duster. By altering the positions of the points of support, the surface may be changed continuously from a part of a sphere to a part of a cylinder.

A **model of an anticlastically curved surface** may be made by placing two glass shades with hemispherical tops at a small distance apart, and spreading a wet duster over them. The duster forms a saddle-shaped surface where it stretches from one shade to the other.

General properties of a curved surface.—Let ABDE (Fig. 145) represent an element of a curved surface bounded by principal sections, and let the radius of curvature of the principal section BD be R_1, while that of the principal section AB is R_2. From any point P draw a normal PQ to the surface. Any section of a surface made by a plane passing through a normal to the surface is called a **normal section.** Let a plane through P cut the surface in the normal section PS; if the length of the section is small it will approximate to a circular arc: let its radius be equal to R. Then, if $PS = a$, it follows that in passing from P to S we travel through a distance h parallel to PQ, where—

$$h = \frac{a^2}{2R}.$$

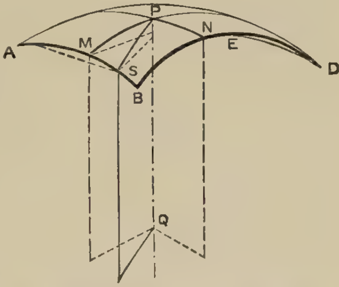

FIG. 145.—Curvature of a normal section of a surface.

Through P draw planes cutting the surface in the principal sections PM and PN; these planes must pass through PQ. Let the plane that cuts the surface in the normal section PS be inclined at an angle θ to the plane that cuts the surface in the principal section PM; then $PM = PS \cos\theta = a\cos\theta$, and $MS = PS \sin\theta = a \sin\theta$. In passing from P to M we travel a distance $(PM)^2/2R_1$ parallel to PQ; and in passing from M to S we travel a distance $(MS)^2/2R_2$ in the same direction. But, in passing from P to S, we must travel through the same distance parallel to PQ, whether we follow the direct path PS or the two paths PM and MS; thus—

$$h = \frac{a^2}{2R} = \frac{a^2\cos^2\theta}{2R_1} + \frac{a^2\sin^2\theta}{2R_2};$$

$$\therefore \quad \frac{1}{R} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2}.$$

This equation gives the radius of curvature R of any normal section, in terms of the principal radii of curvature R_1 and R_2, and the angle θ.

Now let the surface be cut by a plane through PQ, inclined at an angle $\{(\pi/2)+\theta\}$ to that which cuts the surface in the principal section PM; and let the radius of curvature of the normal section thus obtained be denoted by R′. Then—

$$\frac{1}{R'} = \frac{\cos^2\left(\frac{\pi}{2}+\theta\right)}{R_1} + \frac{\sin^2\left(\frac{\pi}{2}+\theta\right)}{R_2} = \frac{\sin^2\theta}{R_1} + \frac{\cos^2\theta}{R_2}:$$

$$\therefore \quad \frac{1}{R} + \frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Thus, **the sum of the curvatures of any two normal sections by mutually perpendicular planes, is equal to the sum of the principal curvatures of the surface.**

Let the surface be cut by two planes through PQ, inclined at the numerically equal angles $(+\theta)$ and $(-\theta)$ to the plane of the principal section PM. Then, since $\cos^2(-\theta)=\cos^2\theta$, and $\sin^2(-\theta)=\sin^2\theta$, it follows that the curvatures of these two normal sections are equal. If the numerical value of θ is very small, $\cos^2\theta=1$, and $\sin^2\theta=0$; thus in this case, the curvatures of the normal sections are both equal to R_1, the curvature of the principal section near to which they lie. If a plane through PQ rotates about that line, the curvature of the normal section in which it cuts the surface will vary; but as the plane approaches the plane of the principal section PM, the curvature of the normal section varies more and more slowly, and becomes constant in the immediate neighbourhood of the principal section. Further, if the curvature increases as the plane of the principal section is approached from one side, the curvature must decrease as the plane of the principal section is left on the other side; or, if the curvature decreases as the plane of the principal section is approached from one side, then the curvature increases as the plane of principal curvature is left on the other side. The same reasoning applies to the plane of the principal section PN. Hence **a principal section is a section of maximum or minimum curvature.** In the case of a synclastically curved surface, one principal section is a section of maximum curvature, and the other principal section is a section of minimum curvature; for, if the plane through PQ rotates through 180° from the plane of the principal section PM, the curvature finally attains the value which it had originally; and if PM is a section of maximum curvature, the curvature must decrease as the plane PM is left, and it must increase as the angular rotation approaches 180°, so that in passing through the principal section PN the curvature must attain a minimum value.

An important distinction between a normal section and a principal section can be understood by reference to Fig. 145. The line PQ is normal to the surface at P; that is, PQ is perpendicular both to PN and to PM, lines drawn on the surface in mutually perpendicular directions. The plane PSQ cuts the surface in the curve PS, and PQ is normal to this curve; that is, PQ is normal both to the surface, and to the curve PS. At the point S, the normal to the curve PS will not generally be a normal to the surface at S. On the other hand, at the point M, the normal to the principal section PM will also be a normal to the surface at M (compare Fig. 143). Thus, **two consecutive normals to a principal section are also normals to the surface; while only one normal to a normal section is a normal to the surface.**

Fig. 146, which represents an anticlastically curved surface, is lettered to correspond with Fig. 145. The principal section PM is concave downwards, its radius of curvature R_1 having a positive value; while the other principal section PN is concave upwards, its radius of curvature R_2 having a negative value. Any normal section cut by a plane through PQ, inclined at an angle θ to the plane of the principal section PM, will possess a radius of curvature R given by the equation

$$\frac{1}{R} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2}.$$

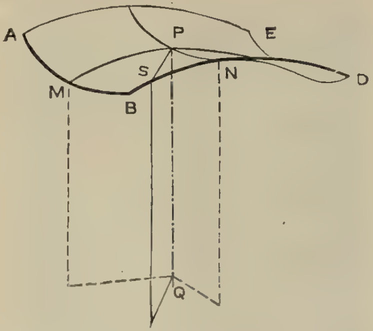

FIG. 146.—Curvature of a normal section of a surface.

The principal section PM possesses the maximum downward curvature, while the principal section PN possesses the maximum upward curvature. As the plane through PQ rotates from the plane of the section PM to that of the section PN, the curvature is at first concave downwards, and is finally concave upwards. Thus, for some section PS between PM and PN, the surface must be cut in a straight line; for this section R = ∞, and—

$$\frac{1}{R} = 0 = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2},$$

$$\therefore \quad \tan\theta = \pm\sqrt{-\frac{R_2}{R_1}}.$$

tan θ has a real value, since if R_1 is positive, R_2 is negative. When R_1 and R_2 are numerically equal, $\tan\theta = \pm 1$, and $\theta = \pm 45°$. Hence, **when the principal radii of curvature of an anticlastic surface are numerically equal, the sections of the surface by planes which bisect the angles between the planes of the principal sections, are straight.** This result has been obtained already in connection with the helicoid (p. 272).

Possible forms of liquid surfaces.—The forms which possibly may be assumed by a liquid surface in a condition of equilibrium must now be studied.

First, let the pressure have the same value on both sides of the surface. In this case the surface may be plane, in which case all radii of curvature are infinitely great, and the curvature is equal to zero. A plane film may be formed by dipping a circular ring, or the mouth of a funnel, into a soap solution,[1] and withdrawing it edgeways.

The curvature of the helicoid is equal to zero, and therefore a helicoidal film will be in equilibrium when the pressure has the same value on both sides of the film; that is, when the film is formed in the open atmosphere.

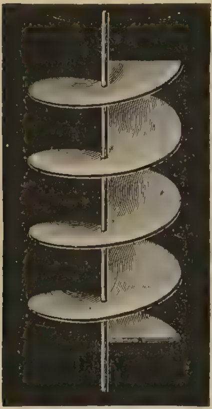
FIG. 147.—Helicoidal soap film.

A helicoidal film may be formed by dipping a wire framework, such as is represented in Fig. 147, into a soap solution, and then withdrawing it. To make the framework, stretch a piece of copper wire so as to stiffen it (p. 224), wind it in the form of a uniform helix around a cylinder, and bend the ends so that they may be soldered to a straight wire which forms the axis. The curvature discussed on pp. 270–272 can be observed without difficulty.

In calculating the pressure inside a spherical soap-bubble, it must be remembered that the soap film has two surfaces, both of which have the same radius R, and each surface exerts a pressure $2S/R$, so that the total internal pressure is equal to $4S/R$.

A spherical soap-bubble may be supported on a wire ring which has been dipped into soap solution; another ring, which has been wetted in a similar manner, may be pressed on the bubble, and thus a number of most interesting surfaces may be formed (Fig. 148). The characteristic of all such surfaces is that the curvature is constant throughout, since the pressure within the bubble is uniform. The pressure within the bubble may be estimated by observing the curvature of the spherical cap which extends over each ring.

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[1] The following soap solution, due to Profs. Reinold and Rücker, is recommended by Prof. Boys. To a litre of cold distilled water, contained in a clean stoppered bottle, add 25 grams of oleate of soda, and let it stand for a day. Then add about 300 c.c. of Price's glycerine, and well shake; let the stoppered bottle stand for a week in a dark place. Then, by means of a syphon, draw off the clear liquid, leaving the scum behind. Add two or three drops of liquid ammonia, and keep the bottle in a dark place. *The liquid must not be warmed or filtered.*

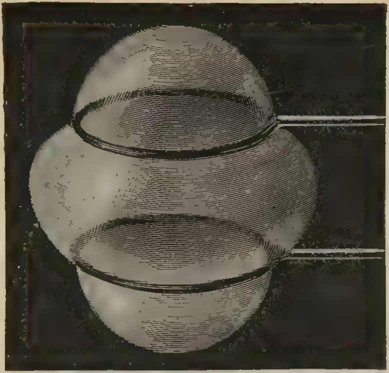

Fig. 148.—Deformed soap bubble.

Let a straight line, drawn through the centre of the two rings, be called the axis of symmetry of the surface. If the bubble is cut by an imaginary plane passing through the axis of symmetry, its section has the form generated by the focus of a conic section when the latter rolls along the axis of symmetry.[1] By pulling the rings apart, the film extending from one to the other may be made cylindrical (Fig. 149); the excess of pressure within the film is equal to 2S/R. When the rings have been drawn apart until the caps which cover them become flat, it is clear that the internal and external pressures must be equal; the section of the surface extending from one ring to the other (Fig. 150) now has the form of a catenary—the curve in which a chain hangs when its ends are supported in a horizontal plane. The plane caps can be broken by touching them with a needle that has been heated in a Bunsen

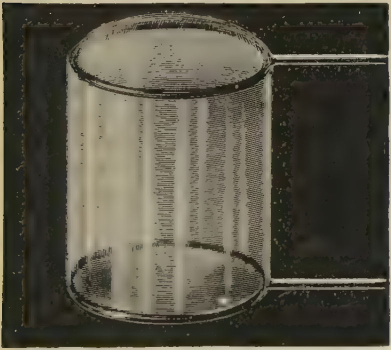

Fig. 149.—Cylindrical soap bubble.

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[1] For practical methods of forming bubbles of various shapes, the student may consult Prof. Boys's volume on *Soap Bubbles* (Society for Promoting Christian Knowledge).

flame, and the remainder of the film still remains in equilibrium. Thus, the anticlastic surface represented in Fig. 150 (called the catenoid) has zero curvature; that is, at any point of the surface the principal radii of curvature are equal in magnitude but opposite in sign.

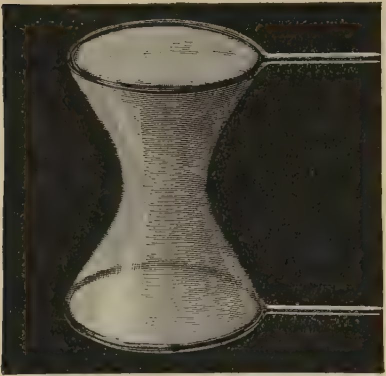


FIG. 150.—Soap bubble, having the form of a catenoid.

Within a liquid drop the hydrostatic pressure varies, and the surface assumes a form which is determined by the condition that the product of the surface tension and the curvature at any point is equal to the hydrostatic pressure at that point. Thus, when a drop hangs from the lower side of a horizontal sheet of glass (Fig. 125), its surface, where it is in contact with the glass, is plane; here the hydrostatic pressure is zero, and therefore the curvature is zero. At a small distance below the glass, the curvature is anticlastic, and has a small positive value. Further down the surface becomes cylindrical, and at this point the hydrostatic pressure is equal to S/R, where R is the radius of the cylinder. Still further down, the curvature of the surface becomes synclastic: at the lowest point of the drop, the surface is spherical; and if h denotes the distance of this point from the glass, and ρ denotes the density of the liquid, the radius of curvature R at the lowest point is given by the equation—

$$\rho g h = 2S/R.$$

Shape assumed by a deformed cylindrical bubble.—It has been proved that a cylindrical bubble is in equilibrium when the internal pressure is equal to 2S/R, where S is the surface tension of the liquid and R is the radius of the cylinder. The question now arises, is the equilibrium stable or unstable?

To explain the character of the problem to be solved, let us consider the nature of the equilibrium of two spherical bubbles communicating with each other. By means of a branched tube provided with the necessary stopcocks (Fig. 151), two spherical soap-bubbles of different sizes can be blown separately and then put in communication, connection with the external atmosphere being cut off. The small bubble has a greater internal pressure than the large one, since the pressures are inversely proportional to the radii of the bubbles ; therefore the air is forced from the small bubble into the large one, and it is clear that the combination of the two is unstable. If the bubbles are blown equal in size, the equilibrium of the combination is still unstable, for the slightest compression of one bubble renders its internal pressure greater than that of the other.

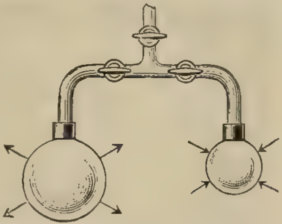


FIG. 151.—Soap bubbles, of different sizes put into communication.

When a spherical bubble contracts or expands, it still remains spherical. In the case of a cylindrical bubble, if its ends are fixed, an expansion or contraction must involve a departure from the cylindrical form ; thus, before we can investigate the nature of the equilibrium of the bubble, we must determine the form assumed when a slight expansion or contraction occurs.

Let AB and CD (Fig. 152) represent the section of a cylindrical surface by a plane passing through the axis XX′ of the cylinder. AB is called a *generating line* of the cylinder; for the cylinder may be generated by the revolution of AB about XX′. If a circle, of radius equal to the distance between AB and XX′, is caused to roll along XX′, its centre will describe the generating line AB. The circle is a conic section of which both foci coincide with the centre, and thus a circular cylinder is one of the forms which may be assumed by a soap-film

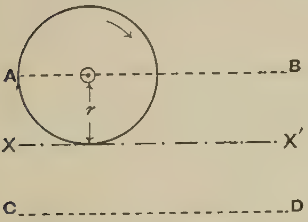


FIG. 152.—Generating lines of a cylindrical surface.

(p. 329). Now, any one form can be changed by insensible gradations into any other form possible to a soap-film. If a cylindrical surface changes slightly, then the form of its generating line must change; and since an ellipse is the only conic section into which a circle can change directly, it follows that the generating line of the deformed cylinder can be described by a focus of a nearly circular ellipse which rolls along XX′. If the deformation of the cylinder is very slight, the periphery of the ellipse will differ very little from a circle, but the focus will be at a small finite distance (say a) from the centre. Hence, to a close approximation, the generating line of the deformed cylinder can be described by a point at a distance a from the centre of a circle which rolls along the axis XX′ (Fig. 153).

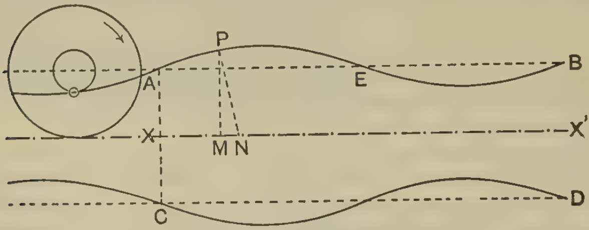


Fig. 153.—Generating lines of a deformed cylindrical bubble.

As the circle rolls along XX′ it rotates, and the tracing point revolves about the centre. Now a uniform revolution is equivalent to two mutually perpendicular simple harmonic motions differing in phase by $\pi/2$ (p. 87). Let one of these simple harmonic motions be performed parallel to the axis XX′; this motion will not affect the form of the generating line to an appreciable extent, since it must be compounded with the uniform linear motion of the centre of the circle parallel to XX′. The other simple harmonic motion, perpendicular to XX′, combined with the uniform motion parallel to XX′, give a wave curve APEB (Fig. 153). One wave length of the curve will be described while the circle completes one rotation, that is, while it rolls through a distance equal to its circumference; therefore, if r is the radius of the rolling circle, the wave length λ of the curve is given by the equation—

$$\lambda = 2\pi r.$$

The centre of the rolling circle describes the straight line AB at a uniform distance r from XX′. If y denotes the distance PM of a point P on the curve, measured from the axis XX′, while x denotes the horizontal distance of P from A, then the equation to the wave curve AEB takes the form—

$$\begin{aligned} y &= r + a\sin\frac{2\pi x}{\lambda} \\ &= r + a\sin(x/r). \end{aligned}$$

This equation determines the shape of the generating line of the deformed cylinder. The surface generated by the revolution of this line about XX′ must possess the property that its curvature (p. 323) has a uniform value for all points on the surface; an independent proof of this property will now be given. In the first place, the value of the radius of curvature at any point on the generating curve will be determined.

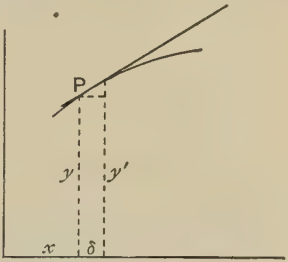


FIG. 154.—Angle of slope of a wave curve.

Curvature of a wave curve. Let a point P on the curve have the co-ordinate x and y, while a neighbouring point has the co-ordinates $(x+\delta)$ and y', (Fig. 154). Then, if θ denotes the angle of slope of the curve between the two points, $\tan\theta = (y'-y)/\delta$. Since the angle of slope is small, $\tan\theta = \theta$, to a sufficient degree of approximation.

Now
$$\begin{aligned} y &= r + a\sin\frac{x}{r} \\ y' &= r + a\sin\left(\frac{x+\delta}{r}\right) \\ &= r + a\left\{\sin\frac{x}{r}\cos\frac{\delta}{r} + \cos\frac{x}{r}\sin\frac{\delta}{r}\right\} \\ &= r + a\left\{\sin\frac{x}{r} + \frac{\delta}{r}\cos\frac{x}{r}\right\}. \end{aligned}$$

since δ is supposed to be so small that $\cos\delta/r = 1$, and $\sin\delta/r = \delta/r$.

$$\therefore \quad \tan\theta = \theta = \frac{y'-y}{\delta} = \frac{a}{r}\cos\frac{x}{r}.$$

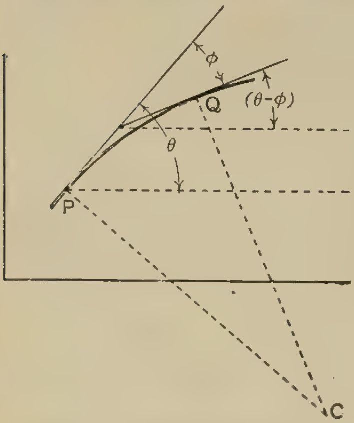


FIG. 155.—Radius of curvature of a wave curve.

Let PQ (Fig. 155) represent a short arc of the curve, for which the radius of curvature R is equal to PC. Draw tangents to the curve at P and Q; then the inclinations of these tangents to the horizontal give the angles of slope of the curve at the points P and Q. At any point of the curve, the radius of curvature and the tangent are mutually perpendicular; therefore, if the radii PC and QC are inclined to each other at an angle ϕ, the tangents at P and Q are also inclined to each other at an angle ϕ; and if θ is the angle of slope of the curve at P, then the angle of slope at Q is $(\theta - \phi)$. The length of the arc PQ is equal to Rϕ; and since the slope of the curve is small, Rϕ is practically equal to the increase in the value of x in passing from P to Q. Thus—

$$\theta = \frac{a}{r}\cos\frac{x}{r},$$

$$\theta - \phi = \frac{a}{r}\cos\frac{(x + \mathrm{R}\phi)}{r}$$

$$= \frac{a}{r}\left\{\cos\frac{x}{r}\cos\frac{\mathrm{R}\phi}{r} - \sin\frac{x}{r}\cdot\sin\frac{\mathrm{R}\phi}{r}\right\}$$

$$= \frac{a}{r}\left\{\cos\frac{x}{r} - \frac{\mathrm{R}\phi}{r}\cdot\sin\frac{x}{r}\right\},$$

$$\therefore\ \phi = \frac{a}{r}\cdot\frac{\mathrm{R}\phi}{r}\cdot\sin\frac{x}{r},$$

$$\text{and}\quad \frac{1}{\mathrm{R}} = \frac{a}{r^2}\cdot\sin\frac{x}{r} = \frac{1}{r^2}(y - r).$$

The curvature of a wave curve may be obtained by a geometrical construction which is very instructive. Let P (Fig. 156) be the tracing point, at a distance CP $= a$ from the centre C of the circle which rolls along the axis XX′; and let Q be the point at which the rolling circle touches the line XX′ at a given instant. Now, the normal to a curve is the perpendicular drawn to a small element of the curve; and since the point Q of the rolling circle is stationary for an instant, the circle as a whole is rotating for an instant about Q; therefore, in the neighbourhood of P the curve described by the tracing point is perpendicular to the straight line joining P and Q, and PQ is the normal to the curve at P. **The centre of curvature of a curve is the intersection of two consecutive normals to the curve.** Let the circle roll along XX′ for a very small distance, so that the tracing point moves to P′, and the rolling circle comes into contact with XX′ at the point Q′; then if a straight line is drawn through the points P′ and Q′, this line will be the normal to the curve at P′, and the point K, at which PQ and P′Q′ intersect, is the centre of curvature of the curve at P. The distance PK gives the radius of curvature of the curve at P. If CP is small, the line PK will be nearly perpendicular to XX′, whatever may be the position of the point P on the curve.

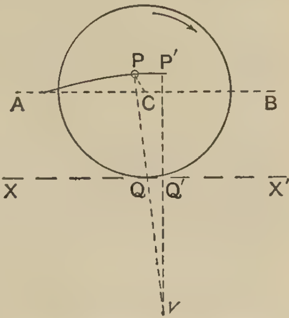


FIG. 156.—Radius of curvature of a wave curve.

As the tracing point moves from P to P′, let the rolling circle rotate through the small angle δ; then if the radius of the rolling circle is equal to r, the distance QQ′ is equal to $r\delta$, and this is also the distance through which the centre of the rolling circle has moved parallel to XX′. The tracing point might have been brought from P to P′ by moving the circle forward through a distance $r\delta$, without allowing it to rotate; and then rotating the circle in a clockwise sense through an angle δ, without allowing its centre to move. In the first of these operations, the tracing point is carried forward through the distance $r\delta$ parallel to XX′. In the second operation, the tracing point moves through a distance $a\delta$ perpendicular to the line CP, and therefore in a direction making an angle $\{(\pi/2) - \angle \text{PCA}\}$ with the line ACB drawn parallel ot XX′; the component displacement of the tracing point parallel to XX′ is equal to $a\delta \cos\{(\pi/2) - \angle \text{PCA}\} = a\delta \sin \angle \text{PCA} = \delta h$, if h denotes $\{\text{CP} \sin \angle \text{PCA}\}$, the distance of the point P above the line ACB. Hence, in bringing the tracing point from P to P′, its total displacement parallel to XX′ is equal to $r\delta + h\delta = (r+h)\delta$. Since PQ is approximately perpendicular to XX′, the angle PKP′ is equal to

$(r+h)\delta/\mathrm{R}$; and this angle is also equal to $\mathrm{QQ}'/\mathrm{QK}=r\delta/\{\mathrm{R}-(r+h)\}$. Thus—

$$\frac{(r+h)\delta}{\mathrm{R}}=\frac{r\delta}{\mathrm{R}-(r+h)};$$

$$\therefore\quad \mathrm{R}(r+h)-(r+h)^2=\mathrm{R}r,$$

so that

$$\mathrm{R}h=(r+h)^2;$$

and

$$\frac{1}{\mathrm{R}}=\frac{h}{(r+h)^2}=\frac{h}{r^2}, \quad . \quad . \quad . \quad . \qquad (1)$$

where h, in the denominator, is neglected in comparison with the much larger quantity r. If we substitute $(y-r)$ for h, we obtain the result already found by another method. Also, since $2\pi r=\lambda$, where λ is the wave length of the curve, we have—

$$\frac{1}{\mathrm{R}}=h\left(\frac{2\pi}{\lambda}\right)^2.$$

Curvature of the surface generated by the revolution of a wave curve. Equation (1) gives the radius of curvature at any point P of the generating curve AEB (Fig. 153). Let this curve revolve about XX′ through a small angle; then a small element of the curve near P will generate a surface element like ABDE, Fig. 143 (p. 323). The plane of the paper (Fig. 153) is one plane of bending for the element, and the radius of curvature R_1 in this plane is equal to R; the other plane of bending is a plane, perpendicular to the plane of the paper, drawn through the normal PN to the curve at P; and the radius of curvature R_2 in this plane is PN, which is practically equal to PM or y. Thus, the curvature of the surface element generated by the curve near P is equal to—

$$\frac{1}{\mathrm{R}}+\frac{1}{\mathrm{PM}}=\frac{h}{r^2}+\frac{1}{y}.$$

Now, $y-r=h$; thus, the curvature at P is equal to—

$$\frac{h}{r^2}+\frac{1}{r+h}=\frac{h}{r^2}+\frac{r-h}{(r+h)(r-h)}$$

$$=\frac{h}{r^2}+\frac{r-h}{r^2-h^2};$$

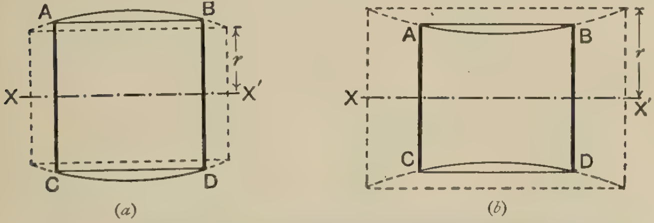
and since h is small, h^2 may be neglected in comparison with r^2 in the denominator of the fraction $(r-h)/(r^2-h^2)$, so that the curvature is equal to—

$$\frac{h}{r^2}+\frac{r}{r^2}-\frac{h}{r^2}=\frac{1}{r}.$$

Thus, the curvature is uniform at all points of the surface, and has the same value as the curvature of the cylinder generated by the revolution of the straight line AB about XX′ (Fig. 153).

Stability of a cylindrical bubble.—Let the length AE or EB (Fig. 153) of the generating curve be called a **loop** of that curve. Then the length of a loop is equal to half a wave length of the curve, or half the circumference of the rolling circle.

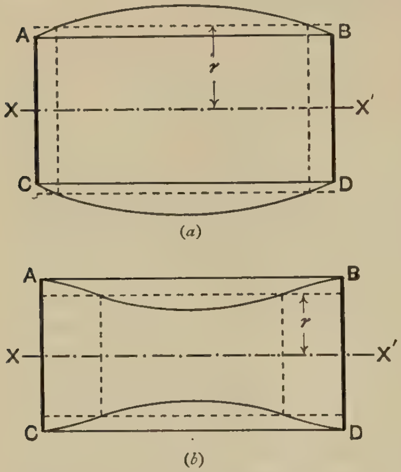
Let ABDC (Fig. 157) represent the section of a cylindrical soap film, the ends AC and BD being fixed. If the film bulges slightly, the generating line of the surface must be part of a wave curve; this curve must pass through A and B, and therefore the crest of the wave curve must lie midway between A and B. Now, a loop of the wave curve is equal to half the circumference of the rolling circle; consequently, if AB is less than $\pi \times (AC/2)$ (that is, if the length of the cylindrical film is less than half its circumference), it is clear that the generating curve of the deformed cylinder must comprise only a fraction of a loop of the wave curve, and the radius r of the rolling circle must be less than AC/2 (Fig. 157 (*a*)). The excess of pressure within the bulging film is equal to $2S/r$, and the excess of pressure within the cylindrical film was equal to $2S/(AC/2)$; therefore, in order to make the cylindrical film bulge outwards, the pressure of the air inside the film must be increased.



(*a*) (*b*)

FIG. 157.—Deformed cylindrical film, of which the length is less than half the circumference.

If the film bends inwards or develops a waist (Fig. 157 (*b*)), its generating curve must be a fraction of a loop of a wave curve described by a rolling circle of which the radius r is greater than AC/2, and therefore the internal pressure must become less in order that the film may develop a waist. Thus, **if the length of a cylindrical film is less than half its circumference, the internal pressure must be increased in order to make the film bulge outwards, while the internal pressure must be diminished in order that the film may develop a waist.**



(*a*)

(*b*)

FIG. 158.—Deformed cylindrical film, of which the length is greater than half the circumference.

Now, let the length of a cylindrical soap film be greater than half the circumference of the cylinder. If the film bulges outwards, the generating curve must comprise more than a loop of the wave curve, and therefore the radius r of the rolling circle must be greater than AC/2 (Fig. 158 (*a*)). Hence, the bulging is accompanied by a decrease in the internal pressure of the film. If the film develops a waist (Fig. 158 (*b*)), the rolling circle must have a radius which is less than AC/2, and therefore the internal pressure must be greater than within the cylinder. Thus, **if the length of a cylindrical film is greater than half its circumference, an increase in the internal pressure will cause it to develop a waist, and a decrease in the internal pressure will cause it to bulge outwards.**

These properties of cylindrical films can be made manifest by a beautiful experiment due to Prof. Boys. A branched tube (Fig. 159) has cylindrical ends of equal radii, and these ends are placed above cylindrical rings which dip into a soap solution. By lowering the branched tube, films may be formed between its ends and the rings below them; and, by separately adjusting the pressures of the two films, by the aid of the stop-cocks shown, one film can be caused to bulge and the other to develop a waist. The upper stop-cock is now closed, and the lower ones are opened so as to put the spaces enclosed by the films into communication. If the length of each film is less than half its circumference, the bulging film forces air into the one with a waist, and both become more nearly cylindrical (Fig. 159 (*a*)). If the length of each film is greater than half its circumference, the film with a waist forces air into the bulging one, and this continues until the film with a waist collapses (Fig. 159 (*b*)).

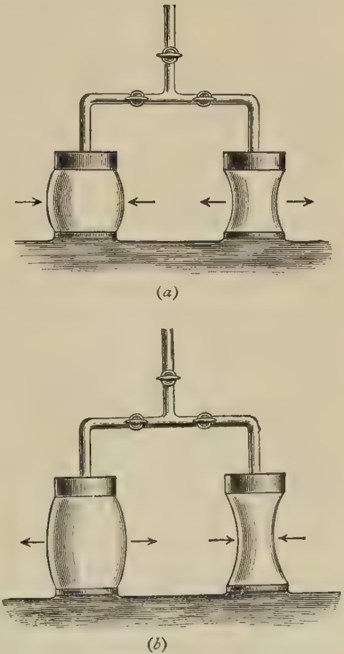


FIG. 159.—Experiments on the stability of cylindrical soap films.

When the two films are put into communication with each other, they virtually constitute a single film. Hence, we may conclude that **if a cylindrical film has a length less than its circumference, the film will be stable**; for, if one end bulges slightly and the other end develops a waist, air will be forced from the bulging to the constricted part of the film and the whole will straighten out (Fig. 160). **If, however, the length of the cylindrical film is greater than its circumference, the film will be unstable**; for, if a constriction is developed at one end and a bulge at the other, the air will be forced from the constricted to the bulging end (Fig. 161), and therefore the constriction and the bulge will both increase until the sides of the constricted part meet.

Similar reasoning shows that any cylindrical surface of a liquid becomes unstable when its length is greater than its circumference. Thus, a jet of water breaks into drops. If a piece of very fine wire be dipped into water, and then withdrawn and held in a horizontal position, it will be found that the water on the wire collects into small drops

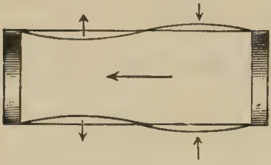


FIG. 160.—Stable cylindrical soap film. (Its length is less than its circumference.)

EXPT 47.—Add carbon bi-sulphide to paraffin oil until the mixture is only slightly less dense than water. Into a tall beaker, pour some water until it stands at a height of an inch or so; then fill the beaker up with the paraffin mixture. Obtain a piece of glass tube about six or seven inches in length and half an inch in diameter; close one end with the finger and plunge the other end below the surface of the paraffin mixture, and into the water. On removing the finger the water rises in the tube. Now rapidly withdraw the tube in the direction of its length; a cylindrical column of water will be left surrounded by the paraffin. If the length of the cylinder is greater than its circumference, constrictions and bulgings appear and develop, and the column finally breaks up into a number of drops. A small drop, called Plateau's spherule, is formed between each pair of larger drops. (Compare Fig. 125, p. 286.)

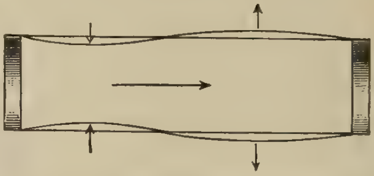


FIG. 161.—Unstable cylindrical soap film. (Its length is greater than its circumference.)

Attraction between small bodies floating on a liquid.—If small light objects, such as match stems, are placed on the surface of water, they collect in groups; if the water does not fill its containing vessel, these groups finally find their way to the edge of the water. If the match stems are coated with paraffin wax, they still form groups, but these groups now avoid the edge of the water. If one match stem is coated with wax while another is wetted by the water, these two exhibit no attraction for each other; on the contrary, if they are placed near to each other, they quickly separate. These phenomena are due to surface tension, and can be explained in accordance with the principles already developed.

Let Fig. 162 represent the section of a flat plate standing upright in a liquid which wets it. At a distance from the plate the surface of the liquid is horizontal, but near to the plate the surface curves upwards and meets the plate tangentially. Let the liquid commence to curve upwards at A; then, if a line be drawn through A, parallel to the plane of the plate, the surface to the left of this line exerts a force of S dynes per centimetre on the liquid to the right of it. Produce the horizontal plane of the liquid surface to meet the plate in a line parallel to the line drawn through the point A; then, the liquid above this plane is in equilibrium, and therefore it must cling to the face of the plate with a force equal to that exerted by the liquid surface to the left of the line through A. Thus, **the direct force exerted on the plate by the liquid, has the same value as if the horizontal surface of the liquid were prolonged to meet the plate perpendicularly.**

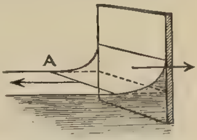


FIG. 162.—Curved surface of a liquid, near to a vertical plate which it wets.

Fig. 163 represents several pairs of parallel plates, standing upright in a liquid. In Fig. 163 A, the plates are wetted by the liquid. Thus the surface of the liquid outside the plates pulls them apart with a force of S dynes for each centimetre of the line of junction between the liquid and the plates. But at some point between the plates, the surface of the liquid must be horizontal; therefore, the surface between the plates pulls them towards each other with a force equal to that with which the outside surface pulls them apart. Hence, the attraction apparently exerted by one plate on the other must be due to some other cause, and this is easily found.

The liquid rises between the plates and virtually hangs from the surface that joins them. Hence, the liquid between the plates is in a state of tension, its molecules being slightly displaced, just as those of a solid would be if its surface were pulled outwards in all directions. Since the liquid molecules attract the glass molecules, a pull will be exerted on the plates by the strained liquid between them. This explains why objects which are wetted by a liquid are attracted when they float near to each other on its surface.

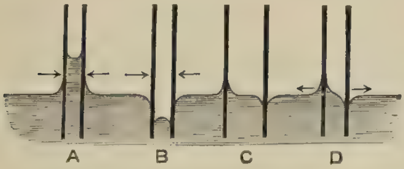


FIG. 163.—Parallel flat plates, floating upright in a liquid.

Fig. 163 B represents two flat plates, floating upright and parallel in a liquid which does not wet them. As before, no resultant force is

produced by the direct action of the surface; but the liquid between the plates stands at a lower level than that outside them (p. 313), and the hydrostatic pressure due to this difference of level presses the plates together.

Now let one plate be wetted by the liquid, while the other is not. If we examine the section of the liquid surface between the plates (Fig. 163 C) it is seen to be concave upwards where it descends from the wetted plate, and convex upwards where it descends to meet the plate that is not wetted. Where the section changes from concave to convex, it must be straight. If the distance between the plates is sufficiently great, the straight part of the section will be horizontal; and in this case there will be no apparent attraction or repulsion between the plates, for the forces exerted on either plate are due merely to the direct action of the liquid surface, and are the same as if the other plate were removed. If the distance between the plates is small, the straight part of the section of the surface between them will be inclined to the horizontal at a finite angle, say θ; and in this case the horizontal force pulling one plate towards the other is equal $S\cos\theta$ dynes per centimetre length of the line of junction of the liquid and solid surfaces, while the force pulling them apart is equal to S dynes per centimetre. Hence, in this case the plates apparently repel each other (Fig. 163 D).

Force exerted by a drop of liquid between two parallel plates.—It is a matter of common experience that two wet sheets of paper cling together with considerable tenacity. If a drop of water is placed between two pieces of plate glass, the resulting attraction is so great that it is difficult to pull one plate away from the other in a perpendicular direction, although there is no difficulty in sliding one over the other.

Let Fig. 164 represent a side view of the plates with a drop of water between them. The water spreads out into a circular disc of considerable area, and the surface of the water is curved anticlastically (p. 324). Let any plane, perpendicular to the glass surfaces, be drawn through the centre of the circular disc of water; this will be one plane of bending, and let the numerical value of the radius of curvature of the corresponding section be r. A plane parallel to the glass

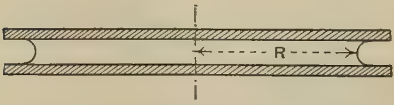


FIG 164.—Parallel plates with a drop of water between them.

surfaces, and midway between them, will be the other plane of bending; let the numerical value of the radius of curvature of the corresponding principal section be R. Thus, the pressure within the drop will be less than that outside it by—

$$S\left(\frac{1}{r} - \frac{1}{R}\right); \text{ (see p. 324).}$$

If the plates are close together, the radius R of the circular disc of water will be very great in comparison with r, and therefore $1/R$ may be neglected. Now the hydrostatic pressure within the drop will not vary appreciably; hence, the pressure inside the drop is uniform, its value being less than the external pressure by S/r. Consequently, r is constant for all points of a section by a plane perpendicular to the glass plates and passing through the middle of the drop; that is, the section by such a plane is a semicircle, and $r = d/2$, where d is the distance between the plates.

Now, the direct pull exerted by the water surface on either plate is equal to $2\pi RS$ (see the last section), and the difference between the external and internal pressures of the drop, acting over a circle of radius R, produces a force equal to $\pi R^2 (2S/d)$, which draws the plates together. Hence, the total force F urging the plates toward each other is given by the equation—

$$F = 2\pi RS + \frac{2\pi R^2 S}{d}.$$

When the plates are very near to each other, the second term on the right hand side of the equation is much greater than the first term, so that, to a sufficient degree of approximation—

$$F = \frac{2\pi R^2 S}{d}.$$

If the surfaces are very nearly truly plane, the plates will approach each other until d becomes exceedingly small, and the force exerted may be sufficient to fracture the plates. In order to understand how such a great force may be exerted by a liquid, it must be remembered that the section of radius $r = d/2$ tends to straighten out, and consequently the water is in a state of tension; hence, the water suffers a dilatational strain, and in attempting to regain its original volume a considerable force is exerted on the surfaces of the glass. This experiment shows that water possesses very great tensile strength, and that the water molecules cling to the glass with great tenacity (see p. 282).

Problem.—*A drop of water weighing* 0·1 *gram is introduced between two plane and parallel metal plates. What force will be exerted when the plates are at a distance of* 0·0001 *cm. apart?*

The volume of the drop is 0·1 c.c. $\therefore \pi R^2 \times d = 0{\cdot}1$.

$$\therefore F = 2 \cdot \frac{\pi R^2 d}{d} \cdot \frac{S}{d} = \frac{2 \times 0{\cdot}1 \times 75}{(0{\cdot}0001)^2} = 1{\cdot}5 \times 10^9 \text{ dynes} = 1{\cdot}5 \text{ tons (nearly).}$$

Force needed to pull a plate away from the surface of a liquid.—Let a plate, similar to that represented in the upper part of Fig. 165, be suspended so that its lower surface is in contact with a liquid which wets it. If the plate is pulled upwards, the liquid clings to its lower surface; and thus an opposing force is called into play which tends to prevent the plate from leaving the surface of the liquid.

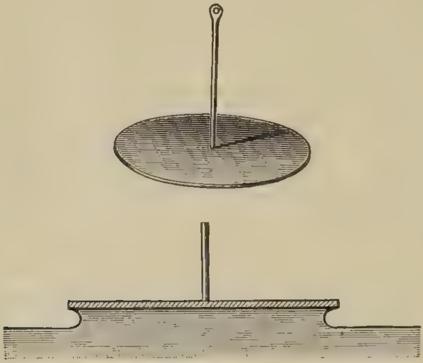


FIG. 165.—Plate being dragged away from the surface of a liquid. (Perspective view of plate above, sectional elevation below.)

At a given instant, let the lower surface of the plate be at a height h above the free surface of the liquid. Then, if P denotes the atmospheric pressure, the pressure on the underside of the plate is equal to $(P - g\rho h)$. The pressure on the upper surface of the plate is equal to P. Thus, if the area of the plate is equal to A, the resultant force which urges the plate downwards is equal to $g\rho hA$. This force, of course, is due to the tensile stress exerted by the liquid which hangs from the lower surface of the plate; its value must now be found, in terms of the surface tension S of the liquid.

Let the circular disc of water that hangs below the plate be divided, in imagination, into two equal parts by means of a vertical plane passing through its centre; and let a slice, of unit width, be cut symmetrically from one half of the disc, by means of two vertical planes perpendicular to the plane passing through the centre of the disc (Fig. 166). This slice must be in equilibrium under the action of the forces exerted upon it.

Since the width of the slice is unity, the horizontal force exerted by the atmosphere on the curved surface is equal to Ph, and the oppositely directed force due to the tension of the surface is equal to $2S$. Hence, the resultant force, tending to urge the slice from right to left, is equal to $(Ph-2S)$.

The plane section at the opposite end of the slice is acted upon by a hydrostatic pressure, which varies from P at its lower edge, to $(P-g\rho h)$ at its upper edge; the average pressure is equal to $\{P-(g\rho h/2)\}$, and this produces a force equal to $\{P-(g\rho h/2)\}h$, tending to urge the slice from left to right. Thus, for equilibrium to be maintained,—

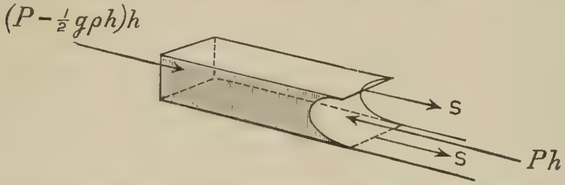


FIG. 166.—Slice, cut from the liquid hanging below the plate in Fig. 165.

$$Ph-2S=Ph-\tfrac{1}{2}g\rho h^2,$$

$$\therefore\quad g\rho h^2=4S,\ \text{and}\ (g\rho h)^2=4g\rho S,$$

$$\therefore\quad g\rho h=2\sqrt{g\rho S}.$$

Thus, the force that must be overcome, in order to pull the plate away from the liquid, is equal to $A(g\rho h)=2A\sqrt{(g\rho S)}$.

Vibrations of liquid jets and drops.—When a liquid issues from a circular orifice, at first its surface is cylindrical; but a cylindrical surface which is longer than its circumference is unstable (p. 340), and therefore the jet ultimately breaks up into a stream of drops. If a musical note is sounded near the jet, the variations in the air pressure start bulgings and constrictions in the surface, and since these occur at regular intervals, the drops ultimately formed are spaced regularly. Fig. 167

represents a jet breaking into drops under the action of a high note produced by blowing across a key.

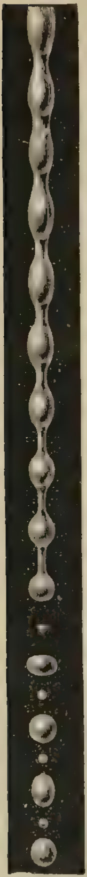


Fig. 167.
Liquid jet breaking into drops.

Before a drop finally breaks away, it is joined by a narrow neck to the liquid column above it. The tension of the neck pulls the drop into a shape resembling that of an egg, but as the neck becomes thinner, the pull exerted by it on the drop diminishes, and the tension of the surface of the drop tends to make it assume a spherical form, since a sphere has a smaller area than any other surface of equal volume. If the drop gains its freedom when its form is spherical, the various particles of the drop are in relative motion, and their kinetic energy carries the drop through the spherical form, and the drop becomes flattened in a vertical direction and expanded laterally. This process continues until the kinetic energy of the particles is used up in increasing the area of the drop, and then the drop once more contracts towards a spherical form and passes through this to the shape of an egg (Fig. 167).

The way in which the period of oscillation t of a drop depends on its radius r, and the surface tension S and density ρ of the liquid, can be determined from the dimensions of the various quantities. Let $t=kr^xS^y\rho^z$, where k is a constant that does not depend on the units of length, mass and time. Then the dimension of t is T, that of r is L; the dimensions of S are M/T^2 (p. 287), and of ρ are M/L^3. Since both sides of the equation must have the same dimensions, and k has no dimensions, it follows that—

$$\mathrm{T} = \mathrm{L}^x\left(\frac{\mathrm{M}}{\mathrm{T}^2}\right)^y\left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^z$$

$$= \mathrm{L}^{(x-3z)}\,\mathrm{M}^{(y+z)}\,\mathrm{T}^{-2y}.$$

$$\therefore\quad -2y=1, \text{ and } y=-\tfrac{1}{2}$$

$$y+z=0;\ \therefore\ z=+\tfrac{1}{2}$$

$$x-3z=0;\ \therefore\ x=+\tfrac{3}{2}.$$

Hence—

$$t = kr^{\frac{3}{2}}\mathrm{S}^{-\frac{1}{2}}\rho^{\frac{1}{2}} = k\sqrt{\frac{r^3\rho}{\mathrm{S}}}.$$

The value of k cannot be determined by the method of dimensions, but it has been found by the use of difficult mathematical analysis that k has the value $\pi/\sqrt{2}$, so that—

$$t = \frac{\pi}{\sqrt{2}} \cdot \sqrt{\frac{r^3\rho}{S}}.$$

Hence, if one drop has four times the radius of another drop of the same liquid, the period of oscillation of the larger drop is eight times as great as that of the smaller drop. For a drop of water 1 inch (2·54 cm.) in radius at 0° C, the time of vibration t is given by the equation—

$$t = \frac{3{\cdot}14}{1{\cdot}415} \sqrt{\frac{(2{\cdot}54)^3 \times 1}{76}} = 1{\cdot}03 \text{ sec.}$$

Lord Rayleigh has determined the surface tension of liquids by measuring the period of oscillation of a jet which issues from an elliptical orifice. The apparatus used is represented in Fig. 168. If the major diameter of the elliptical orifice is horizontal, the jet, when it issues from the orifice, is flattened vertically. The tension of the surface makes the section of the jet change toward the cylindrical form, then to become flattened horizontally, and so on; but as the liquid is moving horizontally with a velocity $\sqrt{2gh}$, where h is the "head" of the liquid (see chapter XII), the jet presents an appearance resembling a string of sausages. One oscillation is completed between two consecutive bulges, and the time required for the liquid to move over the distance between these bulges is known in terms of its velocity. The formula for the period of oscillation of the jet closely resembles that for the period of a drop; the dimensional investigation given on p. 346 applies equally well to the jet, and hence the period of oscillation, t, is given by the equation—

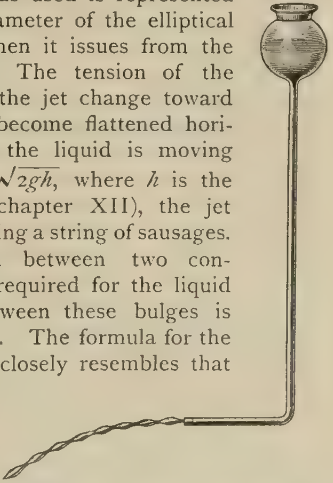


FIG. 168.—Experimental arrangement for the measurement of the surface tension of a jet of liquid.

$$t = k \sqrt{\frac{r^3\rho}{S}},$$

in which r is the radius of the jet where its section is circular.

Stability of any liquid film. When a wire rectangle, such as that represented in Fig. 288, is dipped into and then withdrawn from a liquid, in some cases a stable film is formed (for example, when the liquid is a soap solution), while in other cases a film cannot be formed (for example, when the liquid is water). Let it be supposed that the film is stable when the rectangle is placed in a vertical plane, with one side of the rectangle, of length b, horizontal; then, if S is the surface tension of the liquid, the latter exerts an upward force equal to $2Sb$ on the lower side of the rectangle, and an equal downward force is exerted on the film. Let m be the mass of the film; then, for the film to be in equilibrium, the upward force exerted upon its top horizontal edge must be equal to $2Sb+mg$. But if the value of the surface tension is constant for all parts of the film, the upward force exerted by the top side of the rectangle is $2Sb$; **equilibrium is impossible unless the surface tension is greater at the top than at the bottom edge of the film.** For pure water, the surface tension is very nearly constant, and therefore a water film more than two or three millimetres in length cannot be formed. A slight trace of grease will give the water a variable surface tension; if the surface tension at any point on the film is insufficient to produce equilibrium, the film stretches at this point, and the concentration of the grease is diminished, so that the surface tension increases automatically, and equilibrium is maintained.

If the surface tension of a soap solution is measured by any statical method, it is found to be much less than that of pure water. Lord Rayleigh measured the surface tension of soap solutions by the jet method described in the last section, and obtâined a value exactly equal to that of pure water. Hence it must be inferred that the freshly formed surface of a soap solution has the same tension as a water surface, but exposure to the atmosphere causes the surface tension of the soap solution to fall considerably. Even with pure water, Lord Rayleigh found that a freshly formed surface has a greater tension than one that has been exposed to the atmosphere. The great stability of a soap film is due to the wide variation in surface tension between freshly formed and long exposed parts of the surface; any stretching of the film, due to insufficient strength, immediately increases the surface tension.

Waves and ripples. Let Fig. 169 represent a vertical section of a wave travelling over the surface of a liquid of density ρ, in the direction of the straight line AB; the surface is supposed to be straight in a direction perpendicular to the plane of the paper. Let AQB be the level of the undisturbed surface of the liquid, and let PQ=h be the height of a point P of the wave surface, above the line AB.

If the amplitude of the wave were to increase slightly, so as to raise the point P through an infinitesimal distance δ, a small surface element near P would move upwards against a force aS/R, where a is the area of the element, S is the surface tension of the liquid, and R is the radius of curvature of the surface at P (p. 318); the work done in stretching the surface would be equal to $a\delta$S/R. Let it be supposed that the liquid required to fill the space $a\delta$, swept out by the displacement of the surface element, is obtained from the level AQB (compare p. 303); then the work done against gravity would be equal to $a\delta\rho gh$. The total work done, or the total increase in the potential energy, would be equal to $a\delta(g\rho h+\text{S/R})$. The value of 1/R is equal to $4\pi^2 h/\lambda^2$, where λ is the wave length of the wave curve (p. 336). Thus the increase in the potential energy would be equal to—

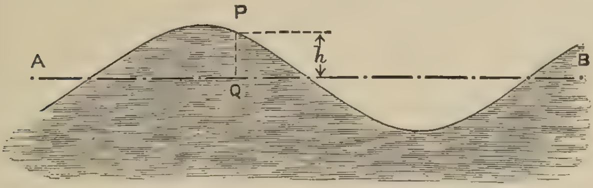


FIG. 169.—A wave on the surface of a liquid.

$$a\delta\left(g\rho h+\frac{4\pi^2\text{S}}{\lambda^2}h\right)=\rho\left(g+\frac{4\pi^2\text{S}}{\rho\lambda^2}\right)a\delta h.$$

When the crest of the wave rises the trough must sink; a small area a of the trough, originally at a distance h below AQB, would sink through a distance δ, and the work done would be equal to the value just found, if the displaced liquid were raised to the level AQB so as to replace the liquid previously removed from that level.

Thus, the effect of the surface tension is to increase the effective value of gravity by $4\pi^2\text{S}/\rho\lambda^2$. It will be proved in

Chapter XIV. that waves of length λ, under the action of gravity alone, travel over the surface of a liquid with a velocity V' given by the equation—

$$V' = \sqrt{\frac{\lambda g}{2\pi}};$$

hence, we may infer that, under the combined action of gravity and surface tension, the velocity V is given by the equation—

$$V = \sqrt{\frac{\lambda}{2\pi}\left(g + \frac{4\pi^2 S}{\rho\lambda^2}\right)}.$$

The quantity under the radical sign is the product of two terms, the first of which (namely, $\lambda/2\pi$) becomes infinitely great when λ is infinite, while the second (namely, $g + 4\pi^2 S/\rho\lambda^2$) decomes infinitely great when λ is infinitely small. Hence, a wave disturbance travels over the surface of a liquid with infinite velocity when the wave length is either infinitely great or infinitely small. Consequently, between these two extreme limits there must be some value, λ_1, of the wave length, for which the velocity has a minimum value; that is, as the wave length is diminished from infinity, the velocity diminishes and reaches its smallest value when the wave length is equal to λ_1; it then increases to infinity as the wave length approaches zero. In the neighbourhood of λ_1, the velocity will change very slowly with the wave length, and therefore the velocity will have equal values for the wave lengths λ_1 and $\lambda_1 + \delta$, if δ is very small. Thus—

$$\frac{\lambda_1}{2\pi}\left(g + \frac{4\pi^2 S}{\rho\lambda_1^2}\right) = \frac{\lambda_1 + \delta}{2\pi}\left(g + \frac{4\pi^2 S}{\rho(\lambda_1 + \delta)^2}\right),$$

$$\therefore \lambda_1 g + \frac{4\pi^2 S}{\rho\lambda_1} = (\lambda_1 + \delta)g + \frac{4\pi^2 S}{\rho(\lambda_1 + \delta)},$$

$$\begin{aligned} \therefore g\delta &= \frac{4\pi^2 S}{\rho} \cdot \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_1 + \delta}\right) \\ &= \frac{4\pi^2 S}{\rho} \cdot \frac{\lambda_1 + \delta - \lambda_1}{\lambda_1(\lambda_1 + \delta)} \\ &= \frac{4\pi^2 S}{\rho} \cdot \frac{\delta}{\lambda_1^2}, \end{aligned}$$

where δ is neglected finally in comparison with λ_1 in the denominator of the fraction on the right-hand side. Thus—

$$g = \frac{4\pi^2 S}{\rho\lambda_1^2}.$$

The minimum velocity of a wave disturbance, travelling over the surface of water, is equal to 23 cm. per sec., and the wave length corresponding to this velocity is 1·7 cm. A wave disturbance with a wave length less than that corresponding to the minimum velocity is said to consist of "ripples"; the propagation of ripples is chiefly due to surface tension.

When ripples travel over the surface of a liquid, their velocity is so great that they cannot be observed by the unaided eye. But, if they are viewed through a small aperture which is opened for an instant at intervals equal to the period of the ripples, then in each interval a ripple will travel over one wave length, and will acquire the position occupied by the preceding ripple at the commencement of the interval, and the ripples, as a whole, will appear to be stationary. Let a piece of tinfoil be fixed to each prong of a tuning fork, so that the two pieces of tinfoil overlap in the space between the prongs. If a small hole is pierced through both pieces of tinfoil where they overlap, this hole will be open whenever the prongs are in their position of equilibrium, but will be closed at other times during the vibration of the fork; that is, the aperture will be open twice in a complete vibration. Now, let ripples, generated on the surface of a liquid by a style which is attached to the prong of a vibrating tuning fork of frequency n, be viewed through holes in two pieces of tinfoil attached to the prongs of a fork vibrating with a frequency $n/2$; the ripples will appear to be stationary, and their wave length λ can be measured. The density ρ of the liquid being known, the value of the surface tension S can be calculated. This method was first employed with success by Lord Rayleigh.

MOLECULAR THEORY OF SURFACE TENSION.

Molecular forces.—Laplace developed a theory which explains surface tension in terms of the attraction exerted on a molecule of a liquid by the molecules in its immediate neighbourhood. A brief sketch of this theory has been given already (p 289). Some of the quantitative relations deduced by Laplace and his followers will now be obtained by simple mathematical methods.

It is assumed that a finite attraction is exerted between two molecules of a substance when they are separated by a distance which is less than a certain small value, say *c*. Hence, if a sphere of radius *c* be drawn around a molecule, this sphere will enclose all the molecules which attract, and are attracted by, the molecule at its centre. Hence, *c* is called the **radius of molecular attraction.** When a molecule is in the interior of a liquid, it is surrounded on all sides by attracting molecules, and therefore no resultant force, tending to move it in any particular direction, is exerted on it. When a molecule is near to the surface of the liquid the conditions are different. Let a molecule be at a point P (Fig. 170) below the surface AB of a liquid, and let its sphere of attraction extend above the surface into space which may be considered to be empty. Then the part of the sphere which lies above the surface contains no attracting molecules. At a distance below P, equal to the distance of P below the surface, draw a plane CD parallel to the surface; then the molecules lying between this plane and the surface exert no resultant force at P, and therefore the resultant force on the molecule at P is due to the attraction of the molecules lying within that part of its sphere of attraction which lies below the plane CD; this force tends to pull the molecule at P away from the surface into the interior of the liquid. Now, a molecule at P′, a point as far above the surface AB as P is below it, will be attracted toward the surface by the molecules which lie in that part of its sphere of attraction which lies below the plane AB; hence, we conclude that a molecule at a distance *d* above the surface of a liquid is pulled towards the surface with a force equal to that which, at a distance *d* below the surface, pulls a molecule into the interior of the liquid. Therefore, **the work done in bringing a molecule from the interior to the plane surface of a liquid, is equal to the work done in carrying a molecule from the surface into the space above the liquid, to a distance exceeding the range of molecular attraction of the molecule.**

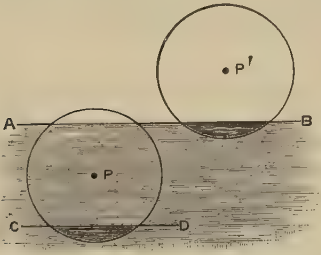


FIG. 170.—Molecules near to the surface of a liquid.

Latent heat of vaporisation.—We know so little about molecular forces, that it is useless to assume any law connecting the attraction between two molecules and the distance between them. If we suppose that the density of the liquid may vary, it is probable that the force acting on a molecule at a given distance from the surface is proportional to the density ρ of the liquid ; for, the number of molecules in the space between the plane CD (Fig. 170) and the sphere of attraction of the molecule at P, will be proportional to ρ. Let a molecule, of mass m, be carried away from the surface into the empty space above it, to a distance exceeding the radius of molecular attraction c; work will be done on the molecule only while it is being carried over the distance c from the surface. Let the distance c immediately above the surface be divided into n elements of length, each equal to c/n ; and let the force, exerted on the molecule while it traverses the first of these elements, be equal to $m\rho f_1$, while in traversing the remaining elements the force has the values $m\rho f_2$, $m\rho f_3 \dots m\rho f_n$, where $f_1, f_2, f_3 \dots f_n$ are in descending order of magnitude, and f_n is practically equal to zero. Then the work done in traversing the path c is equal to—

$$m\rho f_1 \frac{c}{n} + m\rho f_2 \frac{c}{n} + \dots \, m\rho f_n \frac{c}{n}$$

$$= m\rho \, . \frac{f_1 + f_2 + f_3 + \dots + f_n}{n} \, . \, c$$

$$= m\rho fc,$$

where mf denotes the average force exerted on the molecule while it traverses the distance c immediately above the surface. An equal amount of work is done in bringing a molecule from the interior to the surface of the liquid ; therefore, in bringing a molecule from the interior through the surface and beyond the sphere of attraction of the liquid, the work done is equal to $2m\rho fc$.

Let unit volume of the liquid comprise $\mathbf{n}$ molecules ; then the mass of unit volume of the liquid, or its density ρ, is equal to $\mathbf{n}m$; and if K denotes the work done in carrying $\mathbf{n}$ of the molecules from the surface into the space above the liquid, $K = \mathbf{n}m\rho fc = \rho^2 fc$.

In vaporising unit volume of the liquid, the molecules that occupy unit volume in the liquid state, are carried from the interior through the surface and beyond the sphere of attraction of the liquid ; and since these molecules together possess a mass ρ, the work done in the process is equal to $2\rho^2 fc = 2\mathrm{K}$. Then, if L denotes the latent heat per unit mass of the liquid—

$$\rho \mathrm{LJ} = 2\mathrm{K},$$

where J denotes the mechanical equivalent of unit quantity of heat.

Dupré neglected the work done in bringing a molecule from the interior to the surface of a liquid, and thus obtained the equation—

$$\rho \mathrm{LJ} = \mathrm{K};$$

it will be obvious that half the work done was neglected.

The latent heat of water at 100° C. is 537 gram-calories per gram of water evaporated, and the density of water at 100° C. is about 0·96 or, roughly, 1·0 gm. per c.c. From the 537 heat units required to vaporise a gram of water at 100° C. we must deduct[1] 39·7 heat units, which represent the external work done in forcing the atmosphere back, and we then find that the internal latent heat L is equal to 497·3, or roughly 500 gram-calories per gram. Thus—

$$\mathrm{K} = \frac{500 \times 4{\cdot}2 \times 10^7}{2} = 1{\cdot}05 \times 10^{10} \text{ ergs per c.c.}$$

The latent heat of water increases as the temperature of vaporisation falls, so that at 0° C., the value of K is $1{\cdot}26 \times 10^{10}$ ergs per c.c. This probably means that both f and c increase as the temperature falls.

Tensile strength of a liquid.—Let AB (Fig. 171) represent the section of an imaginary plane drawn in the interior of a liquid; the tensile strength of the liquid may be measured by the force, say F dynes per unit area of AB, necessary to separate the liquid above from that below AB.

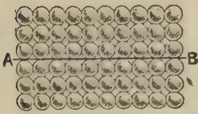


FIG. 171.—Tensile strength of a liquid.

The liquid below AB attracts all molecules which lie within a distance c above AB. The total force exerted across each square centimetre of AB is F, and therefore, if the liquid above AB is displaced upwards through a very small distance δ, the work done per unit area is $\mathrm{F}\delta$; and this is equal to the work done in overcoming the attraction, exerted by the liquid below AB, on the molecules comprised in a slice of unit area and thickness c lying immediately above AB. Let the thickness c of the slice comprise n layers of molecules, each layer consisting of N molecules per unit area. Then, if m is the mass of each molecule, the mass of the slice of unit area is nNm, and this is equal to $\rho \times 1 \times c = \rho c$ grams.

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[1] See the Author's *Heat for Advanced Students* (Macmillan), p. 369.

Let the force, exerted by the liquid below AB on each molecule in the first layer above AB, be equal to $m\rho f_1$, and for the second, third . . . nth layers let the force per molecule have the values $m\rho f_2$, $m\rho f_3$ . . . $m\rho f_n$. While the displacement δ is being produced between the liquid above and that below AB, each layer of molecules is displaced through a distance δ against the force which pulls it toward AB ; hence—

$$\begin{aligned} F\delta &= Nm\rho f_1\delta + Nm\rho f_2\delta + \dots \dots Nm\rho f_n\delta \\ &= nNm\rho \,.\, \frac{f_1+f_2+f_3+\dots\dots f_n}{n} \,.\, \delta \\ &= c\rho^2 f\delta = K\delta, \\ \therefore\ F &= K. \end{aligned}$$

Hence, **the tensile strength of a liquid is numerically equal to half the mechanical equivalent of the internal latent heat of vaporisation per unit volume of the liquid.**

Thus, for water at 100° C. the tensile strength should be equal to $1{\cdot}05 \times 10^{10}$ dynes per square centimetre. If a standard atmosphere is defined as 10^6 dynes per square centimetre, the tensile strength of water is equal to about 10,000 atmospheres. Since a ton is very nearly equal to a million grams, the force exerted by gravity on a ton is 10^9 dynes. Thus, the tensile strength of water should be equal to 10 tons per square centimetre. The tensile strength of steel pianoforte wire is 24 tons per square centimetre, and that of English steel wire about 10 tons per square centimetre. Thus, the tensile strength of water is about equal to that of ordinary steel wire, and half as great as that of pianoforte wire.

Laplace's theory was formulated before the kinetic theory of gases and liquids had been developed, and no account was taken of the motion of the molecules of the liquid. In accordance with the kinetic theory, the molecules of a liquid are not permanently in contact one with another, but each moves to and fro with great velocity, colliding continually with its neighbours. Now, collisions between molecules will occur in the plane AB (Fig. 171), and these collisions will tend to separate the molecules on opposite sides of that plane; that is, the motion of the molecules produces an internal pressure which tends to make the liquid expand, just as the attraction between the molecules tends to make the liquid contract to its smallest possible dimensions. Hence, during the separation of the liquid on one side of the plane AB from that on the other side, the work done is diminished owing to the motion of the molecules; and the higher the temperature of the liquid, the greater will be the velocity of the molecules, and the smaller will be the tensile strength of the liquid. The kinetic theory of liquids will be developed further in the last chapter of the present book; for the moment, it will be sufficient to notice that **the value of K gives the upper limit of the tensile strength of the liquid**, and that at high temperatures the tensile strength must fall short of this value.

Determination of K from van der Waals's equation.—A theory which postulates that the tensile strength of water is comparable with that of steel wire is so much at variance with popular preconceptions, that any independent confirmation of the result obtained is of the greatest value. Now, van der Waals[1] obtained an equation which expresses the relation between the pressure p and the volume v of a substance, not only when it is in the condition of vapour or gas, but also when it is in the liquid state. This equation has the form—

$$\left(p + \frac{a}{v^2}\right)\left(v - b\right) = \mathrm{RT}.$$

The term a/v^2 denotes the attraction, exerted across a plane of one square centimetre area, by the molecules on opposite sides of it; thus, when the substance is in the liquid condition, a/v^2 and K denote the same quantity. If the above equation refers to a gram of a substance, $1/v$ is equal to the density ρ of the substance; and the previous investigation (p. 353) shows that K is proportional to ρ^2 or $1/v^2$.

In order to determine the value of a/v^2 for water, it may be noted that b denotes a magnitude which is comparable with the volume actually occupied by the molecules comprised in one gram of water; hence b must approximate to 1 c.c., and therefore it can be neglected in calculations referring to water vapour.

For saturated water vapour at 100° C., the pressure p is equal to that of 76 cm. of mercury, and its value is therefore equal to $1{\cdot}013 \times 10^6$

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[1] See the Author's *Heat for Advanced Students* (Macmillan), pp. 308–309; also the last chapter of the present book.

dynes per sq. cm.; the volume[1] v of a gram of the saturated vapour is equal to 1672 c.c. Therefore, neglecting b—

$$\left(1{\cdot}013\times 10^6+\frac{a}{(1672)^2}\right)1672=373\text{R}.$$

At 60° C. the pressure of saturated water vapour is 14·96 cm. of mercury, or $1{\cdot}994\times 10^5$ dynes per sq. cm.; the volume of a gram of the saturated vapour is equal to 7671 c.c. Thus—

$$\left(1{\cdot}994\times 10^5+\frac{a}{(7671)^2}\right)7671=333\text{R}.$$

Dividing one equation by the other in order to eliminate R, and solving for a, we find that—

$$a=4{\cdot}7\times 10^{10}.$$

Now a gram of water at 100° C. occupies a volume v of 1·04 c.c. Thus, for water at 100° C.—

$$\text{K}=\frac{a}{v^2}=\frac{4{\cdot}7}{(1{\cdot}04)^2}10^{10}=4{\cdot}35\times 10^{10}\text{ dynes per sq. cm.}$$

This value of K is of the same order of magnitude as that obtained from the latent heat of water, and, everything considered, the confirmation of the theory is very striking. Van der Waals, using other data for water vapour, found the value $1{\cdot}05\times 10^{10}$ dynes per sq. cm. for a/v^2.

Surface tension.—If we separate the liquid above the plane AB (Fig. 171) from that below it, by a distance greater than c, we shall obtain two new surfaces, and the work done for each unit of area of AB will give twice the surface tension of the liquid, since the surface tension of a liquid is equal to the work done in extending its surface by unit area. Now, we have found already, (p. 355) that in separating the two parts of the liquid by a very small distance δ, the work done per unit area of AB is equal to Kδ. We cannot determine the work done in separating the two parts of the liquid by the relatively large quantity c, without a knowledge of the way in which the attraction exerted on a molecule depends on its distance from the plane AB. In the absence of this knowledge, we may assume that the force remains constant until the molecule reaches a distance c from the surface AB, and then falls abruptly to zero; by this means, although we shall be unable to effect an accurate calculation, we shall obtain some valuable information.

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[1] See the table of properties of steam, due to Professor Callendar, in the Author's *Heat for Advanced Students* (Macmillan), p. 469.

As the liquid above AB is displaced perpendicularly to that plane, one layer of molecules after another passes beyond the range of molecular attraction of the liquid below AB, until all the layers pass beyond that range when the displacement amounts to c. Therefore, during the displacement through the distance c, the average number of layers of molecules subject to the attraction of the liquid below AB is $n/2$ (compare p. 355), and therefore the work done is equal to $(K/2) \times c$. Thus, if S denotes the surface tension of the liquid—

$$2S = \frac{Kc}{2},$$

$$\therefore \; c = \frac{4S}{K}.$$

This equation gives the lowest possible limit of c, the radius of molecular attraction; for the molecular attraction unquestionably falls off as the distance from the plane AB increases, and the more quickly it falls off, the greater is the displacement that will give an amount of work equal to 2S. For water at 0° C. the value of S is about 76 dyne/cm., and we have already found that the value of K is $1 \cdot 26 \times 10^{10}$ dyne/(cm.)^2. Hence, for water at 0° C. the radius of molecular attraction is certainly greater than $304 \div (1 \cdot 26 \times 10^{10})$ cm., that is, it is greater than $2 \cdot 4 \times 10^{-8}$ cm., or 0·24 $\mu\mu$ (a micromillimetre, $\mu\mu$, is a millionth part of a millimetre). The above method of determining an inferior limit of c was first used by Young. With reference to the value obtained, it may be remarked that Johannot has measured black soap films which were no more than 6 $\mu\mu$ in thickness. The value 0·24 $\mu\mu$ may be considered to represent the major limit to the diameter of a molecule of water; for if the diameter of a molecule were greater than the radius of its sphere of attraction, there would be no attraction between molecules, even when they were in contact. Other methods of estimating the diameter of a molecule show that this magnitude lies between the limits 0·005 $\mu\mu$ and 0·5 $\mu\mu$.

Velocity of a molecule that can escape from the surface of a liquid.—It is probable that a molecule could not reach the surface of a liquid, from a depth c below it, without colliding a number of times with other molecules; at each collision the velocity of the molecule is changed, and therefore it would be impossible to predict, from the velocity of a molecule at a distance c below the surface, whether that molecule would or would not be able to escape from the surface. If a molecule leaves the surface normally with a velocity v, it will escape if its kinetic energy exceeds the work that must be done in order to carry it beyond the range of molecular attraction of the liquid. Let m be the mass of the molecule ; in carrying unit volume, or a mass ρ of the liquid, from the surface into the space above it, the work done is equal to K (p. 354), and therefore, in carrying a molecule of mass m from the surface, the work done is (m/ρ) K. Thus, the molecule will escape if—

$$\tfrac{1}{2}mv^2 > \frac{m}{\rho}\mathrm{K},$$

that is, if

$$v > \sqrt{(2\mathrm{K})/\rho}.$$

Therefore, for water at 0° C—

$$v > \sqrt{2 \times 1{\cdot}05 \times 10^{10}} = 1{\cdot}45 \times 10^5.$$

The average velocity of a hydrogen molecule[1] at 0° C. is equal to $1{\cdot}8 \times 10^5$ cm. per second. As the velocity of a molecule varies inversely as the square root of its mass, and the molecular weights of H_2 and H_2O are respectively equal to 2 and 18, it follows that the average velocity of a molecule of water vapour at 0° is equal to $(1{\cdot}8 \times 10^5) \div \sqrt{9} = 6 \times 10^4$ cm. per second. Thus, if the velocity of a water molecule has the same value in the liquid as in the gaseous state, it follows that a molecule cannot escape from the surface unless its velocity exceeds the average velocity of its neighbours.

Vapour pressure above the curved surface of a liquid.—When a molecule travels outwards from the surface of a liquid, it will escape if its kinetic energy is sufficient to carry it beyond the range of attraction of the liquid (p. 358) ; otherwise it will be pulled back into the liquid. At any point in its path, the molecule is pulled back towards the surface by those molecules of the liquid which lie within the sphere of its attraction ; hence, at a given distance from the surface, the force pulling the molecule back will depend on the shape of the surface of the liquid. If the surface is convex upwards the force will be less, and if the surface is concave upwards the force will be greater, than if the surface were plane (Fig. 172). Thus, it becomes evident that a molecule, moving with a velocity which would enable it to escape from a plane surface, might be unable to escape from a concave surface; while a molecule moving with a velocity too small to enable it to escape from a plane surface, might escape from a convex surface.

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[1] See the Author's *Heat for Advanced Students* (Macmillan), p. 296.

The rate of escape of molecules from the free surface of a liquid is influenced, in another and more important respect, by the curvature of the surface. On an average, the molecules of a liquid move with a constant velocity which depends only on the temperature of the liquid; therefore, the average frequency with which a molecule collides with its neighbours will be increased if the liquid is compressed, since a compression must diminish the space over which a molecule travels between two successive collisions. Now, the greater the frequency of collision between the molecules, the more frequently will molecules move outward across the free surface of the liquid; and it may be assumed that a certain fraction of the molecules that move outwards will possess a velocity sufficient to enable them to escape, so that it follows that the greater the compression of the liquid, the greater will be the number of molecules that escape in a given time. It must be remembered, however, that a very great pressure is needed in order to compress a liquid to any appreciable extent (p. 281). If the curvature of the surface is sufficient to produce a difference of pressure which compresses or distends the liquid to an appreciable extent, the rate of escape of molecules from the surface will be affected. If the surface is convex, the internal pressure will compress the liquid and augment the number of molecules that escape; if the surface is concave, the internal tension will distend the liquid and diminish the number of molecules that escape.

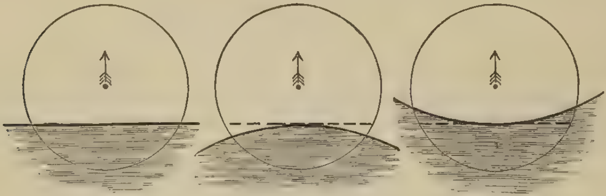

FIG. 172.—The escape of a molecule from the surface of a liquid.

Hence, it may be inferred that, **at a given temperature, the rate of escape of molecules from a liquid with a convex surface is greater than if the surface were plane ; and the rate of escape from a concave surface is less than if the surface were plane.**

If a closed vessel contains a liquid and its vapour, equilibrium will be attained when the number of molecules escaping from the liquid is equal to the number re-entering it per second. At a given temperature, the number of molecules re-entering the liquid per second is proportional to the vapour density above its surface, and the vapour density is proportional to the vapour pressure. Hence, the vapour pressure is proportional to the rate of escape of molecules from the surface of the liquid, and therefore **above a convex surface the vapour pressure must be greater, and above a concave surface it must be less, than that above a plane surface of the liquid.** Thus, if the vapour pressure above a plane surface of a liquid is equal to P, the vapour pressure above a curved surface will be equal to $(P \pm p)$, where the value of p depends on the curvature of the surface.

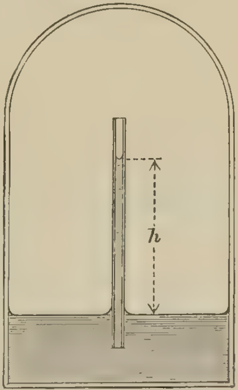

FIG. 173. — Vapour pressure above a curved liquid surface.

The value of p can be determined by the following train of reasoning due to Lord Kelvin. Let the lower end of a vertical capillary tube dip into a liquid contained in a closed vessel (Fig. 173), the space above the liquid being occupied only by the saturated vapour of the liquid. Let the internal radius of the tube be equal to R, and let the liquid wet the tube so that its surface within the tube stands at a height h above the plane surface of the liquid. Owing to the weight of the vapour, its pressure will be greater just above the plane surface of the liquid than at an height h above that surface. Let the pressure just above the plane surface be equal to P, while at a height h above the surface it is equal to $P - p$; then $p = g\sigma h$, where σ is the density of the vapour.

The surface of the liquid within the capillary tube will be practically hemispherical, since the hydrostatic pressure does not vary appreciably over that surface. Hence, since the radius of the hemisphere is equal to the internal radius of the tube, the pressure just below the hemispherical surface is less than that above it by 2S/R, where S is the surface tension of the liquid (p. 314). Thus, the pressure just below the hemispherical surface is equal to $\{P - p - (2S/R)\}$.

At a point within the tube, in the same horizontal plane as the flat surface of the liquid, the pressure must have the same value as in that surface; that is the pressure must be equal to P. But the point in question is at a distance h below the hemispherical surface of the liquid in the tube; therefore at this point the pressure must exceed that just below the hemispherical surface by $g\rho h$, where ρ is the density of the liquid. Hence—

$$P - p - \frac{2S}{R} + g\rho h = P,$$

$$\therefore \ p = - \frac{2S}{R} + g\rho h$$

$$= - \frac{2S}{R} + g\rho \cdot \frac{p}{g\sigma},$$

since $p = g\sigma h$. Therefore—

$$p\left(\frac{\rho}{\sigma} - 1\right) = \frac{2S}{R},$$

and

$$p = \frac{\sigma}{\rho - \sigma} \cdot \frac{2S}{R}.$$

Thus, **for equilibrium to exist between a liquid with a concave surface and the vapour above it, the vapour pressure just above the surface must be less than that above a plane surface of the same liquid by** $\{\sigma/(\rho - \sigma)\}(2S/R)$. If the vapour pressure above the concave surface exceeds $(P - p)$, the rate at which vapour molecules enter the surface will be greater than that at which they leave the surface, and progressive condensation will occur. If the vapour pressure above the surface is less than $(P - p)$, progressive evaporation will take place.

Now let it be supposed that the liquid does not wet the tube; for instance, if the liquid is water or an aqueous solution, it may be supposed that the tube is coated internally with paraffin wax. The surface within the tube will now be a hemisphere with its convex side upwards, and the surface will be at a distance h below the general surface of the liquid; consequently, the vapour pressure above the convex surface will be equal to $(P + p)$, where p has the value already obtained.

Thus, if a liquid has a convex surface, equilibrium between it and its vapours can exist only if the pressure of the vapour is equal to $(P + p)$. Progressive evaporation will occur if the pressure of the vapour is less than $(P + p)$, and progressive condensation will occur if the vapour pressure is greater than $(P+p)$. Thus, if a small drop of water lies just above the plane surface of water in a vessel, equilibrium is impossible; for, if the vapour is in equilibrium with the plane surface of the water, the vapour pressure is not large enough to produce equilibrium with the drop, and therefore the drop will evaporate and condensation will occur at the plane surface. On the other hand, porous substances which can be wetted by water always tend to condense aqueous vapour in their pores; each pore is virtually a very fine tube, and a water surface formed in such a tube will be concave outwards, so that if the surrounding space is saturated with aqueous vapour, the vapour pressure within the pores is too high for equilibrium to exist. The tendency of cotton and linen fabrics to become damp is due to a great extent to this cause.

The vapour pressure necessary to produce equilibrium with a drop of water of any size can be calculated easily. Let the radius of the drop be 0·001 mm., and let the temperature be 0° C.; then the density σ of saturated aqueous vapour at 0° is equal to $4·8 \times 10^{-6}$ gm. per c.c., and the density ρ of water is practically equal to unity. Thus, since S is equal to 76 dynes per cm.—

$$p = \frac{4·8 \times 10^{-6}}{1} \times \frac{2 \times 76}{10^{-4}} = 7·3 \text{ dynes per sq. cm.}$$

The pressure P of saturated aqueous vapour at 0° C. is equivalent to 4·6 mm. of mercury, or 6×10^{3} dynes per sq. cm. Hence, in this case the vapour pressure above the drop is greater than that over a plane surface by about one part in a thousand. If, however, the radius of the drop were 1×10^{-6} mm. (1 $\mu\mu$), equilibrium could only exist if the vapour pressure above it were $7·3 \times 10^{3}$ dynes per sq. cm. in excess of the pressure of normal saturated vapour.

Conditions necessary for the condensation of vapour.—From the reasoning used above, it appears that a very small drop of water would evaporate if it were placed in a space saturated with aqueous vapour; hence, if a space is filled with saturated aqueous vapour, it must be impossible for a few vapour molecules to combine to form a very small drop of water, for if such a drop were formed it would evaporate immediately. Hence, for aqueous vapour to condense, one or other of the following conditions must be complied with:—

1. Condensation may occur in porous bodies, even when the vapour is unsaturated.

2. Condensation may take place on a flat or nearly flat surface when the vapour is saturated.

3. If the initial stage of condensation involves the formation of very minute drops, then the vapour must be super-saturated before such drops can be formed.

If compressed air, contained in a vessel, is allowed to expand suddenly, a fall of temperature is produced; if the air is saturated with aqueous vapour before the expansion, it must be super-saturated afterwards. Aitken found that if the air contains small dust particles, a cloud is formed when the expansion occurs; the dust particles form nuclei around which the vapour can condense to form drops which are initially of finite dimensions. If the mist is allowed to settle, the dust particles are carried down by the water drops formed around them; if the air is again compressed, without allowing dusty air to enter the vessel, another expansion will remove the dust particles left after the first one. When the air is once free from dust particles, a small expansion will produce no cloud: the nuclei being absent, the air is not super-saturated to a sufficient extent for condensation to occur. If the volume of the air is increased suddenly to 1·4 times its original value, condensation is produced even when the air is free from dust; in this case the super-saturation is sufficient to admit of the formation of drops which at first are very small. Clouds and mists are aggregates of minute water drops which, in general, are formed around dust particles as nuclei. When the nuclei are absent, the air may be super-saturated without the formation of a mist; in this case the vapour often condenses as dew on solid bodies.

Relation between surface tension and temperature.—If the molecular weight of a substance be divided by the density of that substance, we obtain a magnitude which is proportional to the average space allotted to each molecule of the substance; let this be denoted by V. Eötvos found the value of the product SV^3 for a number of different substances, and observed that the rate at which this product decreases per unit increase of temperature is constant for all those substances, and has the value 2·1. If we apply this result to water it is found that between 100° and 200° C. the rate of decrease of $SV^{\frac{2}{3}}$ is the same as for other substances, provided we assume the molecular weight of water in the liquid state to be 36 instead of 18. Hence, it was concluded that between 100° and 200° C. each molecule of water in the liquid condition has the composition $2H_2O$. For temperatures between 0° and 100° C., the composition of water must be nH_2O, where n is greater than 2.

From the result obtained by Eötvos, it follows that for any substance—

$$SV^{\frac{2}{3}} = 2\cdot 1(T - t),$$

where t denotes the centigrade temperature of the substance and T is a constant for that substance. When $t = T$ the surface tension S must have zero value; hence, we may conclude that T approximates to the critical temperature of the substance. For ether, the value found for T is 180° C., and van der Waals estimated the critical temperature of ether to be 190° C. For alcohol the value of T is 295° C., and the value of the critical temperature is 256° C. For water the value of T is 560° C., and the critical temperature is 390° C.

QUESTIONS ON CHAPTER X

1. Calculate the difference between the pressures inside and outside a spherical soap bubble of 1 cm. radius, if the surface tension of the soap solution is ·45 dyne/cm.

2. The interiors of a cylindrical and a spherical soap bubble are put into communication; determine the ratio of the radii of the sphere and the cylinder, in order that the bubbles may be in equilibrium.

3. A soap bubble of 10 cm. radius, is filled with air at 20° C., the barometric pressure being equal to that of 76 cm. of mercury. Calculate the mass of air that is expelled when the bubble contracts until its radius is equal to 5 cm.

(Density of dry air at 0° C. and a pressure of 76 cm. of mercury = 0·00129 gm. per c.c. Density of mercury = 13·6 gm. per c.c. Vapour pressure of soap solution at 20° C. = 17·4 mm. of mercury.)

4. One gram of mercury is placed between two plane sheets of glass which are pressed together until the mercury forms a circular disc of uniform thickness, and 5 cm. radius. Calculate the value of the pressure exerted by the mercury on the upper glass plate. (Density of mercury = 13·6 gm. per c.c. Surface tension of mercury = 450 dyne/cm. Angle of contact between mercury and glass = 140°.)

5. A circular flat plate has a radius of 5 cm. Calculate the value of

the force that is needed to pull this plate away from the surface of water.

(Surface tension of water = 70 dyne/cm.)

6. A drop of water of great lateral dimensions, rests on a flat horizontal surface which it does not wet. Determine, by a graphical method, the form of the profile of the drop.

(Surface tension of water = 70 dyne/cm.)

7. A flat plate is placed upright in water. Determine, by a graphical method, the section of the surface of the water in the neighbourhood of the plate, (*a*) when the water wets the plate, and (*b*) when it does not wet the plate.

8. A flat plate is placed upright in a liquid, and is not wetted. Prove that the horizontal force exerted on the plate is exactly the same as if the surface met the plate at right angles.

9. Determine the diameter of the largest steel needle that can be supported by the surface of water, and draw a diagram showing the position of the needle relatively to the surface of the water.

(Density of steel = 7·7 gm. per c.c. Surface tension of water = 70 dyne/cm.)

10. A steel needle, 0·4 mm. in diameter, floats on the surface of water: determine its position relative to the surface of the water.

11. Water trickles slowly over a spherical surface, which it wets, and forms drops which break away below the sphere; by the method of dimensions, determine the way in which the mass of a drop depends on the radius of the sphere.

12. A small drop of water has a radius equal to r cm.; calculate the decrease in the surface energy of the drop when its radius is diminished by a small quantity, δ cm., due to evaporation; and find the condition that the surface energy lost may suffice to provide the latent heat necessary for the evaporation.

13. Calculate the radius of the largest drop of water that can evaporate at 0° C. without heat being communicated to it.

(Surface energy of water at 0° C. = 117 ergs/(cm.)^2. Latent heat of evaporation of water at 0° C. = 606 gram-calories per gram.)

CHAPTER XI

THE MOTION OF FLUIDS

Viscous and inviscid fluids.—When a portion of matter cannot retain its shape without lateral support, it is said to be fluid ; in other words, matter is fluid if it needs a containing vessel to prevent it from flowing. The rate at which a fluid flows, when it is deprived of lateral support, varies with the nature of the fluid. That property of a fluid which retards its flow is called *viscosity*,[1] and a fluid which cannot flow quickly is said to be *viscous* or *viscid.* All material fluids are viscous, to a greater or less degree ; but in some fluids the viscosity is so much less than in others, that it is convenient to divide them into two classes ; one class comprises gases and vapours, as well as the liquids ether, water, mercury, etc., which are said to be *relatively inviscid* : the other class contains substances such as treacle, pitch etc., which are said to be *viscous.* In this and the next chapter the properties of inviscid fluids will be analysed ; later, the properties of viscous fluids will be dealt with. Although no fluid is absolutely inviscid, many fluids possess so little viscosity that their flow differs but little from that of an ideal inviscid fluid ; hence, the results derived in this and the next chapter will apply, with a close degree of approximation, to those material fluids which are classed as relatively inviscid. The approximation will be closest when the fluid is of great volume, and will be less exact when the volume of the fluid is small ; thus, a large piece of pitch flows perceptibly in the course of a day, while a small fragment of pitch can retain its shape almost unaltered for an indefinite time.

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[1] Latin *viscum*, the mistletoe ; the berries of this plant contain a sticky liquid.

Linear motion of a fluid as a whole.—When a closed vessel is entirely filled with a fluid, the motion of the whole in a straight line is subject to the laws of mechanics developed in Chapter I.

EXPT. 48.—Fill a glass flask with water, and, without disturbing the water, introduce a drop of the aniline dye known as soluble blue; then cork the flask, and leave it for some hours. It will be found that the blue colouring matter has spread into a wonderful tree-like form, each "twig" ending in a swelling, shaped like a mushroom. The flask may be moved in a straight line without distorting the tree-like form, provided that the water entirely fills the flask up to the cork. In such cases the liquid behaves like a rigid solid; that is, there is no relative motion of its parts.

Let a liquid be moving in a horizontal straight line, in such a manner that there is no relative motion of its parts, while its velocity as a whole varies from instant to instant. Within the liquid, let a small imaginary circular cylinder be described, the area of its base being a, while its length is l; and let the axis of the cylinder lie in the direction of motion. The pressure acting on the curved surface of the cylinder is perpendicular to the direction of motion, and therefore cannot influence the motion. Let the density of the liquid be ρ; then the mass of the liquid contained within the imaginary cylinder is equal to ρal, and when its acceleration is equal to α, it must be acted upon by a resultant force, in the direction of the acceleration, equal to $\rho al\alpha$ (p. 19). Let p_1 be the pressure on the rear surface and p_2 that on the front surface of the cylinder; then the resultant horizontal force acting in the direction of motion is equal to $(p_1 - p_2)\,a$. Hence—

$$(p_1 - p_2)a = \rho al\alpha.$$

$$\therefore \quad \frac{p_1 - p_2}{l} = \rho\alpha.$$

Thus, **along any straight line in the direction of motion, the pressure must vary; the pressure gradient** (increase of pressure per unit distance) **is equal to $\rho\alpha$, the direction of increase of pressure being opposite to that of the acceleration.**

Let the upper surface of the liquid be free, the atmospheric pressure to which it is exposed being equal to P dynes per sq. cm. Then, at a distance h below the surface, the pressure is equal to $(P+g\rho h)$ (p. 34). For the pressure to vary along a horizontal straight line, the free surface must be inclined to the horizontal; if the end of the cylinder, on which the pressure is p_1, is at a distance h_1 below the free surface, then the opposite end of the cylinder, on which the pressure is p_2, must be at a different distance h_2 below the surface (Fig. 174), and—

$$\frac{p_1-p_2}{l} = g\rho\frac{h_1-h_2}{l} = \rho a.$$

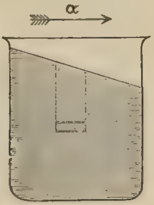

FIG. 174.—Vessel and its contained liquid, moving in a straight line with an acceleration a.

The free surface of the liquid slopes downwards in the direction of the acceleration, and its inclination θ to the horizontal is given by the equation—

$$\tan\theta = \frac{h_1-h_2}{l} = \frac{a}{g}.$$

If the velocity of the liquid is uniform, $a=0$, and the free surface is horizontal. If the velocity is increasing, a is positive, and the surface is inclined to the horizontal, and slopes downwards in the direction of motion; if the velocity is decreasing, a is negative, and the surface slopes upwards in the direction of motion. Thus, a glass of water supported in a railway-carriage will indicate every alteration in the velocity of the train, provided that the jolting of the carriage is inappreciable.

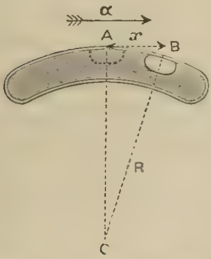

FIG. 175. — Measurement of acceleration by the aid of a spirit level.

An ordinary engineer's spirit level can be used to measure the acceleration of a vehicle, provided that there is little or no jolting. A spirit level consists of a glass tube, bent into an arc of a circle of large radius, say R; the tube is closed, and contains a liquid (such as alcohol, or ether[1]) and a bubble of air (Fig. 175). When the spirit level is stationary, the bubble rises to the highest possible point; let this point be A, in the middle of the tube. If we draw the radius of curvature AC, this will be vertical, and it indicates the direction of maximum increase of pressure within the liquid. The pressure increases in a vertical direction, at the rate of $g\rho$ dynes per unit distance, measured downwards. When the spirit level is moving, with acceleration α, in the direction of the arrow, there will be an additional increase of pressure in a horizontal direction, from right to left, at the rate of $\rho\alpha$ dynes per unit distance; hence the direction of maximum increase of pressure will now be parallel to BC, where—

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[1] These liquids are less viscous than water, and thus the bubble can move very quickly when its position of equilibrium is altered.

$$\tan \angle \mathrm{BCA} = \frac{\rho\alpha}{g\rho} = \frac{\alpha}{g}.$$

The bubble will move to a position B, such that the radius of curvature BC is parallel to the direction of maximum increase of pressure. Hence, if $\mathrm{R} = \mathrm{AC}$ is large, and x is the horizontal component of the displacement of the bubble—

$$\frac{x}{\mathrm{R}} = \frac{\alpha}{g}; \quad \therefore \alpha = g\frac{x}{\mathrm{R}}.$$

Rotation of a fluid as a whole.—Let the flask containing the "tree" of colouring matter described on p. 368 (expt. 48) be rotated slowly about a vertical axis. At first it will be observed that the "tree" remains stationary in space; hence, the water does not at once acquire the rotational motion of the vessel which contains it. After a time, those "twigs" which are nearest to the walls of the flask become bent in the direction of motion of the latter, and this bending gradually increases and extends to twigs nearer to the centre of the flask, until finally the whole of the contained water rotates at the same rate as the flask, when no further deformation of the "tree" is produced. In this condition the water is rotating like a solid body, the relative positions of any two particles remaining unaltered. If the rotation of the flask is now stopped, it will be observed that the contained water continues to rotate for some time, the part which is nearest to the walls coming to rest first, and the central portion last.

When a liquid with a free surface rotates as a whole, the surface becomes curved, its lowest point lying on the axis of rotation. This can be observed when tea in a cup is set in rotation by a spoon; it is due to the centrifugal force which acts on each revolving particle of water, and tends to carry it away from the axis of rotation: equilibrium can be attained only when motion away from the axis is opposed by a pressure gradient, which urges each particle of water toward the axis.

Let Fig. 176 represent a vertical section of a cylindrical vessel which, together with the contained liquid of density ρ, is rotating uniformly about the axis AB of the vessel. Describe an imaginary horizontal tube CD, of very small sectional area a, so that its axis cuts AB at right angles immediately beneath the lowest point of the free surface of the liquid. From this tube, cut off an element by means of planes, perpendicular to CD, at distances x_a and x_b from the axis of rotation; then the mass m of liquid contained by this element is equal to $\rho a(x_b - x_a)$; and in order that this mass may revolve in a circle about AB, it must be urged toward the axis AB by a horizontal force equal to $m(x_b + x_a)\omega^2/2$, where $(x_b + x_a)/2$ is the distance of its centre of gravity from the axis, and ω is the angular velocity of the liquid (p. 74).

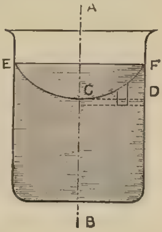

FIG. 176. — Cylindrical vessel and its contained liquid, rotating uniformly about its axis of symmetry.

Let y_a and y_b be the vertical distances of the ends of the element below the free surface of the liquid; then the difference between the pressures on the two ends of the element is equal to $g\rho(y_b - y_a)$, and therefore the resultant force urging the liquid contained by the element toward the axis of rotation is equal to $g\rho(y_b - y_a)a$. Thus—

$$g\rho a(y_b - y_a) = \rho a(x_b - x_a)(x_b + x_a)\frac{\omega^2}{2};$$

$$\therefore\quad y_b - y_a = \frac{\omega^2}{2g}(x_b^2 - x_a^2).$$

Divide the tube CD into elements, by means of planes at distances $x_0, x_1, x_2, x_3, \ldots x_n$ from the axis; and, at these distances from the axis, let the heights of the free surface of the liquid above the tube CD be equal to $y_0, y_1, y_2, \ldots y_n$. Then we have—

$$\begin{aligned} y_1 - y_0 &= \frac{\omega^2}{2g}(x_1^2 - x_0^2), \\ y_2 - y_1 &= \frac{\omega^2}{2g}(x_2^2 - x_1^2), \\ \ldots\ldots &= \ldots\ldots\ldots \\ y_n - y_{n-1} &= \frac{\omega^2}{2g}(x_n^2 - x_{n-1}^2). \end{aligned}$$

Then, (compare p. 49)—

$$y_n - y_0 = \frac{\omega^2}{2g}(x_n^2 - x_0^2).$$

Now, let $x_0 = 0$; then $y_0 = 0$. Writing y for y_n, and x for x_n, we obtain the equation to the section of the surface by the plane of the paper in the form—

$$y = \frac{\omega^2}{2g} \cdot x^2.$$

This represents a **parabola,** of which AB is the axis; hence, since all sections through the axis of rotation will have identical shapes, the surface has the form of a **paraboloid of revolution.** This surface could be generated by rotating the parabola ECF about the axis AB.

Lateral pressure, and longitudinal tension, of a rotating cylinder of liquid.—It is clear from the above investigation, that the pressure acting on the curved walls of the cylinder, in an outward direction, is greater than if the liquid were at rest.

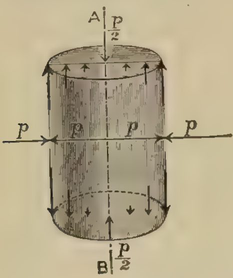

Fig. 177.—Liquid cylinder, rotating uniformly about its axis of symmetry.

Let Fig. 177 represent a cylindrical portion of liquid, rotating as a whole about the axis of symmetry AB, with an angular velocity ω. Describe an imaginary tube, of small sectional area a, which extends perpendicularly from the axis of symmetry to the curved surface of the cylinder. The liquid, contained by an element of this tube which lies between distances x_b and x_a from the axis, must be urged toward the axis by a force equal to $\rho a(x_b^2 - x_a^2)\omega^2/2$. Let p_b and p_a be the pressures on the ends of this element; then—

$$(p_b - p_a)a = \frac{\rho a \omega^2}{2} \cdot (x_b^2 - x_a^2).$$

Divide the imaginary tube into elements, by planes at distances $x_0, x_1, x_2, \ldots x_n$ from the axis, and let the pressures at these sections be equal to $p_0, p_1, p_2, \ldots p_n$. Then—

$$p_1 - p_0 = \frac{\rho\omega^2}{2}(x_1^2 - x_0^2),$$

$$p_2 - p_1 = \frac{\rho\omega^2}{2}(x_2^2 - x_1^2),$$

$$\ldots\ldots = \ldots\ldots$$

$$p_n - p_{n-1} = \frac{\rho\omega^2}{2}(x_n^2 - x_{n-1}^2);$$

$$\therefore\ p_n - p_0 = \frac{\rho\omega^2}{2}(x_n^2 - x_0^2).$$

Now let $x_0 = 0$, while $x_n = r$, the radius of the rotating cylinder of liquid. Then $(p_n - p_0)$ gives the difference between the pressure at a point on the axis of rotation, and that at a point in the same cross section but lying on the curved surface of the cylinder; that is, it gives the additional pressure on the curved surface of the cylinder, due to the rotation of the liquid: call this p. Then—

$$p = \frac{\rho\omega^2}{2} r^2.$$

Hence, **for the rotating cylinder of liquid to be in equilibrium, its curved surface must be subjected to a uniform pressure $p = (\rho\omega^2/2)r^2$, in addition to the pressure that would produce equilibrium if the liquid were at rest.** In other words, the rotation of the liquid generates a lateral pressure $p = (\rho\omega^2/2)r^2$, which tends to make the cylinder expand radially; when the liquid is in equilibrium, this pressure must be balanced by an equal pressure acting inwards on its curved boundary.

If the liquid had a free upper surface, this would be pulled downwards in the neighbourhood of the axis of rotation; hence we may conclude that the rotation of the liquid produces a longitudinal tension which tends to shorten the cylinder.

Let the cylinder of liquid be bounded by plane circular ends, perpendicular to the axis of symmetry. Divide one of these ends into strips by means of concentric circles of radii $r_0, r_1, r_2, \ldots r_n$. Then the strip bounded by the circles of radii r_0 and r_1 has an area equal to $\pi(r_1^2 - r_0^2)$, and is acted upon by an internal pressure equal to $(\rho\omega^2/2)\mathbf{r}^2$, where $\mathbf{r}^2$ denotes the average value of r^2 between the limits r_0 and r_1; we may write $\mathbf{r}^2 = (r_0^2 + r_1^2)/2$ (compare p. 48). Then the force f_1 acting on this strip, tending to lengthen the cylinder, is given by the equation—

$$\begin{aligned} f_1 &= \pi(r_1^2 - r_0^2)(r_1^2 + r_0^2)\frac{\rho\omega^2}{4}, \\ &= \frac{\pi\rho\omega^2}{4}(r_1^4 - r_0^4). \end{aligned}$$

Similarly, if $f_2, f_3, \ldots f_n$, denote the forces acting on the remaining strips, we have—

$$\begin{aligned} f_2 &= \frac{\pi\rho\omega^2}{4}(r_2^4 - r_1^4), \\ \cdot\ \cdot &= \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ f_n &= \frac{\pi\rho\omega^2}{4}(r_n^4 - r_{n-1}^4). \end{aligned}$$

Let F denote the resultant force acting on one of the plane ends of the cylinder, tending to lengthen the cylinder. Then—

$$F = f_1 + f_2 + \dots + f_n,$$

Thus—
$$F = \frac{\pi\rho\omega^2}{4}(r_n^4 - r_o^4) = \frac{\pi\rho\omega^2}{4}r^4,$$

since $r_n = r$, while $r_o = 0$.

Let the cylinder be maintained in equilibrium by applying a uniform external pressure P to its plane ends, which may be supposed to be bounded by rigid circular discs. Then $F = \pi r^2 P$, and therefore—

$$P = \frac{\rho\omega^2}{4}r^2$$

Thus $P = \frac{p}{2}$; that is, the pressure applied to the ends of the cylinder must be only half as great as that applied to its curved surface. If we suppose the rotating cylinder of liquid to be subjected, as a whole, to a uniform external pressure $p = (\rho\omega^2/2)r^2$, while a tensile stress $P = (\rho\omega^2/4)r^2$ is applied to its plane ends, then the cylinder will be in equilibrium; for the curved surfaces are subjected to a pressure equal to $(\rho\omega^2/2)r^2$, while the plane ends are subjected to a pressure $(\rho\omega^2/2)r^2$, together with a tensile stress equal to $(\rho\omega^2/4)r^2$, so that the resultant applied pressure, tending to shorten the cylinder, is equal to $(\rho\omega^2/4)r^2$, and this is equal to the internal pressure tending to lengthen the cylinder.

Vortex filaments.—A vortex[1] filament is a cylindrical portion of fluid, rotating like a solid with uniform angular velocity, in the midst of the surrounding fluid. The surrounding fluid is in motion, but it moves without rotation. The characteristics of irrotational motion will be studied subsequently; at present, we may confine our attention to the conditions essential to the rotational motion of the vortex filament.

In the first place, a vortex filament will be in equilibrium if its boundaries are subjected to a uniform pressure equal to $(\rho\omega^2/2)r^2$ while its ends are subjected to a tensile stress equal to $(\rho\omega^2/4)r^2$. From this it follows that a vortex filament cannot be permanent, unless its ends lie on the bounding surface of the surrounding fluid; for it is only in this case that the tensile stress, necessary to the equilibrium of the filament, can be applied to its ends.

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[1] Latin *vertere*, to turn. Strictly speaking, a vortex filament should be infinitely small in diameter; Lord Kelvin used the expression "columnar vortex" for a vortex of finite diameter. In this book, the expression "vortex filament" will be used for vortices of all diameters.

If one end of a vortex filament lies on the free surface of a liquid, the surface is pulled downwards as we have seen already (p. 371). In rowing, a vortex filament is often formed by a stroke of an oar. The surface of the water is drawn downwards, and air bubbles often lie along the core of the vortex; but the filament is not permanent, since its lower end lies within the water, and thus it is continually drawn upwards until it reaches the surface. When the ends of a vortex filament lie on solids, the adhesion between the liquid and the solids supplies the necessary tension; the longitudinal tensile stress of the filament tends to draw the solids towards each other, and hence the solids apparently attract each other.

According to the modern electro-magnetic theory due to Faraday and Maxwell, the attraction between dissimilarly electrified bodies is due to tubes of force which extend from the positively to the negatively charged body. These tubes of force tend to expand laterally and to contract longitudinally, and so far resemble vortex filaments, which might be supposed to be formed in the ether between the charged bodies. But in the tubes of force, the longitudinal tension is equal to the lateral pressure; or we may consider that the medium is subjected to a general hydrostatic pressure p, combined with a longitudinal tensile stress numerically equal to $2p$. In the case of a vortex filament, if the hydrostatic pressure is equal to p, the longitudinal tensile stress is equal to $p/2$.

The winding of rivers.—When a liquid rotates within a stationary cylindrical vessel, the liquid immediately in contact with the walls is motionless, so that on proceeding outwards from the axis of symmetry, the velocity of the liquid at first increases, reaches a maximum value at some distance from the axis, and then decreases toward zero as the walls of the vessel are approached. Hence, the surface of the liquid will possess a shape somewhat like that represented in Fig. 178; it is lowest where it is cut by the axis, and rises toward the walls of the vessel, but it meets the walls almost at right angles, since the water near the walls

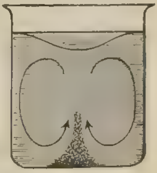
FIG. 178.—Liquid rotating in a stationary cylindrical vessel.

is almost stationary, and no pressure gradient is required to keep it in equilibrium. The shape of the surface indicates how the pressure varies beneath it. Near to the bottom of the vessel the liquid is stationary, while the pressure gradient would suffice to keep it in equilibrium if its motion were the same as that immediately beneath the free surface. Hence, the pressure gradient causes a flow along the lower surface of the vessel, from the walls of the vessel toward the axis; and to supply this flow, the liquid near to the free surface streams away from the axis, and descends along the side walls of the vessel.

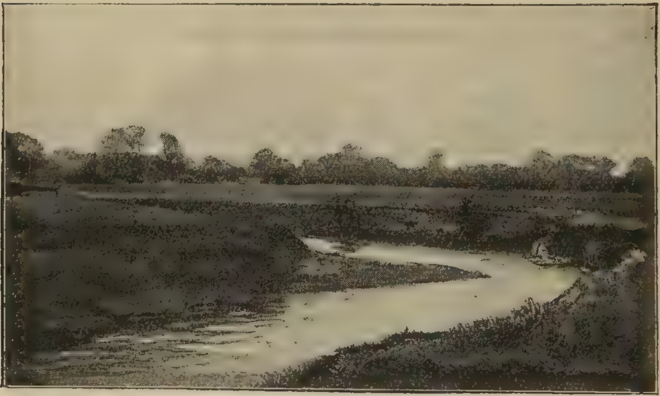

FIG. 179.—A typical river bend in a flat country.

EXPT 49.—Drop some powdered chalk into a beaker which contains water, and then set the water in rotation by stirring it with a rod; it will be seen that the particles of chalk are carried toward the axis by the radial flow described above, and are heaped up there (Fig. 178).

The winding of rivers is proverbial; the sinuous course of the river Meander in Asia Minor, has caused the name of that river to be transformed into a verb expressive of devious motion. Many causes may be assigned for this winding; but in flat countries it is due in great part to the principle explained above.

If there is a small bend in a river, the water must follow a curved path in rounding it, and the surface becomes somewhat higher on the concave than on the convex side of the bend. Near the bed of the river, the water is stationary ; and thus there is a continual flow, downward on the concave side, and across the river bed to the convex side of the bend. This flow carries away the substance of the banks on the concave side, and deposits it as mud on the convex side of the bend. A characteristic form of river bend is represented in Fig. 179 ; the convex side is shallow and muddy, while the concave side is deep and the neighbouring bank is undermined. Any bend in a river tends to increase continually ; and, in a straight part of a river, any accidental circumstance, such as the falling in of the banks, may produce an incipient bend, which becomes more and more pronounced as time goes on.

Sources and sinks.—Let it be supposed that, at certain points in a large volume of fluid, there is a continuous and uniform creation of fluid ; while at other points there is a continuous and uniform annihilation of fluid. The points at which fluid is created are called **sources**, while those at which it is annihilated are called **sinks.**

These conceptions are ideal, since we have every reason to believe that matter can be neither created nor destroyed ; but it is easy to devise a means by which a state of affairs, in all essential points identical with that imagined, may be realised. For instance, by means of narrow tubes, fluid may be conducted from an external reservoir to each source, and discharged there into the surrounding fluid ; and in a similar manner, fluid may be conducted away from each sink and discharged outside the volume of fluid which we are considering. In this case, the tubes will modify, to a slight extent, the motion of the surrounding fluid ; but if the tubes are very narrow, the modification will be insignificant.

The creation of fluid at the sources, and its annihilation at the sinks, will produce motion in the surrounding fluid ; and at any given point, the velocity of the fluid will have a definite value. The conditions which govern this flow must now be considered.

In the first place, within any element of volume which does not contain a source or a sink, there will be neither creation nor annihilation of fluid. Hence, **in an incompressible fluid, the volume of fluid that enters an element of volume that does not contain a source or sink, is equal to the volume that simultaneously leaves it.**

Imagine an incompressible fluid to be flowing uniformly along a cylindrical straight tube; then if the area of the tube is a, and the uniform velocity of the fluid is V, it follows that the volume of the fluid that passes through any cross section in a second is equal to Va. For, in a second, each particle of the fluid moves through a distance V, and therefore every particle that was at a distance less than V behind the section at the beginning of the second, will pass through that section during the second.

Now imagine an incompressible fluid to be flowing along a tube of variable cross-section; in this case the velocity of the fluid will have different values as it crosses different sections. Let two imaginary parallel planes cut the tube transversely, the distance l separating the planes being so small that between them the cross-sectional area of the tube is sensibly constant and equal to a. Let the fluid, which is found in the space between these planes at a given instant, just leave the space in a time t; then, in the time t, a volume al of fluid passes through the plane section by which the fluid leaves the space, and therefore the fluid flows across the section at the rate of (al/t) units of volume per second. But l is the distance through which each particle of the fluid has travelled in the time t, and therefore (l/t) is the velocity V with which the fluid is flowing in the neighbourhood of the section; thus the volume of fluid that flows across the section in a second is equal to Va.

Thus we obtain the general law: **if the velocity of a fluid, perpendicular to any element of area a, is equal to V, then the volume of fluid that crosses the element of area per second is equal to Va. If the density of the fluid where it crosses the element of area is equal to ρ, then the mass of fluid that crosses the element per second is equal to ρVa.**

Let an imaginary surface enclose an element of volume within an incompressible fluid; and let the flow of the fluid be such, that fluid enters this element through an area a_1 with a velocity V_1, and leaves the element through an area a_2 with a velocity V_2. Then, there is neither destruction nor creation of fluid within the element, and no accumulation of fluid can take place within it, since the fluid is incompressible; thus it follows that—

$$V_1 a_1 = V_2 a_2.$$

If the fluid were compressible, it would be possible for the mass of fluid that enters an element to be unequal to that which leaves it in unit time;

in this case, the fluid within the element would become progressively denser or rarer as time went on, and a progressive change of this character could not be permanent. When the flow becomes permanent, the fluid within the element acquires a permanent density, and the mass of fluid that enters the element is equal to the mass that leaves it per unit time. Thus, if a compressible fluid enters an element through an area a_1, its density being ρ_1 and its velocity V_1; and leaves the element through an area a_2, its density being ρ_2 and its velocity V_2; it follows that—

$$\rho_1 V_1 a_1 = \rho_2 V_2 a_2.$$

If a fluid is enclosed within rigid walls, the flow at any point depends, not only on the sources and sinks within the fluid, but also on the shape and position of the walls; for there can be no flow across the walls, and this condition modifies the flow throughout the fluid. But if the enclosure is very large, the flow at any point at a great distance from the walls will depend, to a close approximation, only on the sources and sinks within the enclosure. If we are concerned with the flow near to a source or sink which is not only far from the walls, but also far from any other sources or sinks, then this flow can be determined, to a close approximation, without reference to the distant sources or sinks.

Strength of sources and sinks.—Let us consider a source which is at a great distance from the boundaries of the fluid and from any other sources or sinks. The fluid must travel away from the source as fast as it is created there, and when the flow is permanent, the mass of fluid which enters any element of volume near to the source, must be equal to the mass that leaves the same element in unit time. The fluid must travel away from the source uniformly in all directions, for there is no reason why the flow in one direction should differ from that in any other direction.

The **strength of a source or sink may be defined as the mass of fluid created or annihilated per second.** A positive sign is prefixed to the strength of a source, and a negative sign to that of a sink.

If an imaginary surface encloses any combination of sources and sinks, it follows that the net mass of fluid that flows across the surface per second is equal to the sum of the strengths of the enclosed sources and sinks. The net flow across the surface will be from within outwards, or from without inwards, according as the sum of the strength of the sources and sinks is positive or negative.

Describe a sphere of unit radius about a source as centre; this sphere has an area 4π, and if q denotes the strength of the source, q units of mass must pass outwards through the surface of the sphere during each second. Since the flow is similar in all directions, it follows that $q/4\pi$ units cross each unit of area of the spherical surface during each second; hence, if ρ_1 is the density and V_1 the velocity of the fluid as it passes through the spherical surface, it follows that—

$$\rho_1 V_1 \times 1 = \frac{q}{4\pi}.$$

Now describe another sphere, of radius r, with the source as centre. This sphere has an area $4\pi r^2$, and therefore $q/4\pi r^2$ units of mass cross each unit of its area per second; and if ρ is the density and V the velocity of the fluid as it passes through the spherical surface, it follows that—

$$\rho V = \frac{q}{4\pi r^2} \quad \therefore V = \frac{q}{4\pi\rho r^2}.$$

If the fluid is incompressible, ρ is constant. In this case, **the velocity of the fluid varies inversely as the square of the distance from the source**, its direction at any point being away from the source, along the straight line drawn from this to the point. If we prefix a negative sign to q, we obtain the velocity at a distance r from an isolated sink of strength $(-q)$; in this case, the direction of the velocity is along the straight line drawn from the point to the sink.

Lines and tubes of flow.—If we draw a number of straight lines radiating from an isolated source or sink, each of these lines will represent a **line of flow** or **stream-line**; that is, a line along which a particle of the fluid travels. If a particle commences to travel along a given straight line drawn from an isolated source, we assume that it will continue on that straight line; this condition, which is not exactly complied with in practice, is one condition that the fluid shall be what is called "perfect"; hence, **in a perfect fluid, all particles that commence to travel along any line of flow, continue on that line throughout their subsequent course.** As we shall see presently, the lines of flow are curved when the flow is due to two or more neighbouring sources or sinks; but if the fluid is perfect, the lines of flow are definite, and each line is traversed from end to end by all particles that commence to travel along it. Two lines of flow can never cross each other; for the line of flow passing through a point indicates the direction of motion at that point, and it is obvious that the resultant motion at a point cannot have two directions. Thus **there can be no flow across a line of flow.**

We may select a number of lines of flow which together bound a tube, called a **tube of flow.** Thus, if a conical surface be described, with an isolated source or sink as apex, any straight line drawn on the conical surface will radiate from the source, and constitute a line of flow. Since there is no flow across any line of flow, **there can be no flow across the lateral boundaries of a tube of flow.**

If a fluid is bounded by rigid walls, and the flow has become permanent, the mass of fluid within the walls must be constant; hence, whatever quantity of fluid is created at the sources, an equal quantity of fluid must be annihilated at the sinks within the boundaries. A tube of flow must start from a source and end at a sink; for there can be no flow across the lateral boundaries of a tube, and there can be no progressive accumulation of fluid within a tube. Thus, if a given quantity of fluid enters one end of a tube at a source, an equal quantity must leave the other end of the tube at a sink. It follows, of course, that a tube of flow can never extend from a source to a source, or from a sink to a sink. When we speak of an isolated source or sink, we mean that it is at so great a distance from other sources or sinks, and from the boundaries of the fluid, that its tubes of flow are conical within the part of the fluid which we are studying.

Velocity-potential.—At any point within an incompressible fluid, the velocity due to an isolated source varies inversely as the square of the distance from that source. In this respect the velocity resembles the gravitational attraction at a given distance from an isolated mass of matter (p. 192) or the electric field at a given distance from an isolated electric charge. Now, just as problems connected with gravitation can be simplified by introducing the idea of gravitational potential (p. 195), and electrical problems can be simplified by introducing the idea of electric potential, so problems connected with the flow of fluids can be simplified by introducing the idea of velocity-potential. **The difference of velocity-potential between two neighbouring points is defined as the product of the component velocity along the line joining the points, and the distance separating the points.** Further, the velocity-potential diminishes along the direction of motion. Thus, if the component velocity along a short straight line AB is equal to V, and the distance AB is equal to *d*, the difference of velocity-potential between A and B is equal to V*d*; and if the direction of the component velocity is from A to B, the velocity-potential at A is higher than that at B. If, from a point A in the neighbourhood of a source, we draw a line to an infinite distance (where the velocity will be equal to zero), divide this line into elements, calculate the difference of velocity-potential between the extremities of each element, and then sum the results, we obtain **the velocity-potential at the point** A; the velocity-potential at a point is generally indicated by the Greek letter ψ. We agree that the velocity-potential at an infinite distance is equal to zero. By a method similar to that employed on p. 193, it can be proved that, in an incompressible fluid, the velocity-potential at a distance r from a source of strength q is equal to $(q/4\pi\rho r)$. Hence, if we draw a spherical surface of radius r about an isolated source of strength q as centre, the value of ψ at any point on this surface is equal to $(q/4\pi\rho r)$. The spherical surface is thus a surface of equal velocity-potential, or, in short, an **equipotential surface** (Fig. 180). There can be no flow along any line drawn on an equipotential surface; for, when there is any velocity along a line, the extremities of the line are at different

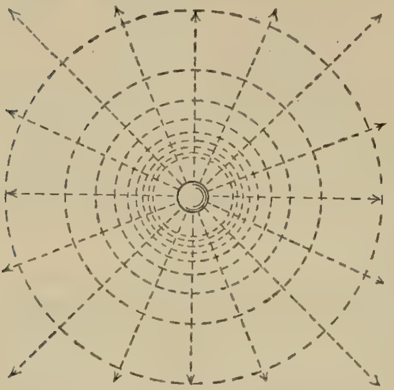

FIG. 180.—Lines of flow and equipotential surfaces, due to an isolated source. (The equipotential surfaces near to the source are omitted, to avoid overcrowding.)

velocity-potentials. From this it follows that **the flow at any point in a fluid is perpendicular to the surface of equal velocity-potential drawn through that point**; in other words, a line of flow, or a stream-line, must pass perpendicularly through all equipotential surfaces which it penetrates.

Let a point A lie on a surface of constant potential ψ_1, while a neighbouring point B lies on a surface of constant potential ψ_2; then if $AB = d$, it follows that the component velocity V along AB, from A to B, is given by the equation—

$$\psi_1 - \psi_2 = Vd; \quad \therefore\ V = (\psi_1 - \psi_2)/d.$$

Hence, if surfaces of constant velocity-potential can be described throughout a fluid, the velocity of the fluid at each point can be determined with ease.

Principle of superposition.—No material fluid possesses properties in exact agreement with those already assigned to a perfect fluid; but the agreement is so close, that many results determined for a perfect fluid can be used to predict, to a close approximation, the motion of a material fluid. A further property of a perfect fluid must now be postulated.

At any point within a perfect fluid, a definite velocity of flow is produced by each source or sink, independently of the remaining sources and sinks; and the resultant velocity is found by combining the component velocities just as in the cases of forces, etc. (p. 22). This is called the principle of superposition: it is realised in material fluids to about the same degree of approximation as the principle of lines of flow (p. 380).

An expression for the velocity-potential at any point in an incompressible fluid can now be obtained, in terms of the strengths and positions of the sources and sinks within the fluid. Call the point P, and draw a line from P to an infinite distance; then if V_1, V_2, V_3 . . . etc., are the component velocities (due respectively to sources at points A,B,C, . . . etc., within the fluid), resolved along an element of length d of the line, the difference of potential between the ends of the element is equal to $(V_1 + V_2 + V_3 + \ldots)\, d$. Hence, the resultant difference of potential between the ends of a line-element, is equal to the sum of the differences of potential between the ends of the element, due to the various sources taken singly. From this it follows at once that the resultant potential at the point P is equal to the sum of the potentials that would have been produced there by the various sources taken singly. Hence if $AP = r_1$, while the strength of the source at A is q_1, and $BP = r_2$, while the strength of the source at B is q_2, etc., etc., then the resultant potential ψ at P is given by the equation—

$$\psi = \frac{1}{4\pi\rho}\left\{\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ . \ . \ . \right\}.$$

Sinks are treated merely as negative sources.

From the reasoning explained above, it follows that when the flow is due entirely to source and sinks, there will be one, and only one, value of ψ at each point of the fluid: this is generally expressed by saying that the velocity-potential due to any combination of sources and sinks is a single-valued function of the position of the point at which the potential is measured.

Lines of flow due to particular combinations of sources and sinks.—The electric potential ψ' at a point P, at distances $r_1, r_2, r_3 \ldots$ from electric charges of which the magnitudes are $Q_1, Q_2, Q_3 \ldots$ (measured in electrostatic units), is given by the equation—

$$\psi' = \frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \frac{Q_3}{r_3} + \ . \ . \ . \ .$$

Hence it appears that, if we know the electric potential ψ' at a point in the neighbourhood of any combination of electric charges, we can determine the velocity-potential ψ at a point in an incompressible fluid, situated similarly with respect to a combination of sources of which the magnitudes are numerically equal to the electric charges; for $\psi = (1/4\pi\rho)\psi'$. In the electric problem, the lines of force cut the lines of equal potential at right angles; and in the problem in fluid motion, the lines of flow cut the lines of equal velocity-potential at right angles. Thus, the lines of force due to any combination of electric charges are precisely similar to the lines of flow (or stream-lines) due to a corresponding combination of sources and sinks.

Fig. 181 represents[1] the lines of force due to two dissimilar charges of equal magnitudes, or the lines of flow due to a source and a numerically equal sink. In the electrical problem, the shaded circles represent spheres on which the charges reside; in the problem on fluid motion, the shaded circles represent spherical surfaces through which fluid is extruded (in the case of the circle marked +) or absorbed (in the case of the circle marked −). Thus, the positively charged sphere corresponds to a finite spherical source, while the negatively charged sphere corresponds to a finite spherical sink. The lines of force correspond, in shape and in direction, to the lines of flow. In a similar manner, Fig. 182 represents the lines of flow due to two finite spherical sources of equal magnitudes. The lines of flow due to two finite spherical sinks of equal magnitudes would be similar in shape to those represented in Fig. 182, but the direction of flow would be opposite to that indicated by the arrowheads.[1]

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[1] See *Magnetism and Electricity for Students*, by H. E. Hadley (Macmillan), p. 136.

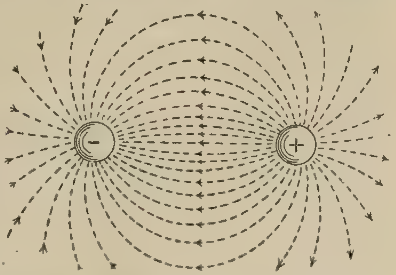

FIG. 181.—Lines of flow due to a source and a numerically equal sink.

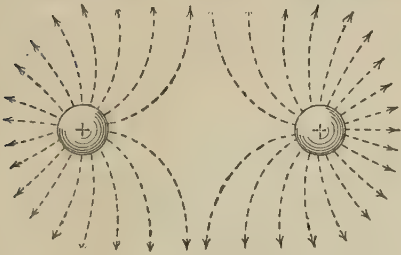

FIG. 182.—Lines of flow due to two equal sources.

"Curl."—Describe any closed linear circuit within a fluid, divide this circuit into elements, and, taking each element in turn, let the difference of potential between its ends be determined; then, if the differences of potential are measured so that the circuit is traversed continuously in one direction, their sum gives the difference of potential all around the circuit: this sum is called **the curl of the velocity around the given circuit.**

---

[1] For a geometrical method of drawing lines of force and equipotential surfaces, see Hadley's *Magnetism and Electricity for Students* (Macmillan), pp. 51, 57, and 150.

In the case of fluid motion due to any combination of sources and sinks, the potential at any given point has a definite value (p. 384), and therefore if we start from a point and traverse any closed circuit, and so return to the point from which we started, the total difference of potential around the circuit must be zero; that is, the curl of the velocity around the circuit is equal to zero.

Now let us suppose that a vortex filament is cut by an imaginary plane perpendicular to the axis of the filament, and let us determine the curl of the velocity around the circuit which is formed by the circular boundary of the section. If r is the radius, and ω the angular velocity of the filament, the velocity at each point of the circle is equal to $r\omega$, and at any point this velocity is parallel to the element of the circle passing through that point. Hence, in passing once around the circular circuit in a direction opposite to that in which the fluid is moving, the potential increases by $2\pi r \times r\omega = 2\pi r^2\omega$; hence, the curl of the velocity around the circuit is equal to $2\pi r^2\omega$. But πr^2 is the cross-sectional area of he filament; hence, if the cross-sectional area of a vortex filament is denoted by a, the curl of the velocity around the circular boundary of the cross-section is equal to $2a\omega$. **The quantity $2a\omega$ is called the strength of the vortex filament.** Hence, **the curl of the velocity around the circle which bounds the cross-section of a vortex filament is equal to the strength of the filament.**

Let it be agreed that, in obtaining the curl, a circuit shall be traversed always in a certain sense, say in an anti-clockwise sense. When the rotation of a vortex filament is in a clockwise sense, the difference of velocity-potential due to traversing the circular boundary of the cross-section of the filament in an anti-clockwise sense is positive, and the curl is therefore positive. When the direction of rotation is anti-clockwise, the circuit is traversed in the direction of the flow, and the difference of velocity-potential is negative, so that the curl is negative. Hence, when one end of a vortex filament is viewed normally, the strength of the vortex filament is positive or negative, according as the rotation observed is in a clockwise or an anti-clockwise sense. From this it follows, that a vortex filament will have a positive or negative strength according to the end of the filament which is viewed; but when we deal with a number of vortex filaments, all must be viewed in the same direction.

When the motion of a fluid is such that the curl of the velocity around every small closed circuit that may be drawn in the fluid is equal to zero, the motion is said to be irrotational. It has been proved already that the motion due to any combination of sources and sinks complies with this condition.

Experiments made on material fluids show that there is never any abrupt change of velocity in passing across the imaginary boundary which separates two contiguous layers of fluid; in other words, **there is no slip between contiguous layers of fluid.** Applying the same principle to a perfect fluid, we see that the fluid immediately surrounding a vortex filament must be in motion.

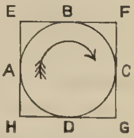

FIG. 183. Square circuit, enclosing a vortex filament.

Let Fig. 183 represent the circular cross section of an isolated vortex filament enclosed by a square circuit which touches the filament at the points A, B, C, and D. By definition, the motion inside the vortex filament is rotational; outside the filament the motion must be irrotational, for if it were rotational there would be more vortex filaments, which is supposed not to be the case. Then, in passing round the circuit ABEA, the curl is equal to zero; and similarly with the circuits BCFB, CDGC, DAHD. Hence—

$$\text{Curl } ABEA + \text{curl } BCFB + \text{curl } CDGC + \text{curl } DAHD = 0.$$

But, in obtaining this sum, we have passed once completely round the circle ABCDA, in a clockwise sense, and once along the square EHGFE in an anti-clockwise sense. Hence—

$$\text{curl } ABCDA + \text{curl } EHGFE = 0.$$

In reversing the direction in which a circuit is traversed, we obviously reverse the sign of the curl; hence—

$$\text{curl } ABCDA = \text{curl } EFGHE.$$

Thus, if a small square be drawn so that its sides touch the boundaries of the cross section of a vortex filament, the curl of the velocity along the square circuit is equal to the curl along the circular cross section of the enclosed vortex filament. A similar result would have been obtained if we had drawn a triangular or, indeed, any other form of circuit, around the cross section of the filament.

Draw any circuit ABCA (Fig. 184) within a fluid, and divide this circuit into meshes by means of two sets of mutually perpendicular lines. If we find the value of the curl around each mesh, traversing all meshes in the same sense, and sum up the results, it is clear that this will give us the curl around the boundary ABCA. For, in traversing any two contiguous meshes, the straight line that separates them from each other is traversed, first in one and then in the opposite direction, so that the net result is equal to zero; the lines which are traversed in one direction only, are the elements into which the boundary is divided by the two sets of straight lines. Hence, the curl along the boundary is equal to the sum of the curls around the meshes into which the enclosed area has been divided. Now, if any mesh encloses a vortex filament, the curl around the mesh is equal to the strength of the enclosed filament; if no filament is enclosed, the curl around the mesh is equal to zero. Hence we arrive at this important law: **the curl of the velocity around any closed circuit in a fluid, is equal to the algebraic sum of the strengths of the vortex filaments which are encircled by the circuit.**

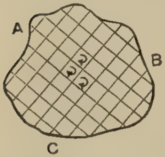

FIG. 184.
Curl around any closed circuit.

Constant strength of a vortex filament.—Let l be the length, r_1 the radius, and ω the angular velocity of a vortex filament in a perfect inviscid fluid. The strength of the filament is equal to the curl of the velocity around the circular cross-section of the filament; thus if the strength is denoted by m, we have—

$$m = r_1\omega \times 2\pi r_1 = 2\pi r_1^2\omega.$$

It will now be proved that the strength of the filament cannot be altered by human agency. For simplicity, the proof will be confined to the case where the fluid is incompressible.

If we could apply the necessary forces to the individual particles of a fluid, we could modify their motion at pleasure; but in practice we can apply mechanical forces only to the boundaries of a fluid, and thus the modifications which we can produce are limited. It is impossible to apply any tangential force to the bounding surface of an inviscid fluid: and even if the particles which lie in the boundary could be set in motion at pleasure, no tangential forces would be transmitted to the contiguous layers of the fluid. Hence, tangential forces cannot be applied to the boundaries of a vortex filament formed from an inviscid fluid. The moment of momentum of the filament about its axis cannot be altered without the application of a torque about the axis (p. 61), and the application of such a torque is impossible without using tangential forces ; therefore **the moment of momentum of the filament about its axis cannot be altered.**

It is possible, however, to elongate or shorten the filament ; the effects produced by a small elongation will now be studied.

The energy possessed by the filament is wholly kinetic, and is due to the rotation of the filament about its axis. Since the filament rotates like a solid body, its kinetic energy is equal to $(1/2)\mathrm{I}\omega^2$, where I denotes the moment of inertia of the rotating cylinder about its axis. Now, the moment of inertia of a cylinder of radius r_1 and length l, about its axis, is equal to $(1/2)\pi r_1^2 l\rho r_1^2 = (1/2)\pi\rho l r_1^4$, where ρ is the density, and $\pi r_1^2 l\rho$ is the mass of the cylinder (p. 51). Thus, the kinetic energy of the rotating cylinder is equal to—

$$\tfrac{1}{4}\pi\rho l r_1^4\omega^2.$$

Now let it be supposed that the filament suffers a small elongation δ ; during the elongation, the average value of the external tensile stress acting on each end of the filament is equal to $(1/4)\rho r_1^2\omega^2$, (p. 374), and this stress acts through a distance δ ; hence the work done by the agent which produces the elongation is equal to $\pi r_1^2 \times (1/4)\rho r_1^2\omega^2 \times \delta = (1/4)\pi\rho\delta r_1^4\omega^2$, and this expression gives the increase in the energy of the filament due to the elongation δ. Since the fluid is supposed to be incompressible, the volume of the filament cannot change, and therefore no work is done by the pressure acting normally over the whole surface of the filament (p. 374). Thus, after the extension, the kinetic energy of the filament must be equal to—

$$\tfrac{1}{4}\pi\rho l r_1^4\omega^2 + \tfrac{1}{4}\pi\rho\delta r_1^4\omega^2 = \tfrac{1}{4}\pi\rho(l+\delta)r_1^4\omega^2 ;$$

consequently, **the elongation of the filament by δ, increases its kinetic energy in the ratio** $(l+\delta)/l$: let this ratio be denoted by k.

As to the condition of the filament after the elongation, it is conceivable that it might no longer be rotating like a solid body : the various particles might be revolving about the axis, each with its own angular velocity. Let the filament be divided into particles of masses

m_1, m_2, m_3, ... and let the radii of the circles in which these particles revolve be respectively equal to R_1, R_2, R_3 . . . . before the elongation, and to $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, ... after the elongation. Before the elongation, all of the particles revolve with the common angular velocity ω; after the elongation, let them revolve with the respective angular velocities Ω_1, Ω_2, Ω_3, ... Then, before the elongation, the kinetic energy of the filament is equal to

$$\tfrac{1}{2}\{m_1\mathrm{R}_1{}^2\omega^2 + m_2\mathrm{R}_2\omega^2 + \ldots\ldots\} = \tfrac{1}{2}\Sigma m\mathrm{R}^2\omega^2,$$

where the symbol Σ is an abbreviation for the expression "the sum of terms of the type."

After the elongation, the kinetic energy of the filament is equal to

$$\tfrac{1}{2}\{m_1\mathbf{R}_1{}^2\Omega_1{}^2 + m_2\mathbf{R}_2\Omega_2{}^2 + \ldots\ldots\} = \tfrac{1}{2}\Sigma m\mathbf{R}^2\Omega^2.$$

The condition that **the increase in the kinetic energy of the particles is equal to the work done on the filament during its elongation,** is expressed by the equation

$$\Sigma m\mathbf{R}^2\Omega^2 = k\Sigma m\mathrm{R}^2\omega^2 \quad . \ . \ . \ . \ . \ . \ . \quad (1)$$

where k denotes the ratio of the final length to the original length of the filament.

The condition that **the moment of momentum of the filament is left unchanged by the elongation** is easily expressed. Before the elongation, the momentum of the particle m_1 was equal to $m_1\mathrm{R}_1\omega$, and its moment of momentum about the axis was equal to $m_1\mathrm{R}_1{}^2\omega$. After the elongation, the moment of momentum of the same particle is equal to $m_1\mathbf{R}_1{}^2\Omega_1$. Then

$$m_1\mathbf{R}_1{}^2\Omega_1 + m_2\mathbf{R}_2{}^2\Omega_2 + \ldots = m_1\mathrm{R}_1{}^2\omega + m_2\mathrm{R}_2{}^2\omega + \ldots$$

$$\therefore \quad \Sigma m\mathbf{R}^2\Omega = \Sigma m\mathrm{R}^2\omega \quad . \ . \ . \ . \ . \ . \ . \quad (2)$$

Since the fluid is incompressible, the volume of the filament is left unchanged by the elongation. If r_1 and r_2 denote the original and final radii of the filament, we have

$$\pi r_1{}^2 l = \pi r_2{}^2(l+\delta)\,; \quad \therefore\ r_1{}^2/r_2{}^2 = (l+\delta)/l = k.$$

If M denotes the total mass of the filament, its moment of inertia is equal to $\mathrm{M}r_1{}^2/2$ before the elongation. The moment of inertia of the filament is merely the sum of the terms obtained by multiplying the mass of each particle by the square of its distance from the axis; since this sum is independent of the angular velocities of the particles, it follows that the moment of inertia of the filament after its elongation is equal to $\mathrm{M}r_2{}^2/2$.

Then, since

$$\frac{\dfrac{\mathrm{M}r_1^2}{2}}{\dfrac{\mathrm{M}r_2^2}{2}} = \frac{r_1^2}{r_2^2} = \frac{l+\delta}{l} = k,$$

it follows that

$$k\{m_1\mathbf{R}_1^2 + m_2\mathbf{R}_2^2 + \ldots\} = m_1\mathrm{R}_1^2 + m_2\mathrm{R}_2^2 + \ldots$$

$$\therefore\quad k\Sigma m\mathbf{R}^2 = \Sigma m\mathrm{R}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equations (1), (2), and (3) suffice to determine the exact condition of the filament after its elongation.

To solve these equations, multiply equation (2) by a, and equation (3) by b, (where a and b may have any values except zero), and add the results to equation (1).

Then

$$\Sigma m\mathbf{R}^2\Omega^2 + a\Sigma m\mathbf{R}^2\Omega + bk\Sigma m\mathbf{R}^2$$

$$= k\Sigma m\mathrm{R}^2\omega^2 + a\Sigma m\mathrm{R}^2\omega + b\Sigma m\mathrm{R}^2\,;$$

$$\therefore\quad \Sigma\{m\mathbf{R}^2(\Omega^2 + a\Omega + bk)\} = \Sigma\{m\mathrm{R}^2(k\omega^2 + a\omega + b)\}$$

$$= \Sigma\left\{m\mathrm{R}^2k\,.\left(\omega^2 + \frac{a}{k}\omega + \frac{b}{k}\right)\right\} \quad . \quad . \quad . \quad . \quad (4)$$

The meaning of equation (4) must be remembered carefully. The left-hand side of the equation represents a series of terms of the type

$$m_1\mathbf{R}_1^2(\Omega_1^2 + a\Omega_1 + bk) + m_2\mathbf{R}_2^2(\Omega_2^2 + a\Omega_2 + bk) + \quad . \quad . \quad . \quad .$$

where Ω_1, Ω_2, Ω_3, ... may possibly have unequal values. On the right-hand side of the equation, ω is constant for all particles.

Equation (4) is true for all finite values of a and b. Choose the values of a and b so that the right-hand side of (4) becomes equal to zero. This condition will be complied with if

$$\frac{a}{k} = -\,2\omega, \text{ and } a = -\,2\omega k,$$

while

$$\frac{b}{k} = \omega^2, \text{ and } b = k\omega^2\,;$$

for then

$$\left(\omega^2 + \frac{a}{k}\omega + \frac{b}{k}\right) = \omega^2 - 2\omega^2 + \omega^3 = 0.$$

Substituting these values of a and b in the left-hand side of (4), we obtain the equation

$$\Sigma\{m\mathbf{R}^2(\Omega^2 - 2\omega k\Omega + k^2\omega^2)\} = \Sigma m\mathbf{R}^2(\Omega - k\omega)^2 = 0.$$

This result means that

$$m_1R_1^2(\Omega_1-k\omega)^2+m_2R_2^2(\Omega_2-k\omega)^2+\ldots\ldots=0.$$

Each term in the series comprises two factors, neither of which can have a negative value; for $m\mathbf{R}_1^2$ is essentially positive, while $(\Omega_1-k\omega)^2$ is a square and cannot, therefore, be negative. But the sum of a series of terms, none of which is negative, cannot be equal to zero unless each term is equal to zero. Hence we see that each term must be equal to zero, and since the factors $m_1\mathbf{R}_1^2$, $m_2\mathbf{R}_2^2$, etc., cannot all be equal to zero, it follows that

$$\Omega_1-k\omega=0,$$
$$\Omega_2-k\omega=0, \ \&c.$$

Hence, **the effect of stretching the filament to k times its original length, is to increase its angular velocity of rotation k times**; after the elongation the filament still rotates like a solid, since $\Omega_1=\Omega_2=\ldots=k\omega$. Thus, **for a given vortex filament, the angular velocity of rotation is directly proportional to the length of the filament.**

Since the moment of momentum is unchanged by the elongation, we have—

$$M\frac{r_1^2}{2}\cdot\omega=M\frac{r_2^2}{2}\cdot\Omega,$$

where M denotes the mass of the filament, r_1 and r_2 its radii before and after the extension, while ω and Ω denote its angular velocities before and after the extension.

$$\therefore\quad 2\pi r_1^2\omega=2\pi r_2^2\Omega,$$

so that **the strength of the filament is not altered by the elongation.** Hence, **the strength of the filament cannot be changed by human agency.**

In an inviscid fluid, we could neither produce nor destroy vortex motion; that is, a vortex filament would be uncreatable and indestructible. A vortex filament once in existence, we could alter its angular velocity, but not its strength, by elongating or shortening the filament; but we could not impress an initial angular velocity on the stationary fluid. All material fluids are viscous, and their viscosity enables us to set them in rotation by means of tangential forces applied to their boundaries; hence, we can produce vortex motion in a viscous fluid. Viscosity, which enables us to produce vortex motion, ensures the decay and final disappearance of this motion on the cessation of the forces that produced it.

Stresses in a fluid which contains vortex filaments.

Problem.—*A cylindrical volume of fluid contains a number of similar infinitely thin vortex filaments, which are distributed uniformly with their axes parallel to the axis of the cylinder. Determine the velocity of flow at a distance r from the axis of the cylinder, and from this result calculate the value of the forces that must be applied to the ends of the cylinder in order to produce equilibrium.*

The strength m of a vortex filament is equal to $2\pi r^2\omega = 2a\omega$, where a is the sectional area, r is the radius, and ω is the angular velocity of the filament. If a is extremely small, the filament may still possess a finite strength if ω is sufficiently large. The force acting on the end of each filament must be equal to $(1/4)\pi\rho r^2 r^2\omega^2 = \rho a^2\omega^2/4\pi$, and the stress on the end of the cylinder, due directly to the enclosed vortex filaments, will be equal to $N\rho a^2\omega^2/4\pi = N\rho m^2/16\pi$, where N denotes the number of vortex filaments that cross each unit of area of the cross section of the cylinder.

In any imaginary cross-section of the cylinder, draw a concentric circle of radius r_1; there can be no average flow, either outwards or inwards, across the circumference of this circle, for there are neither sources nor sinks within the cylinder. The flow along the circumference of the circle may vary slightly, according as it is measured at a point near to or far from a vortex filament; but if the vortex filaments are numerous, the velocity will have a constant *average* value along each short element into which the circle may be divided. Let the average velocity along the circumference of the circle be denoted by V. Then the curl of the velocity around the circle is equal to $2\pi r_1 V$. The sum of the strengths of the filaments enclosed by the circle is equal to $Nm \, . \, \pi r_1^2$, and thus—

$$2\pi r_1 V = Nm \, . \, \pi r_1^2 ;$$

$$\therefore \quad V = (Nm/2) r_1 .$$

Thus, the average velocity at a distance r_1 from the axis is proportional to r_1; that is, the velocity is the same as if the cylinder were rotating like a solid with an angular velocity equal to $(Nm/2)$.

From reasoning precisely similar to that used on p. 373, it follows that, for the fluid within the cylinder to be in equilibrium, it must be subjected to a general pressure p given by the equation—

$$p = \frac{\rho}{2}\left(\frac{Nm}{2}\right)^2 R^2 = \frac{\rho}{8} N^2 m^2 R^2 ,$$

where R denotes the radius of the cylinder; together with a longitudinal tensile stress of which the average value is equal to $p/2$.

Hence, the ends of the cylinder must be subjected to an average stress equal to—

$$\frac{\rho}{16\pi}\mathrm{N}m^2 + \frac{\rho}{16}\mathrm{N}^2m^2\mathrm{R}^2.$$

If N is very large, the second term (which is proportional to N^2) will be very large in comparison with the first (which is proportional to N); in these circumstances we may neglect the first in comparison with the second term.

The tensile stress in an electric field is proportional to the square of the number of lines of force per unit area,[1] and to this extent resembles the tensile stress obtained above, which is proportional to N^2 when R is constant. But, in the problem just solved, the average stress is proportional to $\mathrm{N}^2\mathrm{R}^2$, and if a similar condition applied to the electric field, the pull per unit area on either plate of a charged condenser would depend, not alone on the surface density of the charge, but also on the area of the charged plate, and this is not the case. Here again, therefore, the vortex theory of the electric field breaks down signally (compare p. 375). For similar reasons, the properties of the magnetic field cannot be explained by assuming that magnetic lines of force are vortex filaments.

There is yet another respect in which the properties of an electric field are incompatible with the hypothesis that electric lines of force are vortex filaments. Imagine an air condenser to consist of two plane parallel circular plates at a small distance apart. When the condenser is charged, lines of force extend from the positively to the negatively charged plate. If these lines of force were vortex filaments, all of them would be revolving about the axis of the discs, due to the rotational motion imposed on the surrounding medium. But the late Prof. Rowland proved experimentally, that when electric lines of force are set in motion in a direction perpendicular to their lengths, they produce a magnetic field; and no magnetic field has been observed in the space between the stationary plates of a charged condenser.

Flow of the fluid which surrounds a vortex filament.

Problem.—*Determine the velocity of flow at a point in the fluid which surrounds a long isolated vortex filament.*

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[1] See *Magnetism and Electricity for Students*, by H. E. Hadley (Macmillan), p. 144.

Within a vortex filament the motion is rotational, while the motion of the surrounding fluid is irrotational. In a plane, perpendicular to the axis of the filament, draw a circle of radius r, with its centre on the axis of the filament. By reasoning similar to that employed previously, the circle will be a stream-line: let the velocity tangential to the circle be equal to V. The curl of the velocity around the circle is equal to $2\pi r$V, and since only one vortex filament of strength m (say), is encircled, it follows that—

$$2\pi r V = m;$$

$$\therefore \quad V = \frac{m}{2\pi r}.$$

Thus, outside the filament, the velocity varies inversely as the distance from the axis of the filament.

At a distance r from the axis of a long straight wire along which flows an electric current C (measured in c.g.s. electro-magnetic units), the magnetic field is equal to ($2C/r$) dynes per unit pole. Hence, the lines of flow around a long, straight, isolated vortex filament, are similar to the lines of force around a long, straight conductor along which an electric current flows.

The lines of flow depicted in a diagram obviously represent the sections of tubes of flow by the plane of the paper. In dealing with circular motion about an axis perpendicular to the plane of the paper, we may postulate that two sides of a tube of flow are generated by adjacent lines of flow when the diagram is displaced through unit distance perpendicular to the plane of the paper, the remaining sides being flat and coincident respectively with the original and final planes of the diagram. In this case, if r_1 and r_2 are the radii of two successive circular lines of flow, the cross-sectional area of the corresponding tube of flow is equal to $(r_2 - r_1) \times 1$. Now, the volume of incompressible fluid that passes per second through any cross-section of a single tube of flow is constant, (p. 378), and it is convenient to choose the boundaries of all tubes of flow so that the flow of liquid per second through each and every cross-section is constant. Subject to this convention, the lines of flow in the fluid surrounding a linear vortex filament can be represented by concentric circles of which the radii increases in geometrical progression. To prove this, let the line of flow immediately encircling the vortex filament have a radius r, and let the remaining lines of flow have radii kr, k^2r, k^3r, . . , where $k > 1$. Then the average velocity in the nth tube of flow, counting the tube nearest to the vortex filament as the first, will be equal to—

$$\frac{1}{2} \cdot \frac{m}{2\pi}\left(\frac{1}{k^n r} + \frac{1}{k^{n+1} r}\right) = \frac{1}{4}\,\frac{m}{\pi}\,\frac{1}{k^n}\left(\frac{k+1}{k}\right) \cdot \frac{1}{r},$$

and the volume of incompressible fluid which flows per second through any cross-section of the tube is equal to the product of the average velocity and the area of the cross-section (p. 378), that is, to—

$$\frac{m}{4\pi}\cdot\frac{1}{r}\frac{1}{k^n}\left(\frac{k+1}{k}\right)\times k^n(kr-r)=\frac{m}{4\pi}\left(\frac{k^2-1}{k}\right).$$

Since this value is independent of n, it follows that equal volumes of fluid flow through all of the tubes; and by choosing a suitable value of k, the diagram can be drawn to represent the sections of tubes through each of which a stated volume flows in unit time. Fig. 185 represents the lines of flow in the fluid surrounding a vortex filament, drawn according to this convention.

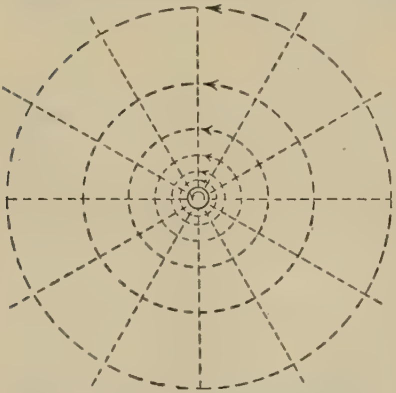

FIG. 185.—Lines of flow, and equipotential surfaces, in the fluid surrounding a vortex filament. (To obtain the tubes of flow, imagine the diagram to be displaced through unit distance, perpendicular to the plane of the paper.)

The lines of equal velocity-potential are obviously straight and radiate from the axis of the vortex filament, since there is no radial flow (p. 383). The velocity in any tube of flow is inversely proportional to the length cut off from an equipotential line, by the two lines of flow which form the sections of the boundary of the tube.

Single and multiple-valued potentials.—If we start from a point in a fluid, and traverse any closed circuit which encircles a vortex filament, the difference of potential all round the circuit is equal to the strength of the encircled filament. But in traversing a closed circuit, we finally return to the point from which we started; therefore, after traversing a closed circuit, the potential of the point, from which we started and at which we stopped, is increased by the strength of the vortex filament encircled. Hence, at any point in a fluid in which there are vortex filaments, the potential is indeterminate. When the flow of a fluid is due entirely to sources and sinks, the potential at each point is definite and single-valued (p. 384).

For somewhat similar reasons, in travelling from one point to another in a plane, we suffer a resultant displacement definite in magnitude and direction, and the positions of all points in the plane may be defined by their vector distances from some fixed point; but if we start from a point on the equator of the earth, and travel continuously in a westerly direction, we shall arrive at the point from which we started after traversing the equatorial circumference of the earth.

QUESTIONS ON CHAPTER XI.

1. As the earth travels around its elliptic orbit, its velocity alternately increases and decreases. Would it be possible to measure the corresponding accelerations by the aid of a suitable spirit level?

2. A closed cylindrical vessel with plane ends is filled with water in which are immersed several small bodies, some denser, and others less dense than water. Prove that when the cylinder is rotating steadily about its axis, the lighter bodies will take up positions near to the axis of rotation, while the denser bodies will take up positions as far from the axis as possible.

3. A cylindrical beaker containing water is supported concentrically on a table which can be rotated about a vertical axis. Some powder lies at the bottom of the beaker; prove that this powder will move away from the axis of the beaker when the speed of rotation of the table increases; and will move towards the axis when the speed of rotation decreases.

4. A uniform straight tube is immersed in a liquid; and when the tube is filled with the liquid, its ends are closed by means of plane diaphragms. The tube is constrained to move uniformly parallel to its length until a given instant, when it is brought to rest, the diaphragms at its ends being simultaneously removed. Prove that, if the liquid is inviscid, and can slip freely over the surfaces of the tube, the ultimate flow in the surrounding liquid will be the same as if there were a source at one end, and a sink at the other end, of the tube.

5. Tubes of flow are described in a liquid, in such a manner that their cross-sectional areas are everywhere inversely proportional to the velocity of flow. Surfaces of equal velocity-potential are described in such a manner that the velocity-potential increases by a constant amount when we pass from any equipotential surface to the next. Prove that the equipotential surfaces divide the tubes of flow into cells which contain equal amounts of kinetic energy.

6. A liquid is bounded above and below by plane surfaces, and extends to an infinite distance in all other directions. A vortex filament

extends vertically from one plane bounding surface to the other; prove that the kinetic energy of the liquid as a whole is infinitely great.

7. Two straight and parallel vortex filaments, separated by a distance b, are situated in the midst of an infinite ocean of incompressible fluid. The strengths of the filaments are numerically equal, but they differ in sign. A line is drawn perpendicularly from one filament to the other, and on this line produced, a point is chosen, at a distance d from the nearer filament; finally, a circle of radius R is described about this point as centre. Prove that the circle will be a line of instantaneous flow, if $R^2 = d(d+b)$.

8. A sphere, situated in an infinite ocean of incompressible fluid, is expanding uniformly. Prove that the flow in the surrounding fluid is the same as if the sphere were removed, and a source of strength $q = 4\pi\rho r^2 v$ were placed at the point previously occupied by the centre of the sphere; where r denotes the instantaneous radius of the sphere and v denotes the velocity with which each element of the surface of the sphere is moving outwards.

9. A line is drawn from a doublet in an incompressible fluid, in a direction making an angle θ with the axis of the doublet; and a point P is chosen on this line, at a distance r from the doublet. Prove that the component velocity at P, resolved along the chosen line, is equal to $2s \cos \theta / 4\pi\rho r^3$, while the component velocity at P, resolved perpendicular to the line, is equal to $s \sin \theta / 4\pi\rho r^3$; where s denotes the strength of the doublet, and ρ denotes the density of the fluid.

(A **doublet** is defined as the combination of a source and a numerically equal sink, situated at a very small distance apart. The **axis of the doublet** is the straight line joining the source and sink; and the strength of the doublet is the product of the strength of the source (p. 379) and the distance between the source and the sink.)

10. Prove that, at a distance r from a long linear source of strength q, the velocity is equal to $q_1 / 2\pi\rho r$.

(A **linear source** consists of a number of contiguous point sources arranged in a straight line. The **strength of a linear source** is equal to the mass of fluid created per second per unit length of the line.)

11. Prove that, at a distance r from a linear doublet, the radial velocity is equal to $s_1 \cos \theta / 2\pi\rho r^2$, while the velocity perpendicular to the radius is equal to $s_1 \sin \theta / 2\pi\rho r^2$; where s_1 is the strength of the linear doublet, and θ is the angle that the radial line makes with the axis of the doublet.

(A **linear doublet** consists of a linear source, say of strength q_1, placed parallel to and at a small distance δ from a numerically equal sink. The **strength of the linear doublet** $= q_1\delta$. The **axis** is a straight line drawn perpendicularly from the source to the sink.)

CHAPTER XII

TRANSFER OF ENERGY BY THE STEADY MOTION OF A FLUID

Comparison between perfect and material fluids.—A fairly clear idea of the properties of a perfect fluid can now be formed.

(1) A perfect fluid is inviscid.

(2) The motion of a perfect fluid, when steady, occurs along definite and permanent lines of flow, or stream-lines (p. 380).

(3) At any point in a perfect fluid, the component velocity due to each source, sink, or vortex filament is equal to the velocity that would have been produced at that point by the source, sink, or vortex filament in question, if there had been no other sources, sinks, or vortex filaments in the fluid. The resultant velocity at the point is obtained from the component velocities by the general method used in connection with displacements, velocities, accelerations, and forces (p. 9).

(4) There is no slip between contiguous layers of a perfect fluid. Hence, a layer never glides with finite velocity over another layer which is at rest; and the velocity changes gradually in passing from one point to another.

No material fluid complies exactly with the ideal conditions prescribed for a perfect fluid. For instance, no material fluid is absolutely inviscid; and, further, we must not assume that the fluids which are most nearly inviscid are those which comply most nearly with conditions (2), (3), and (4) above. The molecular structure of material fluids prevents condition (2) from being fulfilled, for the straying of molecules, due to diffusion, renders the formation of perfect stream-lines impossible. In a liquid, viscosity tends to retard diffusion, and therefore tends to make the stream-lines more perfect; thus, Prof. Hele Shaw has obtained perfect stream-lines by forcing glycerin to flow, in a very thin layer, between two sheets of glass; coloured glycerin, introduced at suitable points, was drawn out into lines which indicated the stream-lines.

When the velocity in a fluid is very great, there is a tendency for the motion to become unstable. Prof. Osborne Reynolds has compared the motion of a material fluid to that of a regiment of imperfectly trained soldiers, who can execute slow manœuvres in perfect order but fall into confusion when quick manœuvres are attempted.

The laws already derived from the properties of a perfect fluid are for the most part purely geometrical, and are useful in the same sense, and subject to the same limitations, as other geometrical laws. Their usefulness is proved by the fact that by their aid we can, in many instances, predict the motion of a material fluid, and afterwards verify our prediction experimentally.

Many attempts have been made to explain the phenomena of magnetism and electricity on the assumption that the ether is a perfect fluid, and that the phenomena in question are due to its motion; several of these attempts have been alluded to in the last chapter, and some will call for discussion subsequently; but it may be said here, that all attempts made up to the present time have proved unsuccessful.

We shall now determine the dynamical conditions which must hold when a perfect fluid flows steadily in accordance with the geometrical laws already derived.

Transfer of energy along tubes of flow.—Since the lateral boundaries of a tube of flow are permanent, and there is no flow across them, it follows that the surrounding fluid might become solid without disturbing the motion within the tube, provided that there were nothing resembling friction between the fluid and its solid boundaries. If the surrounding fluid were solidified, no mechanical energy could be transmitted through it, for work entails motion; and since the motion within the tube is not modified by the solidification of the surrounding fluid, it follows that **there is no transfer of mechanical energy across the lateral boundaries of a tube of flow.**

If the fluid within the tube is acted upon by gravity, the potential energy of a particle of the fluid will increase or decrease according as the particle rises from, or falls towards the earth, and this increase or decrease in the potential energy of the particle must be balanced by a loss or gain of some other kind of energy, such as kinetic energy.

Let fluid enter one end of a tube of flow at a pressure p_1 and with a velocity V_1 (Fig. 186); then the fluid just about to enter the tube acts like a piston urged forward by a force p_1a_1, where a_1 is the area of the orifice of the tube. In a very short interval of time t, the piston moves forward through a distance V_1t, and the work done on the fluid within the tube is equal to $p_1a_1V_1t$, so that, due to this cause, energy enters the tube at the rate of $p_1a_1V_1$ units per second.

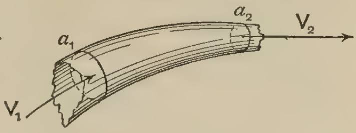

FIG. 186.—Part of a tube of flow.

Further, the fluid which enters the tube possesses kinetic energy. Since the mass of fluid which enters per second is equal to $\rho_1a_1V_1$, (p. 379) where ρ_1 is the density of the fluid which enters, it follows that kinetic energy is entering the tube at the rate of $(1/2)\rho_1a_1V_1V_1^2$. Hence, the total rate at which energy is being transferred across the mouth of the tube is equal to $a_1V_1\{p_1+(1/2)\rho_1V_1^2\}$ units per second.

When p_1 is measured in dynes per sq. cm., ρ_1 in grams per c.c., V_1 in cm. per sec., and a_1 in sq. cm., the rate at which energy enters the tube is obtained in ergs per sec.

Let the fluid leave the other end of the tube with a velocity V_2, a pressure p_2, and a density ρ_2; the area of the tube where the fluid leaves being a_2. Then the rate at which energy leaves the tube is equal to $a_2V_2(p_2+(1/2)\rho_2V_2^2)$.

After the flow has become steady, the amount of energy within the tube must remain constant. For, if the energy within the tube were to increase, it would continue to increase at a constant rate, since no alteration occurs in the flow; and in the end there would be an infinite accumulation of energy within the tube. If the energy within the tube were to decrease, it would continue to decrease at a constant rate, and this could occur only if the original store of energy within the tube were infinitely large.

As the fluid traverses the tube, it may gain energy in two ways. If the fluid is descending under the action of gravity, its gravitational energy diminishes, and an equivalent quantity of some other kind of energy must be gained by the fluid.

Let it be supposed that no heat passes through the lateral walls of the tube, that is, that the flow is adiabatic ; then the fluid may gain energy at the expense of some of its own thermal energy which disappears.

The rate at which mechanical energy leaves one end of the tube can exceed the rate at which mechanical energy enters the other end, only by the rate at which the fluid gains energy in passing through the tube. Hence—

$a_2V_2(p_2 + \frac{1}{2}\rho_2V_2^2) - a_1V_1(p_1 + \frac{1}{2}\rho_1V_1^2) =$ *energy gained by the mass* $\rho_1a_1V_1 = \rho_2a_2V_2$, *at the expense of gravitational or thermal energy which disappears.*

This equation expresses the dynamical condition which the motion of a perfect fluid must comply with. In certain cases some of the terms of the equation become equal to zero. For instance, if the fluid is incompressible, its density cannot alter ; that is, the fluid can neither expand nor contract, and in this case its thermal energy cannot alter when the flow is adiabatic. Again, if the fluid is a gas, its gravitational energy is inconsiderable, and any change in this can be neglected. In the case of a liquid, there can be no net gain or loss of gravitational energy, if the entrance and exit of the tube are in the same horizontal plane.

It is obvious that the results of the above investigation will apply, not only to the whole, but to any part of a tube of flow cut off by two equipotential surfaces, if a_1 and the other quantities with the subscript $_1$ refer to the section where the fluid enters the portion of the tube, and a_2 together with the other quantities with subscript $_2$ refer to the section where the fluid leaves the portion of the tube.

In considering the motion of a material fluid, it must be remembered, in the first place, that some mechanical energy is transformed into heat, owing to the viscosity of the fluid ; in other words, the thermal energy of the fluid within the tube increases at the expense of its mechanical energy. If it be remembered that a gain is equivalent to a negative loss, it will become clear that the mechanical equivalent of the rate of gain of thermal energy must be subtracted from the right-hand side of the fundamental equation.

In the second place, the motion of a material fluid does not, in all circumstances, occur along definite stream-lines; when the motion does not occur along stream-lines, it is said to be **turbulent.** Some idea of the nature of the turbulent motion of a material fluid within a rigid tube may be gained as follows. Imagine a rubber ring, such as is used on umbrellas, to be fitted within a tube, the plane of the ring being perpendicular to the axis of the tube, and the outer periphery of the ring being in contact with the tube. Now, imagine a long rod to be fitted within the rubber ring; if the rod is pushed through the tube, the rubber ring will roll along the inside of the tube, and there will be no slip at the line of contact of the ring and the tube, and no slip at the line of contact of the ring and the rod (Fig. 187). The ring as a whole will advance along the tube at a rate equal to half the velocity of the rod; for at any instant the ring is rolling with a definite angular velocity about its line of contact with the tube, and if the centre C of the cross-section of the ring is at a distance r from the surface of the tube, the line of contact of the ring and rod will be at a distance $2r$ from the surface of the tube. The turbulent motion of a fluid within a tube is somewhat similar in character; the fluid in contact with the tube forms rings which roll along the tube, while the fluid that travels along the axis follows an approximately straight path. The kinetic energy of the fluid within the tube will be due, partly to translation along the tube, and partly to rotation; if the rolling motion is generated within the tube, the average translational velocity along the tube will be less than that calculated on the assumption that there is no rotation, just as the velocity of a body that rolls down an inclined plane is less than that of a body which slides freely down the same plane (p. 59).

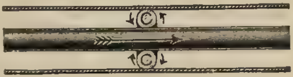

FIG. 187.—To explain the nature of turbulent motion in a tube.

Flow of an incompressible fluid along a uniform tube, of which the axis is horizontal.—Since the bore of the tube is uniform, and the product of the velocity and the sectional area is constant (p. 378), it follows that the velocity is constant.

If the fluid is perfect, there is no loss or gain of thermal energy in traversing the tube; since the tube is horizontal, the gravitational energy of the fluid is constant. Substituting $a_1V_1=a_2V_2$ and $V_1=V_2$ in the fundamental equation, and remembering that the right-hand side of the equation is equal to zero, we have—

$$p_2-p_1=0.$$

Hence, **there is no pressure gradient along the tube.**

If the fluid is material, there will be a certain loss of energy due to viscosity and turbulence. Let e denote the loss of energy sustained by unit volume of a liquid in traversing unit length of a tube; then, if the subscript numbers $_1$ and $_2$ refer to sections at a distance l apart, the loss of energy of a volume $a_1V_1=a_2V_2$, in traversing the length l of the tube, is equal to ea_1V_1l. Hence—

$$a_2V_2\{p_2+(1/2)\rho V_2^2\}-a_1V_1\{p_1+(1/2)\rho V_1^2\}=-ea_1V_1l;$$

And since $V_2=V_1$, and $a_2V_2=a_1V_1$, we have—

$$p_2-p_1=-el;$$

$$\therefore\quad \frac{p_1-p_2}{l}=e.$$

In this case, **there is a pressure gradient which slopes downwards in the direction of the flow; the value of the pressure gradient is equal to the loss of mechanical energy sustained by unit volume of the liquid in traversing unit length of the tube.** If vertical glass tubes are fixed into apertures in the horizontal tube, the pressure gradient along the tube will be indicated by the heights at which the liquid stands in the glass tubes (Fig. 188).

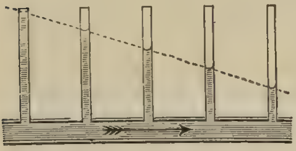

FIG. 188.—Pressure gradient, due to the flow of a liquid along a uniform horizontal tube.

When an electric current flows along a uniform wire, there is a dissipation of electrical energy which results in the production of heat, and the energy dissipated per unit quantity of electricity in traversing unit length of the wire gives the potential gradient, or the fall of potential per unit length of the wire. In this respect, the flow of electricity and the flow of a material fluid resemble each other; but we shall see that the resemblance between the two extends very little further.

Flow of an incompressible fluid in a constricted tube.—Let the axis of a constricted tube be horizontal, and let Fig. 189 represent its vertical axial section. Let the transverse area of the tube diminish regularly from a_1 at the section A, to a_2 at the section B, and then increase regularly to a_1 at the section C. Let the pressure and velocity of the fluid which traverses the tube be equal respectively to p_1 and V_1 at A, and p_2 and V_2 at B. Then for there to be no progressive increase or decrease in the volume of fluid between the sections A and B, we must have (p. 378)—

$$V_1a_1 = V_2a_2.$$

Since a_2 is less than a_1, it follows that the velocity V_2 at the section B is greater than the velocity V_1 at the section A. Consequently, as a particle of fluid travels from the section A, to the section B, its velocity must increase; that is, the motion of the particle is accelerated. Hence, the pressure behind the particle must exceed that in front of it, (compare p. 368), and we conclude that **the pressure must diminish as we pass from A to B.**

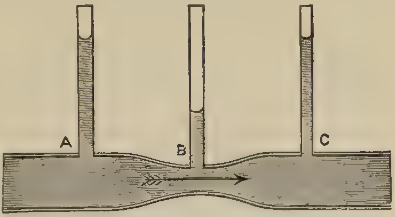

FIG. 189.—Flow of an incompressible fluid in a constricted tube.

Now, the velocity of the fluid at the section C must be equal to that at the section A, since the tube has equal cross-sectional areas at these two sections; therefore, as a particle travels from the section B to the section C, its velocity must decrease from V_2 to V_1; and for this to be possible, the pressure in front of the particle must be greater than that behind it. Hence, we conclude that, in passing from the section A to the section C, the velocity of the fluid increases and its pressure decreases, until B, the section of minimum area, is reached; and then the velocity decreases and the pressure increases until the section A is reached. From this we may make the important generalisation: **when a fluid traverses a constricted tube, and there is no loss or gain of gravitational or thermal energy, the pressure is greatest where the velocity is least.**

Writing zero for the quantities on the right hand side of the fundamental equation (p. 402), we have—

$$a_2V_2(p_2 + (1/2)\rho V_2^2) - a_1V_1(p_1 + (1/2)\rho V_1^2) = 0.$$

Since $a_2V_2 = a_1V_1$, we have $V_2 = (a_1/a_2)V_1$. Thus—

$$p_1 - p_2 = \tfrac{1}{2}\rho V_1^2 \left\{ \left(\frac{a_1}{a_2}\right)^2 - 1 \right\}.$$

This equation serves to determine the velocity V_1 in the wide section of the tube, in terms of the differences of pressure between the wide and constricted sections, and the areas of these sections.

This result is utilised commercially in the **Venturi water meter**; water flows through a horizontal constricted tube, and the pressures at the wide and constricted sections are determined by means of vertical gauge tubes (Fig. 189) in which the water rises to heights proportional to the pressures at the corresponding sections.

The investigation of the flow of a compressible fluid, such as a gas, is more difficult than that of the flow of an incompressible fluid; a particular problem will be solved subsequently, but at present it will suffice to say that when the variation of pressure is not very great, the general conclusions arrived at above still hold. The student will be familiar with many appliances which depend for their action on the principle that the pressure is least where the velocity is greatest. For instance, in the Bunsen burner, the gas escapes from a narrow tapering jet into a chamber below the tube at the top of which the gas is burnt. The pressure of the gas where it leaves the jet is very low, owing to the high velocity attained; consequently, air is drawn in through the lateral apertures in the chamber, and thus a mixture of air and gas, which can burn with a smokeless, non-luminous flame, reaches the top of the tube.

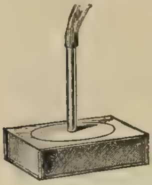

FIG. 190. — Match box, supported by blowing downwards upon it.

EXPT. 50.—Fit a piece of glass tubing, of about 4 mm. diameter, in a hole bored through the centre of a flat circular piece of wood. Arrange that the end of the glass tube does not project through the wood, but is just flush with the surface. Place the wood just above the upper surface of a match-box which is laid on the table, and blow downwards through the tube. The match-box will be drawn upwards, and supported so long as the blowing continues. (Fig. 190.)

In this case the air travels horizontally outwards from the aperture of the tube, through the narrow space between the wood and the match-box. Since the flow is radial, the tubes of flow must increase in area from the centre to the circumference of the air space, and therefore the velocity diminishes from the centre to the circumference. At the circumference the pressure is equal to that of the atmosphere; therefore, nearer to the centre the pressure is less than that of the atmosphere.

EXPT. 51.—Obtain a piece of glass tube, about 6 inches long and of about ½ inch internal diameter, and draw this out near its middle to form a constriction of about $\frac{1}{16}$ inch internal diameter. Connect one end of the glass tube to the water supply, by means of rubber tubing. Hold the glass tube with its open end inclined upwards, and slightly open the water tap, so as to fill the tube with water. Now turn the tap fully on, and place the open end of the glass tube below the surface of the water in a beaker. (Fig. 191.)

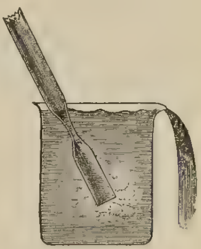

FIG. 191.—Water, at the temperature of the room, boiling in the constriction of a glass tube.

A hissing sound is heard, and the water which enters the beaker presents a foggy appearance. The pressure in the constricted section of the tube is so much reduced that the air dissolved in the water separates out, just as it would if the water were placed in a partial vacuum; the fogginess of the water which issues from the tube is due to minute bubbles of air distributed through it. The hissing sound is due, partly to the formation of the bubbles of air, and partly to the boiling of the water at the constricted section, and may be compared with the singing of a kettle. The hissing of water which issues from a partially-opened tap is due to similar causes. When the velocity of the water is very high, the phenomena described above are not observed; in this case the motion becomes turbulent (p. 403) and the results deduced from the theory of motion along stream-lines become inapplicable.

EXPT. 52.—Select a child's rubber balloon which, when inflated, is approximately spherical in shape; strike it with the hand so as to project it through the air with combined rotatory and translational velocities.

It will be observed that the path described depends on the magnitude and direction of the rotatory motion imparted to the balloon. If the balloon is struck below its horizontal diametral section (that is, if it is under-cut) it is set in rotation about a horizontal diameter perpendicular to its direction of translation, the motion being forwards on the lower surface of the balloon, and backwards on the upper surface. In this case the balloon soars upwards, and afterwards may travel backwards over the experimenter's head, and fall behind him. Golf and tennis balls exhibit similar peculiarities of movement when they have been set in motion by a "cutting" stroke.

Fig. 192 *a* represents a sphere rotating about a diameter perpendicular to the plane of the paper; the air surrounding the sphere is set in motion, its stream-lines being concentric circles in planes perpendicular to the axis of rotation. Fig. 192 *b* represents a sphere travelling in a straight line from left to right, without rotation; the air, pushed forwards in front of the sphere, flows backwards on either side, and ultimately travels up towards the rear surface of the sphere. Fig. 192 *c* represents a sphere which is rotating about a diameter perpendicular to the plane of the paper, and, in addition, is moving bodily from left to right; the stream-lines of the surrounding air are the resultants of those depicted in Fig. 192 *a* and *b*. On the side of the sphere where the flow, indicated in Fig. 192 *a*, is opposite to that indicated in Fig. 192 *b*, there is but little motion in the air; while on the opposite side of the sphere the velocity of flow of the air is considerable. The pressure is least where the velocity is greatest; thus the pressure of the air, on the side of the sphere where the component velocities reinforce each other, must be less than that on the side where they wholly or partially neutralise each other, and a resultant difference of pressure must act on the sphere tending to urge it from the side on which the velocity is least to that on which the velocity is greatest. Consequently, the path of the sphere is curved, being convex toward the side where the pressure is the greater.

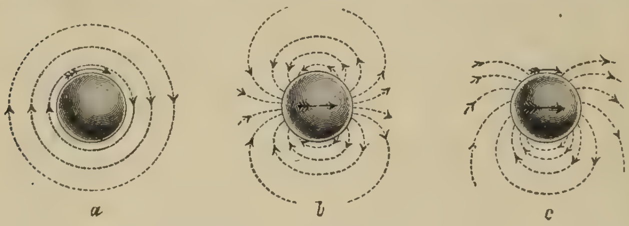

FIG. 192.—Stream lines in air, due to the motion of a sphere.

Let it be supposed that a spherical bullet is fired from a gun with a barrel bent downwards, as in Fig. 193. In traversing the barrel, the bullet is deflected downwards, and rolls along the concave side of the barrel; it issues from the muzzle with a rotatory motion which causes its subsequent path to curve upwards. Even if the barrel were straight, some accidental circumstance might impart a rotatory motion to the bullet, and then the path of the bullet through the air would be curved, and accurate marksmanship would be impossible. For this reason, spherical bullets have been discarded in favour of cylindrical projectiles which are set in rotation about their axes by a screw thread cut inside the barrel of the gun; in this case, the rotation is about an axis which coincides with the direction of motion, and thus produces no lateral difference of pressure on opposite sides of the bullet; the gyrostatic property (p. 71) of the rotating projectile ensures that the axis of rotation shall not change in direction.

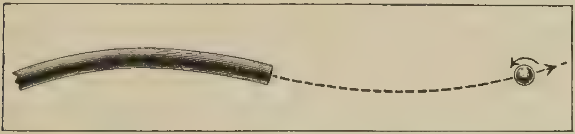

FIG. 193.—Path of a spherical bullet fired from a gun with a bent barrel.

When an electric current flows along a constricted conductor, there is no rise of potential in passing in the direction of the current from the constricted to the wider section of the conductor.

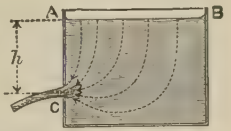

FIG. 194.—Liquid flowing from a small orifice in a tank.

Flow of a liquid from an orifice in a tank.—Let Fig. 194 represent a tank from which a liquid is issuing by way of a small orifice C, at a distance *h* below the free surface AB of the contained liquid. If the flow is not too rapid, it will occur along permanent stream-lines; these stream-lines must all start from the free surface of the liquid, for no sources are present: a continual flow away from the bounding walls of the tank is impossible, since if it occurred the space near the walls would be left empty. Outside the orifice, where the section of the issuing jet has become uniform, the internal pressure is equal to the atmospheric pressure P; for the curved boundary of the jet is flexible and extensible, and any difference between the internal and the external pressures would produce lateral contraction or expansion. (It is assumed that the section of the jet is so large that the internal pressure due to surface tension (p. 345) is negligible.) At the free surface of the liquid within the tank, the pressure is also equal to P.

From the form of the stream-lines, it is obvious that, near the surface AB, the section of a tube of flow is very great in comparison with the section of the same tube at the orifice C; and as the velocity is inversely proportional to the cross-section of a tube of flow (p. 378), it follows that the velocity at the surface AB is inappreciable in comparison with that at the orifice C. If a is the area of a tube of flow at the orifice, where the velocity is equal to V, we may write a for a_2 and V for V_2 in the fundamental eqation on p. 404. Also $a_1V_1=a_2V_2=aV$, and $p_1=p_2=P$, while the gravitational energy lost by a volume aV of the liquid, in its descent from the free surface AB to the orifice C, is equal to $g\rho aVh$. Thus—

$$aV(P+(1/2)\rho V^2)-aV(P+0)=g\rho aVh;$$

$$\therefore\quad (1/2)V^2=gh, \text{ and } V=\sqrt{2gh}.$$

Owing to the viscosity of the liquid, some mechanical energy is converted into heat within the tube of flow, and this will cause the velocity of the issuing jet to be somewhat less than that obtained above.

The cross-section of the issuing jet, at a small distance beyond the orifice, will not be exactly equal to the cross-section of the orifice; this is due to the fact that lines of flow must be continuous, and cannot be deflected abruptly, so that the flow from the orifice resembles that represented in Fig. 195. The contraction of the jet on leaving the orifice shows that the velocity in the plane of the orifice must be slightly less, and the pressure slightly greater, than at a short distance beyond the orifice where the section of the jet has become uniform.

The external pressure on the jet is everywhere equal to that of the atmosphere; thus, near to the orifice the pressure inside the jet is greater than that outside, and the excess of the internal over the external pressure tends to make the jet expand laterally; but the momentum of those parts of the liquid which are moving towards the axis of the jet, carries them on, until that part of their kinetic energy which is due to their radial velocity has been lost, and meanwhile the axial velocity of the liquid has increased owing to the excess of the pressure behind it. When the liquid reaches the section beyond which no further contraction occurs, the internal pressure is equal to the external pressure; it is to this section that the value of V obtained above refers.

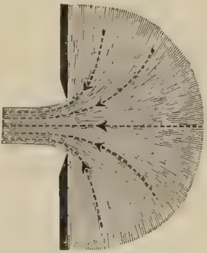

FIG. 195.—Jet of liquid issuing from an orifice in the wall of a tank.

From the equation—

$$\tfrac{1}{2}V^2=gh.$$

which applies to an inviscid liquid, it appears that **the work done on each unit mass of liquid before it issues from the orifice, imparts to it an amount of energy which would suffice to raise it to a height *h* above the orifice.** Hence, if the orifice is fitted with a nozzle which is turned vertically upwards, the liquid jet rises to a height *h* above the orifice (Fig. 196). The viscosity of a liquid will prevent the jet from rising exactly to the level of the free surface of the liquid within the tank.

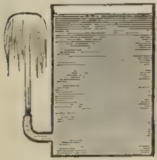

FIG. 196. — Vertical jet of liquid issuing from a tank.

Now let it be supposed that a long vertical tube is connected with the orifice, as shown in Fig. 197. In the first place, let the tube be empty, and let the orifice be closed. On opening the orifice, the liquid commences to rise in the tube, and will continue to rise until the gravitational energy of the liquid within the tube is just equal to the work done on that liquid before it issued from the orifice; when this condition has been attained, the liquid in the tube will be stationary for an instant. Let a be the sectional area of the tube, and H the height to which the liquid rises in it; then the gravitational energy of the liquid within the tube is equal to $\rho a\mathrm{H}\times(g\mathrm{H}/2)$, since the centre of gravity of the liquid in the tube is at a height $(\mathrm{H}/2)$ above the orifice. The work done on the mass $\rho a\mathrm{H}$ before it issued from the orifice is equal to $\rho a\mathrm{H}\times gh$, and thus—

$$\frac{g\rho a\mathrm{H}^2}{2}=g\rho a\mathrm{H}h;$$

$$\therefore\quad \mathrm{H}=2h,$$

and the liquid rises in the tube to a height h above the free surface of the liquid in the tank (Fig. 197). The liquid then flows back into the tank until the tube is emptied; and in the absence of viscosity, the tube would be alternately filled to a height H, and then completely emptied, at regular intervals. This reasoning applies to a perfect inviscid liquid; the losses of energy which always occur during the flow of a material liquid, damp the oscillations down, and the liquid in the tube finally comes to rest with its free surface at the same level as the surface of the liquid in the tank. In this case the gravitational energy of the liquid in the tube is equal to $\rho ah \times (gh/2)$, and this is only one half of the work done on the same liquid before it issued from the orifice: hence, half of the energy imparted to the liquid which finally remains in the tube, has been dissipated in the form of heat.

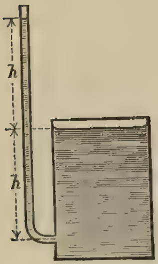

FIG. 197.—Oscillatory motion of a liquid in a vertical tube connected to a tank.

An analogous phenomenon occurs in connection with the charging of an electric condenser from a battery of constant E.M.F. Let the condenser require a charge Q to raise its potential to that of the battery, E; then the energy lost by the battery during the charging of the condenser is equal to EQ, and the energy finally stored in the charged condenser is equal to (EQ/2), so that half of the energy supplied by the battery has been dissipated. During the charging of the condenser, electrical oscillations are produced, which bear a general resemblance to those discussed above: that is, the potential of the condenser varies between zero and a value a little less than 2E; but the oscillations have a very high frequency, and die down rapidly.

Devices for obtaining a constant flow of water.—In many physical experiments a constant flow of water is required. If the flow occurs from an orifice below the surface of water in a vessel, the velocity of efflux decreases as the surface of the water in the vessel descends; the surface may be maintained in a stationary position by allowing water to flow into the vessel from the supply mains, the superfluity being discharged by a spout, or a tube near the top of the vessel; this arrangement ensures a very steady outflow from an orifice near to the bottom of the vessel.

Another device is represented in Fig. 198. Water is contained in a large bottle provided with a discharge tap; the neck of the bottle is closed with a cork bored to receive an open tube which descends to a point at a short distance above the level of the tap. When the flow has become steady, air is sucked into the bottle by way of the tube, and the water at the bottom of the tube is at atmospheric pressure; hence, the flow depends only on the vertical distance from the bottom of the air-inlet tube, to the bottom of the tube attached to the tap.

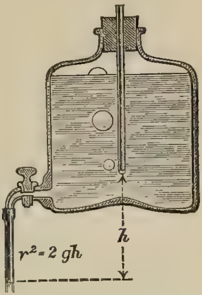

FIG. 198. — Arrangement for obtaining a steady flow of water from a bottle.

Radial flow of a liquid.—Let a liquid flow between two parallel horizontal plates, toward a hole in the middle of the lower plate, where it is discharged by a vertical pipe (Fig. 199). The flow must be radial, from the circumference to the centre of the space between the plates; hence, the lines of flow are radial, and if d is the distance between the plates, and θ is the angle between two neighbouring lines of flow, the sectional area of the corresponding tube of flow, at a distance r from the centre of the outlet hole, is equal to $dr\theta$. Let R be the radius of either plate, and let P be the pressure along the circumference of the space between the plates; then, if p is the pressure at a distance r from the middle of the outlet hole, we have (p. 402)—

$$a_2V_2(p + \tfrac{1}{2}\rho V_2^2) - a_1V_1(P + \tfrac{1}{2}\rho V_1^2) = 0.$$

Further $a_2V_2 = a_1V_1$, and $a_2 = dr\theta$, while $a_1 = dR\theta$. Therefore–

$$P - p = \tfrac{1}{2}\rho(V_2^2 - V_1^2) = \tfrac{1}{2}\rho V_1^2\left(\frac{R^2}{r^2} - 1\right).$$

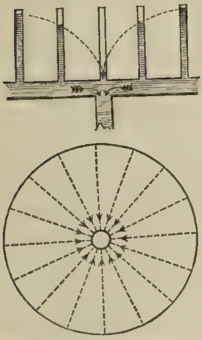

FIG. 199.—Radial flow of a liquid. (Vertical section above, plan below.)

Thus, the pressure falls as the outlet orifice is approached; and if vertical gauge tubes were fitted into the upper plate, the water in these would stand at heights corresponding with the pressures calculated from the equation just obtained.

When water is allowed to escape from the central hole in a lavatory basin, the surface of the water is drawn downwards over the hole, and sometimes air is sucked violently into the discharge pipe. Occasionally the water in the basin is set in rotation, but the negative presssure at the hole is due essentially to the radial flow of the water. As the liquid descends in the vertical discharge pipe, it fills less and less of the cross-section of that pipe; for the velocity of the liquid increases as kinetic energy is gained at the expense of gravitational energy which disappears.

If, in Fig. 199, the flow had been up the vertical pipe, and radially outwards between the plates, a similar negative pressure would have been produced in the middle of the space between the plates. (Compare exp. 50, p. 406.)

Force transmitted across the cross-section of a tube of flow.—The fluid, just about to pass through any cross-section of a tube or flow, acts like a piston, and exerts a force equal to pa on the fluid on the opposite side of the section, where p is the pressure at the section and a is the area of the section.

Further, if the fluid is flowing across the section with a velocity V, a mass $\rho a\mathrm{V}$ of fluid, possessing a velocity V, crosses the section in a second (p. 378), and the momentum carried per second through the section is equal to $\rho a\mathrm{V} \times \mathrm{V} = \rho a\mathrm{V}^2$. Now, a change of momentum at any section is equivalent to a force equal to the rate at which the momentum changes (p. 19); therefore a force equal to $\rho a\mathrm{V}^2$ acts, in the direction of the flow, on the fluid that has crossed the section already; and an equal force (the reaction) acts, in a direction opposite to that of the flow, on the fluid that is just about to cross the section.

Hence, **on either side of any cross-section of a tube of flow, the fluid is acted on by a resultant force of which the magnitude is equal to** $a(p+\rho\mathrm{V}^2)$; the fluid that has crossed the section already is urged onwards, and the fluid that is just about to cross the section is urged backwards, by a force of this magnitude.

The forces acting across the transverse section of a tube of flow, due respectively to **the pressure of the fluid** and to **the transfer of momentum across the section**, must be clearly distinguished from each other. The pressure acts equally in all directions (p. 34); but the force due to the transfer of momentum has a value which depends on the inclination of the section to the direction of flow of the fluid. If a tube of flow is cut obliquely by a plane, and the area of the section is equal to a, the force acting normally across this section, due to the pressure p of the liquid, is equal to pa. The normal component of the force due to the transfer of momentum across the section, is equal to

$$\rho a\mathrm{V}\cos\theta\times\mathrm{V}\cos\theta=\rho a\mathrm{V}^2\cos^2\theta,$$

where V denotes the resultant velocity of the fluid, and θ denotes the angle which the resultant velocity makes with the normal to the section. The tangential component of this force is equal to

$$\rho a\mathrm{V}\sin\theta\times\mathrm{V}\cos\theta=\rho a\mathrm{V}^2\sin\theta\cos\theta.$$

Let us apply this result to the flow of a liquid from a small orifice of area A in the wall of a tank (Fig. 195). Neglecting the effects of the viscosity of the liquid, the velocity V of the issuing jet, where its section has become uniform, is equal to $\sqrt{(2gh)}$ (p. 410), where h is the height of the free surface of the liquid within the tank above the orifice. Let a be the area of the jet where its section has become uniform; then, at this section, the pressure within the jet is equal to the atmospheric pressure P, and the liquid that has just crossed the section is urged forwards with a force equal to $a(\mathrm{P}+\rho\mathrm{V}^2)=a(\mathrm{P}+2g\rho h)$, and the liquid just about to cross the section is urged backwards with an equal force. The jet contracts, from the orifice where its area is A, to the section where its area is equal to a; and the atmospheric pressure acting on the tapering surface of the jet produces a resultant force $\mathrm{P}(\mathrm{A}-a)$, tending to urge the liquid back into the tank. Hence, at the orifice, the liquid is urged backwards into the tank with a resultant force equal to—

$$\mathrm{P}(\mathrm{A}-a)+a(\mathrm{P}+2g\rho h)=\mathrm{PA}+2g\rho ah.$$

Now, whether the orifice is opened or closed, the liquid within the tank, as a whole, is in equilibrium; that is, the forces applied to the liquid at its boundaries produce equilibrium. The resultant force applied to the liquid by the wall of the tank opposite to the orifice, is the same whether the orifice is opened or closed; hence, the resultant force, applied to the liquid across the plane of the wall which contains the orifice, must have the same value whether the orifice is opened or closed. When the orifice is closed, the force, acting across its plane on the liquid in the tank, is equal to $\mathrm{A}(\mathrm{P}+g\rho h)$; this force no longer acts when the orifice is opened, but its place is taken by a force equal to $(\mathrm{PA}+2g\rho ah)$; thus, *if we assume that the pressure over the wall which contains the orifice is not changed by opening the orifice*, it follows that no other change is produced in the forces acting on the liquid contained in the tank. Hence—

$$\mathrm{A}(\mathrm{P}+g\rho h)=\mathrm{PA}+2g\rho ah;$$

$$\therefore\quad 2a=\mathrm{A};$$

that is, the cross-sectional area of the jet where it has become uniform, is equal to half the area of the orifice. Now experiments show that when a liquid issues from a small orifice with sharp edges (Fig. 195) the cross-sectional area of the jet where it has become uniform is equal to about 0·64 times the area of the orifice. The reason for this discrepancy between theory and practice is to be found in the untruth of the assumption which has been italicised on p. 415. There is a radial flow towards the orifice along the circumjacent wall, and this radial flow produces a negative pressure over the wall (p. 413). Since the total force exerted upon the liquid within the tank, across the plane of the wall that contains the orifice, must be unaltered by the opening of the orifice, and the radial flow over the wall that surrounds the orifice produces a negative pressure, it follows that the force transmitted across the plane of the orifice itself must be increased by opening the orifice, and thus—

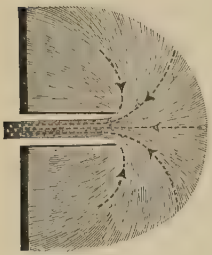

FIG. 200. — Borda's re-entrant mouthpiece.

$$PA + 2g\rho ah > A(P + g\rho h),$$

and therefore
$$2a > A.$$

The radial flow over the wall in which the orifice is pierced, may be eliminated by using a **Borda's re-entrant mouth-piece** ; this consists of a tube which projects some distance into the tank (Fig. 200) ; using this, it is found that when the tube is more than twice as long as the diameter of the orifice, the area of the issuing jet is approximately equal to half the area of the orifice.

The flow of a liquid through a flexible inextensible tube, has no tendency to move the tube laterally, provided that the ends of the tube are fixed.—This important law can be proved by the following train of reasoning. Let ABCDA (Fig. 201) represent an endless tube made of some flexible and inextensible material, and let a perfect inviscid liquid be flowing along this tube in the direction from A through B, C, and D, to A. In this case, the energy of the liquid is wholly kinetic ; if the cross-sectional area of the tube is uniform, the velocity

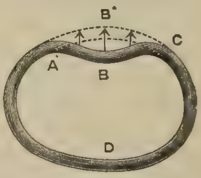

FIG. 201.—Flow of a perfect fluid in an endless tube.

is uniform at all points in the tube of flow, and there is no change of pressure along the tube (p. 403). If the flow of the liquid tends to move any part of the tube laterally, let the part ABC tend to move to the position AB′C when the remainder CDA of the tube is held stationary. During this motion, the parts of the liquid between A and C are subjected to lateral accelerations, and the acceleration of each part must be associated with a pressure-gradient in the direction of the acceleration (p. 368). Thus, as the liquid flows from A to B, it is flowing down the lateral pressure-gradient; while from B to C it is flowing up the pressure-gradient. If the fall of pressure from A to B is less than the rise of pressure from B to C, the liquid will lose energy in flowing from A to C, and the energy lost might be utilised in displacing the portion ABC of the tube and its contents. But, in this case, the pressure at C must be higher than that at A, and therefore there must be a fall of pressure in the stationary part CDA of the tube. This fall of pressure must entail an increase in the velocity of the liquid, and this involves a *gain* instead of a *loss* of energy, which is contradictory. Hence, the pressure at C must be equal to the pressure at A, and no energy can be lost by the liquid flowing from A to C along the tube: therefore no energy is available to set the portion ABC of the tube in motion laterally.

It may be left as an exercise for the student to prove that there is no tendency for the flow of a liquid to produce lateral motion in a tube with fixed ends, even when the section of the tube varies from point to point. In the first place, suppose that the section of the tube is constant along CDA, but varies in any manner along ABC, subject to the condition that the sections at A and C are equal. Next, suppose that the section at C is greater or less than that at A, but that the tube CDA tapers rapidly from C until a uniform section equal to that at A is reached.

Now suppose the liquid to possess viscosity; in this case the inner surface of the tube will be subjected to tangential forces parallel to the direction of flow, and it is clear that these forces can have no tendency to set the tube in motion laterally (*i.e.*, perpendicular to the direction of flow).

EXPT. 53.—Connect one end of a long rubber tube to the water supply, and coil the tube up in any manner; it will be found that when the flow is steady, the tube shows no tendency to move so long as its free end is held stationary. When the free end is released, it moves violently in a direction opposite to that in which the water is issuing from it; this is due to the force which acts across any section of a tube (p. 414).

When an electric current flows along a flexible conductor, the conductor tends to move to a position in which it forms an arc of a circle. In this respect, as in that mentioned on p. 409, the flow of an electric current differs radically from the flow of a fluid.

QUESTIONS ON CHAPTER XII

1. A liquid of density ρ flows along a uniform straight tube, and loses energy (due to viscosity, etc.) at the rate of e units per unit volume for each unit distance traversed. What must be the inclination of the tube to the horizontal, in order that the pressure of the liquid may be uniform throughout the tube?

2. A jet of liquid impinges normally on a plane surface, and the liquid flows tangentially along the surface away from the point of impact. Prove that the force exerted on the surface is equal to $\rho a V^2$, where a is the sectional area of the jet before it impinges on the surface, ρ is the density of the liquid, and V is the velocity of the jet. In what units is this force measured?

3. A tank is provided with an outlet nozzle which is shaped so that its internal surface merges, without any abrupt change of curvature, into the internal surface of the tank. Water escapes from the tank by way of the nozzle, which it completely fills. Prove that the force exerted across the external orifice of the nozzle is twice as great as the statical force that would be exerted on a plug that closed the orifice of the nozzle.

4. Kerosene is contained in a vessel under an internal pressure of two atmospheres. With what velocity will the kerosene escape into the atmosphere from an orifice in the tank?

{Density of kerosene $= 0{\cdot}83$ gm./(cm.)^3.
Pressure of an atmosphere $= 10^6$ dyne/(cm.)^2}.

5. Railway locomotive engines, travelling express over long distances, can fill their water tanks while in motion by means of the following device. The tank is provided with a vertical pipe, which extends downwards so that it can dip into a reservoir situated between the rails; the lower end of the pipe is bent at right-angles, so that it projects horizontally in the direction in which the locomotive is moving. If the upper end of the pipe is at a height h above the

surface of the water in the reservoir, and the velocity of the locomotive is equal to V, with what velocity will the water flow into the tank? Neglect viscosity, etc.

6. A locomotive engine, travelling at a speed of 40 miles per hour, scoops up water from a reservoir between the rails in the manner indicated in question 5. If the delivery pipe is 4 inches in diameter, and its upper end is 8 ft. above the surface of the water in the reservoir, what time will be required to fill the tank if its volume is equal to 200 c. ft.? Assume that the energy dissipated in the pipe, in the form of heat, is equal to $\frac{1}{3}$ of the work done in lifting the water.

7. A tube comprises two limbs inclined to each other at a right angle. One limb is placed in a vertical position, and the other is placed horizontally below the surface of a stream with its end pointing in a direction opposite to that in which the stream is flowing. To what height will the water rise in the vertical limb of the tube?

8. The sectional area of a constriction in a tube is equal to half the sectional area of the remainder of the tube. Water enters one end of the tube under a head of 10 ft., and escapes from the other end into the atmosphere, the motion within the tube being along stream-lines, from end to end. Neglecting loss of energy due to viscosity, calculate the pressure of the water in the constriction, and show that the water will be on the point of boiling in the constriction if its temperature is equal to 21·5° C. (Atmospheric pressure = 76 cm. of mercury, or 10^6 dyne/(cm.)^2 Vapour pressure of water at 21·5° C = 19·05 cm. of mercury.)

9. A heavy spherical body, suspended by a fine wire, is immersed in water contained in a beaker. When the body is set rotating about the wire as axis, and is brought near to the side of the beaker, it appears to be attracted by the part of the beaker nearest to it. Explain this phenomenon.

10. Water escapes from an orifice in the floor of a tank, and flows vertically downwards by way of a uniform pipe. Water is introduced into the tank at such a rate that its surface remains constantly at a small distance above the floor of the tank. Prove that it is impossible for the vertical pipe to be filled from end to end with the water which flows through it.

CHAPTER XIII

EXAMPLES TO ILLUSTRATE THE LAWS OF MOTION OF FLUIDS

In the present chapter, detailed attention will be devoted to a number of problems which can be solved by the aid of the laws derived in the two preceding chapters. These problems are all of considerable importance, either in their theoretical or their practical aspect.

SOURCES AND SINKS

Mutual attraction of two sources or two sinks.—The general character of the lines of flow due to two neighbouring sources is exhibited in Fig. 202; the lines of flow are precisely similar to the lines of electric force due to two positive charges.

It is a property of the electric field that the tension at any point, measured along the direction of the lines of force, is equal to $(N^2/8\pi)$, where N denotes either the number of tubes of force per sq. cm. at the place, or the force in dynes that would be exerted on a unit positive charge (measured in electrostatic units) at that place. Since N similar tubes of force together possess a cross-sectional area of one sq. cm., it follows that each tube has a cross-sectional area a equal to $(1/N)$ sq. cm. Hence, in the electric field, the tension (force per unit area) is equal to $(1/8\pi a^2)$ and the force across the section of a tube of force of area a is equal to $a(1/8\pi a^2)$, or $(1/8\pi a)$. Thus, the force acting across any cross-section of a tube of force is inversely proportional to the area of the cross-section.

A tube of electric force must leave a conductor normally; thus, if the area of the tube where it leaves the conductor is equal to a, it follows that a force equal to $(1/8\pi a)$ is exerted by the tube on the conductor. When a charged sphere is placed in the middle of a large room, the tubes of force leave the sphere uniformly in all directions ; therefore all of the tubes are equal in area where they leave the sphere, and their action on the sphere is to make it expand, but not to displace it bodily in any direction. When two similarly charged spheres are placed at a moderate distance apart, the tubes of force become distributed in a manner that may be inferred from Fig. 202. A lateral pressure is exerted between neighbouring tubes which extend in the same general direction, and this lateral pressure pushes the tubes away from the space immediately between the spheres, with the result that comparatively few tubes of force leave the sides of the spheres which are turned towards each other ; on the sides of the spheres which are turned away from each other, the tubes of force are both numerous and of small sectional area. This inequality in the distribution of the tubes of force causes the spheres to be pulled apart, thus producing an apparent repulsion between the spheres ; and the inequality increases as the spheres are brought closer together, thus giving rise to a repulsive force which varies inversely as the square of the distance between the centres of the spheres.

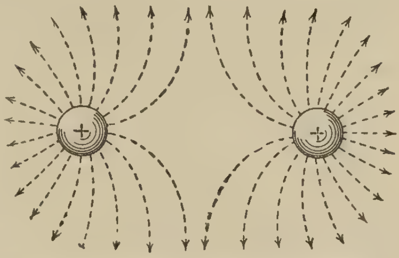

FIG. 202.—Lines of flow due to two sources.

Let us now examine the dynamical action on a source, due to the flow of fluid away from it. If we suppose the source to be surrounded by a spherical surface through which the fluid flows, then the flow across this surface into each tube of flow will produce a reaction, by which the fluid that is about to cross the surface is urged backwards towards the centre of the sphere. When the source is isolated, the forces exerted on the fluid which is about to cross the spherical surface have no tendency to displace the source bodily. But when two sources are placed, at a moderate distance apart, in an infinite ocean of fluid, the tubes of flow on the sides of the spheres which are turned away from each other are narrower and more numerous than those on the sides of the spheres which are turned towards each other; and as the reactions, tending to drive the fluid backwards towards the sources, increase with the velocity of flow of the fluid, we may infer that the sources will be subjected to resultant forces which tend to drive them towards each other; that is, **the sources will appear to attract each other.**

Let it be supposed that two equal sources are placed at a moderate distance apart, in an ocean of incompressible fluid of density ρ; and let the fluid created at the sources escape at the infinitely distant boundaries of the fluid. Let it be supposed that the fluid is created within small spherical surfaces, and let the area of any tube of flow which leaves one of these spherical surfaces be a, while the pressure and velocity of the issuing fluid are denoted respectively by p and V. Then (p. 414) the fluid about to cross the spherical surfaces is urged backwards, toward the centre of the sphere, with a force f given by the equation—

$$f = a(p + \rho V^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

At the infinitely distant boundaries of the fluid, let the pressure be uniform and equal to P; and let the tube of flow under consideration end there with an area equal to A, the velocity of flow being equal to v. Then A will be infinitely great, and v infinitely small, but $aV = Av$. Since there will be no creation or destruction of energy within the tube of flow, we have (p. 402)—

$$aV(p + \tfrac{1}{2}\rho V^2) = Av(P + \tfrac{1}{2}\rho v^2).$$

Dividing the left-hand side of the equation by aV, and the right-hand side by the equal quantity Av, and then writing $v = 0$, we obtain the equation—

$$p + \tfrac{1}{2}\rho V^2 = P.$$

Substituting in equation (1), we find that the fluid about to cross the sphere which surrounds the source is urged backwards with a force f given by the equation—

$$f = a(P + \tfrac{1}{2}\rho V^2).$$

The advantage gained by this substitution is due to the fact that p may vary over the surface of the sphere, but P, which denotes the uniform pressure at the boundary of the fluid, is constant.

The force f comprises two components. The first, equal to Pa, is equivalent to a uniform hydrostatic pressure P acting over the area a, and a uniform pressure has no tendency to displace the sphere bodily. The second component is equal to $(1/2)\rho a V^2$; and since the lateral boundaries of the tubes of flow have been chosen so that the velocity across any section is inversely proportional to the area of that section (p. 395), it follows that V is inversely proportional to a, and therefore aV^2 is inversely proportional to a. This component of f will vary from point to point of the spherical surfaces from which the fluid is issuing, in a manner similar to that in which the force varies over the surfaces of the electrically charged spheres.

The resultant force, acting on either of the charged spheres, varies inversely as the square of the distance between the spheres; and this force is due to the tubes of force leaving the sphere, the force exerted by each tube being inversely proportional to the sectional area of the tube where it leaves the sphere. The force exerted on either of the spherical sources, due to a particular tube of flow, is inversely proportional to the sectional area of the tube where it leaves the source; and the distribution of the tubes of flow around the spherical sources is precisely similar to the distribution of the tubes of force around the charged spheres. Hence, the forces acting on the spherical sources must vary in the same way as the forces acting on the charged spheres. Thus, **two sources must attract each other with a force inversely proportional to the square of the distance between them.**

It may be left as an exercise to the student to prove that two sinks also attract each other, while a source and a sink repel each other, with a force inversely proportional to the square of the distance between them.

Electric, magnetic, and gravitational attractions.—The similarity between the lines of flow due to two sources or two sinks, and the lines of force due to two similar electric charges, has suggested the hypothesis that lines of electric force are lines of flow in the ether, while electric charges are sources or sinks in the ether. But the utility of the conception of lines of force depends on the fact that it enables us to explain the force exerted by one electric charge on another; and if electric charges were sources or sinks in the ether, it follows that two similar electric charges must be two sources or two sinks, and then they would attract, while experiment shows that they repel each other. Hence, the suggested hypothesis breaks down at the very outset. Hypothetical explanations of the force exerted by one magnetic pole on another, in terms of sources and sinks in the ether, break down in a similar manner.

With regard to gravitation, the case at first sight appears to be more hopeful. Every particle of matter attracts every other particle of matter in the universe, and we might assume that all atoms are sources, or that all atoms are sinks in the ether, while the ether itself is an incompressible fluid. The creation of ether at the sources, or its destruction at the sinks, is not essential to this hypothesis; if we suppose that all atoms are small elastic spheres, which are expanding and pushing the ether away from their centres, they will produce a motion in the surrounding ether identical with that which would result if the atoms were sources in the strict sense of the term. This leads us to the hypothesis, due to Prof. Hicks, that atoms are elastic spheres which expand and contract rhythmically, subject to the condition that all are in similar phases of vibration at any given instant. As to the period of vibration, either this must be extremely small in comparison with the shortest observable interval of time, or it must be extremely large —greater, in fact, than the immense period of time in which, we may infer, the state of the universe has subsisted without an appreciable alteration in the gravitational attraction of matter. It is scarcely necessary to pursue this speculation further, in view of the fact that the properties of the ether must serve to explain, not only gravitation, but also the phenomena of electricity and magnetism; and if the ether is an incompressible fluid, the only form of its motion which might explain electric and magnetic phenomena is vortex motion, and this we have seen (p. 394) is incapable of furnishing the required explanation.

VORTEX MOTION

Properties of vortex filaments.—It now becomes necessary to discuss certain peculiarities of vortex motion which as yet have not been alluded to; these may be deduced without trouble from the properties of vortex filaments which have been determined already. In the first place, it has been proved that, in a perfect fluid, the strength of a vortex filament is constant and cannot be changed (p. 392). The fluid surrounding the filament moves irrotationally (p. 387); if the filament is straight, each particle of the surrounding fluid describes a circular path of which the centre is on the axis of the filament: the velocity at a distance r from the axis of the filament is equal to $m/2\pi r$, where m denotes the strength of the filament, (p. 395). The boundaries of the tubes of flow are chosen so that the product of the velocity and the cross-sectional area of a tube has a constant value (p. 395) ; hence, the cross-sectional area of a tube is inversely proportional to the average velocity across the section, and since the velocity is inversely proportional to the distance from the axis of the filament, it follows that the cross-sectional area of a tube is directly proportional to its distance from the axis of the filament. Further, the length of a tube of flow is directly proportional to its distance from the axis of the filament, since the tube forms a circular ring surrounding the filament ; hence, if the fluid is incompressible, the mass of fluid contained within a tube of flow is directly proportional to the square of its distance from the axis of the filament. The kinetic energy possessed by the fluid within a tube of flow is equal to half the product of the mass and the square of the velocity of the fluid ; and since the mass is directly proportional to the square of the distance from the axis of the filament, while the velocity is inversely proportional to the first power of the same distance, it follows that the kinetic energy possessed by the fluid which circulates in any tube of flow is independent of the distance of the tube from the axis ; that is, the kinetic energy within a tube of flow has an identical value for all tubes. Since an infinite number of tubes of flow may be described about a straight filament, immersed in an ocean of fluid which extends to an infinite distance on all sides, it follows that **an infinite store of energy is associated with a straight vortex filament immersed in an infinite ocean of perfect fluid.**

It has been pointed out already (p. 392) that vortex motion could neither be created nor destroyed in a perfect fluid. In a fluid possessing viscosity, we could generate a straight vortex filament by rotating two parallel and co-axial solid discs, which, for simplicity, might be supposed to lie in the plane and parallel boundaries of the fluid. The filament would stretch from one disc to the other, and the circulatory motion would spread out from the filament into the surrounding fluid ; but an infinite time would be needed for the motion to spread to an infinite distance from the filament, and during this time a continual expenditure of energy would be required to keep the discs rotating against the viscous drag exerted on them by the fluid.

Two similar vortex filaments (*i.e.*, two filaments rotating in similar directions) in an infinite ocean of perfect fluid must possess an infinite store of kinetic energy. But two dissimilar filaments of equal strengths must possess a finite store of energy; for the curl of the velocity along a circle enclosing both filaments will be equal to zero (p. 388), and therefore the average velocity, at a moderate distance from the point midway between the filaments, will be approximately equal to zero; that is, no part of the fluid will possess any appreciable amount of energy except those parts which are within a moderate distance from the filaments.

Many problems on the motion of fluids may be simplified by the application of a principle which now must be discussed. The earth is moving through space with a considerable velocity, and all solids and fluids which we can study at close quarters are moving with it. When a fluid, together with a body immersed in it, are merely moving with the earth, we commonly speak of them as stationary; and we know that no forces tending to set the body in motion relatively to the fluid will be called into play by the common motion of both with the earth. If the body is in motion relatively to the fluid, forces may be called into play which tend to modify this motion; but these forces are independent of the velocity common to the body and the fluid due to their motion with the earth, and this is true although the velocity of the earth is changing from instant to instant. Hence, it may be concluded that **the forces called into play by the motion of a body through a fluid will not be modified by impressing any linear velocity on the body, provided that the same velocity is impressed on the fluid.** In particular, it may be supposed that the body is brought to rest by impressing on it a velocity equal and opposite to its actual velocity, provided that the same velocity is impressed on the fluid; in this case the motion of the fluid relatively to the body will be identical with that which would be perceived by an observer who travelled through space with the body. This principle will now be used in deducing some interesting properties of vortex filaments.

Let it be supposed that a straight vortex filament is set in translatory motion in a direction perpendicular to its length, and that an observer travels through space with the filament. Then, to the observer, the filament will appear to be rotating about a stationary axis, while the surrounding fluid streams past in a direction opposite to that in which the filament and the observer are actually moving. The appearance presented can be determined by superposing a uniform linear motion, equal in magnitude but opposite in direction to the actual velocity of translation of the filament, upon the circular motion due to the filament when at rest; the resultant stream-lines are similar to the magnetic lines of force due to a straight electric current flowing perpendicular to a uniform magnetic field. Fig. 203 represents the apparent lines of flow of a vortex filament rotating in a clockwise direction, and travelling bodily from right to left.

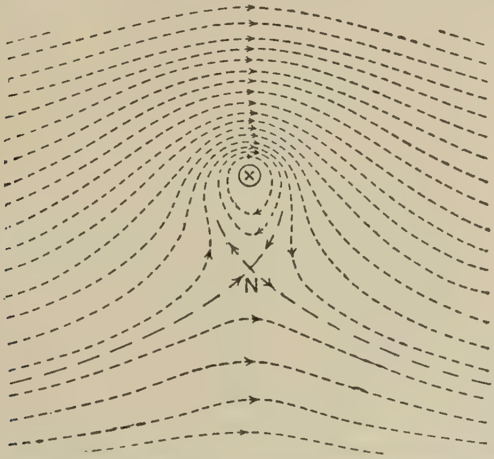

FIG. 203.—Relative lines of flow, due to a vortex filament travelling from right to left.

(To obtain the tubes of flow, imagine the diagram to be displaced through unit distance perpendicular to the plane of the paper.)

Two points of importance at once become apparent. In the first place, certain stream-lines form closed curves encircling the filament, and since there is no flow across any stream-line, it follows that the fluid enclosed by the outermost of these curves remains permanently within that curve; that is, the fluid in question travels bodily with the filament through the surrounding fluid.

The outermost stream line which forms a closed curve about the filament, is the one that passes through N (Fig. 203), the point where the resultant velocity is equal to zero. Let N be at a distance L from the axis of the filament; then the velocity at N, due to the rotation of the filament, is equal to $(m/2\pi L)$, where m is the strength of the filament (p. 395); and this velocity is equal in magnitude, but opposite in direction to the velocity V due to the uniform linear motion imparted to the surrounding fluid, so that—

$$\frac{m}{2\pi L} = V, \text{ and } L = \frac{m}{2\pi V}.$$

If we imagine that a number of filaments of equal strengths are set in translatory motion with different velocities, then each filament will carry with it a mass of fluid which is directly proportional to L^2, and therefore inversely proportional to the square of the velocity of translation V; and the kinetic energy due to the translatory motion of this fluid will have equal values for all the filaments.

Again, it becomes apparent from Fig. 203 that the velocity of the fluid is equal to zero at N, while it has a large value on the opposite side of the filament; hence, on the side of the filament which faces towards N, the pressure must be greater than on the opposite side of the filament (p. 405), and therefore the filament cannot travel in a straight line. If the filament was originally set in motion in a straight line, it will finally revolve in a circular orbit to which its original direction of motion is tangential, the centre of the circular orbit being on the side of the filament remote from N. The excess of pressure, on the side of the filament which faces away from the centre of its orbit, supplies the centripetal force necessary to keep the filament moving in that orbit. (Compare with expt. 52, p. 407).

The **stream-lines due to two straight and parallel vortex filaments** are similar to the magnetic lines of force due to two straight and parallel electric currents. Fig. 204 represents the stream-lines due to two equal and similar vortex filaments, on the supposition that the axes of the filaments are stationary. It will be noticed at once that the stream-lines are further apart on the side of the filaments which face each other than on the opposite sides of the filaments; hence, on the sides of the filaments which face each other the velocity is less, and the pressure is greater, than on the sides of the filaments which face away from each other, and the filaments apparently repel each other. Thus the application of forces is necessary to keep the axes of the filaments stationary.

Let it be supposed that the flow around the filaments is established while the axes of the filaments are held stationary, and that subsequently the filaments are set free. On gaining their freedom, the filaments commence to move away from each other: to determine their ultimate motions, let it be supposed that an observer accompanies one of the filaments, and always faces in the direction in which the filament is moving. At the moment of starting, the observer will see the fluid moving backwards on his right-hand side, and forwards with an equal velocity on his left hand side (Fig. 204). Directly the filament commences to move forwards, it will appear to the observer that an equal backward motion is imposed on the whole of the surrounding fluid; thus, the relative velocity of the fluid is increased on the right-hand side, and diminished on the left hand side of the observer, and therefore the pressure acting on the left-hand side is greater than that acting on the right-hand side of the filament. Consequently, the filament does not move along the straight line drawn through the positions of the filaments at the moment of starting, but its path curves away to the right; and the

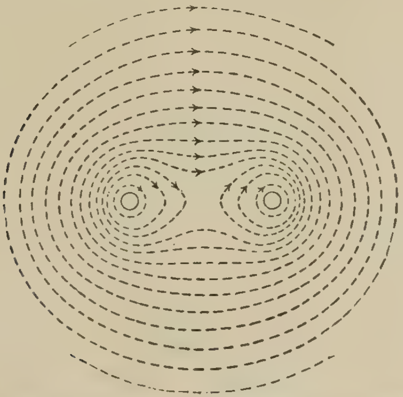

FIG. 204.—Lines of flow due to two similar straight vortex filaments in fixed positions.

(To obtain the tubes of flow, imagine the diagram to be displaced through unit distance perpendicular to the plane of the paper.)

faster the filament moves, the greater is the curvature of its path. Ultimately the filaments will travel along a circular path described about the point midway between them as centre; if, as in Fig. 204, the filaments are positive (that is, if the filaments rotate about their axes in a clockwise direction) they will travel around their common orbit in a clockwise direction. The velocity with which each filament travels can be found without difficulty; for the velocity at any point in the fluid is the resultant of the velocities produced at that point by the two filaments (p. 383), and at a point on the axis of a straight filament there is no linear velocity due to the rotation of that filament, while the velocity due to the other filament, which is at a distance 2R(say), is equal to $m/(2\pi \times 2R)$, (p. 395), where m is the strength of either filament. In this case the relative velocity of the fluid is greater on the sides of the filaments which face each other than on the opposite sides, and the consequent excess of pressure urges each filament towards the centre of its circular orbit with a force just equal to the centripetal force necessary to keep the filament moving in its circular path.

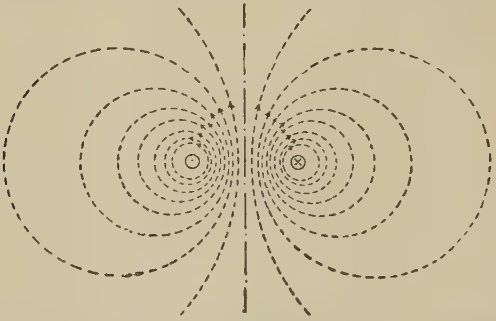

FIG. 205.—Lines of flow due to two dissimilar straight vortex filaments in fixed positions.

(To obtain the tubes of flow, imagine the diagram to be displaced through unit distance perpendicular to the plane of the paper.)

Fig. 205 represents **the stream-lines due to two equal but dissimilar vortex filaments,** on the supposition that the axes of the filaments are stationary. The filament on the right-hand side rotates in a clockwise sense, while the other filament rotates in an anti-clockwise sense; in the space between the filaments the stream-lines are closer together than elsewhere, and thus on the sides of the filaments which face each other the pressure is less than on the sides of the filaments which face away from each other, and the filaments appear to attract each other.

It can be proved without difficulty that, if the filaments represented in Fig. 205 are set free, they will commence to move towards each other, but their paths will be deflected towards the top of the page. Ultimately the two filaments will travel abreast with equal velocities, their common direction of motion being perpendicular to the line that joins them, and from the bottom to the top of the page. When they are at a distance 2R apart the linear velocity of either filament must be equal to $m/(2\pi \times 2R) = (m/4\pi R)$, where m is the numerical value of the strength of either filament. Hence, the closer the filaments are to each other, the greater is their common translatory velocity; and if their translatory velocity is reduced (as, for instance, if they approach a plane wall normally) the filaments must separate, and they must be at an infinite distance apart before they can become stationary.

The similarity between the lines of flow in the fluid surrounding two vortex filaments, and the lines of magnetic force in the space surrounding two conductors through which electric currents are flowing (Figs. 204 and 205), has suggested the hypothesis that electric currents are really ethereal vortex filaments, while the lines of magnetic force in the surrounding space are lines of irrotational flow in the ether. But it is known that two straight and parallel electric currents, flowing in the same direction, attract each other; and it has been proved that two similar vortex filaments repel each other. Consequently, this attempt to explain electro-magnetic phenomena, in terms of lines of flow in the ether, breaks down at the outset (compare p. 394).

Vortex rings.—It has been proved (p. 375) that a vortex filament cannot terminate within a fluid. In the preceding discussion some of the most important properties of vortex filaments which terminate at the boundaries of a fluid have been deduced; it remains to devote some attention to endless vortex filaments. One of these could be constructed, in imagination, by uniting the ends of a filament such as has been discussed already. An endless vortex filament is called a vortex ring; in the form most commonly met with, the filament resembles a curtain ring in shape.

EXPT. 54.—Obtain an ordinary cardboard collar box, and bore a clean circular hole, of about one to two cm. diameter, in the lid. Place a piece of smouldering brown paper in the box, and fit the lid in place. On striking a sharp blow on the back of the box, a beautifully formed vortex ring will issue from the hole in the lid, and travel for a considerable distance through the air.

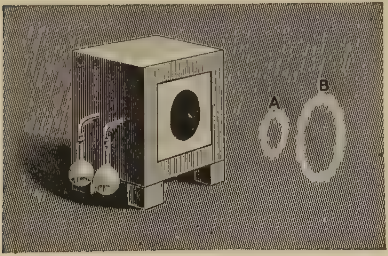

Fig. 206.—Method of producing large vortex rings.

Vortex rings on a larger scale can be obtained by aid of the arrangement, due to the late Prof. Tait, represented in Fig. 206. The back of a large wooden box is covered with canvas, and a circular hole is cut in a piece of tin plate which is fitted over a large hole in the front of the box. Vapours of ammonia and hydrochloric acid are introduced into the box, by way of tubes connected to glass flasks which contain the corresponding liquids ; a brisk evolution of the vapours can be ensured by heating the flasks. Dense clouds laden with sal-ammoniac are formed within the box ; on striking the canvas back of the box, a large vortex ring issues from the hole in the tin plate, and travels for a distance of 20 to 30 ft. through the air.

As already explained, the axis of a vortex filament is the line about which the filament rotates. In a circular vortex ring, the axis of the filament takes the form of a circle ; each particle of the filament revolves about an element of this circle as axis, and so continually crosses and recrosses the plane which contains the circle. The connection between the direction of rotation and the direction of translation of the filament can be seen from Fig. 207. The ring as a whole always travels in a direction perpendicular to the plane that contains the circular axis of the filament ; and each particle of the filament revolves about the circular axis, so that it moves forwards when on the

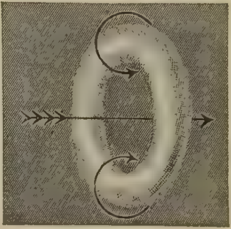

Fig. 207.—A vortex ring.

concave side and backwards when on the convex side of the circular axis. If a plane, perpendicular to the plane of the circular axis, be drawn through a diameter of the circle, the lines of flow in this plane will have the form represented in Fig. 208, if the vortex ring is stationary in space. It at once becomes apparent that a vortex ring cannot be in equilibrium when stationary, just as two parallel straight vortices of opposite signs cannot be in equilibrium when their axes are stationary (p. 431). The ring represented in Fig. 208 can be in equilibrium only when it is moving from the bottom to the top of the diagram; for the motion at any point on the circular axis is due to the circulatory flow derived from the circular filament as a whole.

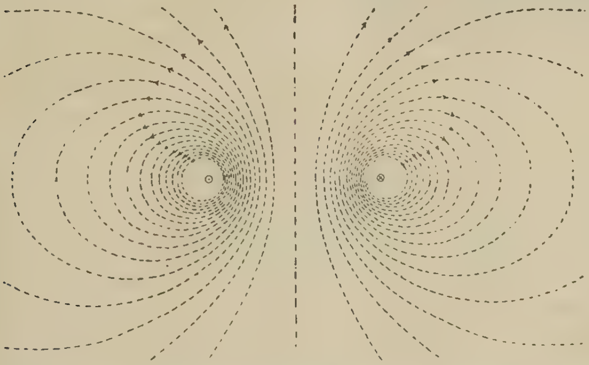

FIG. 208.—Lines of flow due to a stationary vortex ring.

(To obtain the tubes of flow, imagine the diagram to be rotated through a small angle about the middle dotted line extending from top to bottom of the diagram.)

The calculation of the translational velocity or a vortex ring is a matter of some difficulty, but the following argument will explain the conditions which determine the direction and magnitude of this velocity.

Let Fig. 209 represent part of an endless vortex filament, and let it be required to determine the velocity with which any point P on the axis of the filament is moving. Divide the filament into short elements; then the element within which the point P lies will produce no linear velocity at P, for the element produces a circulatory motion only about its axis, and therefore about P. Any other element will produce a motion at P in the direction of the arrow.

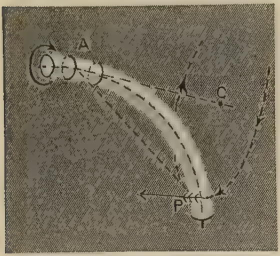

FIG. 209.—Part of a vortex ring.

(The dotted circle shows the direction of the velocity due to the element at A.)

The axis of an element A approximates to a short straight line; at any point in the surrounding fluid, the element produces a circulatory motion about this axis, produced if necessary. The further the point P is from the element A, the smaller will be the velocity at P due to the element; at a given distance from the element, the velocity produced is greatest when the line PA joining the point to the element is perpendicular to the element, and the velocity falls off as the angle PAC between the line and the axis of the element diminishes. If the vortex filament is strongly curved, an element very near to P will add considerably to the velocity at P; firstly, on account of the nearness of the element, and secondly, because the line joining P to the element makes a fairly large angle with the axis of the element. On the other hand, if the curvature of the filament is small, the velocity at P will be small.

Thus, **if a vortex ring is of large diameter, its translational velocity will be small**; for, if we choose any point on the circular axis of the ring, the line joining this point to any adjacent element will make a very small angle with the axis of that element, while any element whose axis is inclined at a large angle to the line joining it to the point will be far distant. Conversely, vortex rings of small diameter will move with great velocity. When a vortex ring is projected normally towards a wall, the translational velocity of the ring must diminish as the wall is approached, and this entails an increase in the diameter of the ring, and an increase in the length of the filament; also, as the length of the filament increases, the angular velocity of rotation of the filament increases (p. 392).

If a vortex ring is projected from an elliptical hole, it exhibits a very interesting type of motion. Let it be supposed that the major diameter of the elliptic hole is horizontal, and that the vortex ring is projected in a horizontal direction. The shape of the ring, as it leaves the hole, is that of an ellipse with its major diameter horizontal; but the parts of the ring near to the ends of the major diameter are most strongly curved, and therefore these parts move forward more quickly than the remainder of the ring. Hence the ring very quickly acquires a shape similar to that represented in Fig. 210. The upper and lower sides of the ring become concave in front, and this curvature causes the upper side of the ring to move upwards, and the lower side downwards. Consequently, after a certain lapse of time, the ring acquires the shape of an ellipse with its major axis vertical; and then the horizontal diameter begins to increase until the ring has the shape of an ellipse with its major axis horizontal. The ring wriggles forwards in a grotesque manner, the most strongly curved parts always moving forward with the greatest velocity.

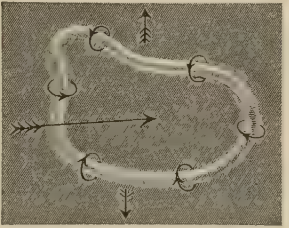

FIG. 210.—The motion of an elliptical vortex ring.

When two vortex rings (B and A, Fig. 206) are projected in rapid succession from a circular hole, the one in advance, B, slows down and expands laterally, while the one in the rear, A, increases in speed and contracts laterally. In this case, the motion of a point on the axis of either ring is due, not only to the circulatory motion derived from that ring, but also to that derived from the other ring. Presently A passes bodily through B and shoots ahead; then the speed of A diminishes and that of B increases, until B passes bodily through A, and so on. In this manner each vortex filament contrives to revolve about the other (compare p. 431).

It has been proved (p. 374) that a vortex filament tends to contract longitudinally; the question now arises: how is it that a vortex ring maintains its size so long as its translatory velocity remains constant, instead of becoming smaller and smaller? The answer to this question becomes obvious, when the distribution of the lines of flow about the moving vortex ring is realised. Let it be supposed that an observer travels with the vortex ring; then the appearance presented to the observer will be the same as if the ring and the observer were stationary.

and a uniform velocity, equal in magnitude but opposite in direction to their actual translatory velocity, were imposed on the surrounding fluid. Now, the lines of flow due to a stationary vortex ring are similar to the lines of magnetic force due to an electric current flowing along a conductor bent into a circle (compare p. 428), and the lines of flow due to a uniform velocity are similar to the lines of magnetic force due to a uniform magnetic field, such as the horizontal component of the earth's field. Hence, the lines of relative flow of the moving vortex ring will be in agreement with Fig. 211, which also represents the lines of force due to an electric current flowing around a circular

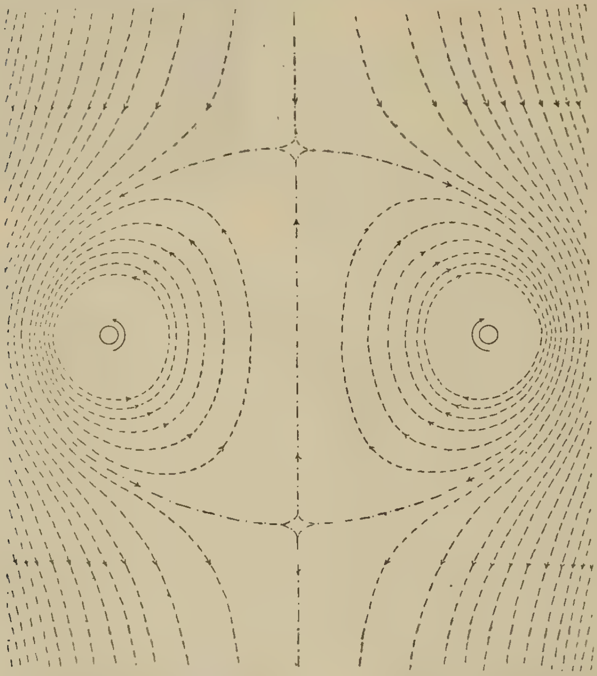

FIG. 211.—Lines of relative flow, due to a vortex ring moving from the bottom to the top of the page.

(To obtain the tubes of flow, imagine the diagram to be rotated through a small angle about the straight line extending from top to bottom of the diagram.)

coil of which the plane is perpendicular to the horizontal component of the earth's field, the direction of the current being such that its field at the centre of the coil is opposed to the horizontal component of the earth's field. It will be seen at once that the velocity of flow across the plane which contains the circular axis of the ring is less where the fluid is flowing forwards through the ring than where it is flowing backwards outside the ring; the consequent difference of pressure tends to make the ring expand, and this neutralises the natural tendency of the filament to contract.

Reasoning similar to that used on p. 427 shows that an approximately spheroidal volume of the fluid is carried onwards by the vortex ring, and this endows the ring with considerable kinetic energy. This explains why an obstacle may be moved by the impact of a vortex ring projected against it.

The strength of a vortex filament in a perfect fluid could not be altered by any human agency (p. 392), and therefore a vortex ring in a perfect fluid would be uncreatable and indestructible. The late Lord Kelvin formulated an hypothesis which accounts for the indestructibility of matter, by assuming that atoms are vortex rings in the ether, the latter being supposed to be a perfect fluid. This hypothesis suggests no explanation of gravitational attraction, and it possesses other disadvantages which have never been overcome; for instance, a vortex ring cannot become stationary unless its diameter is infinitely large, and therefore at the absolute zero of temperature, where molecules are stationary, vortex ring atoms would be infinitely large. In the laboratory it is possible, at the present time, to obtain temperatures only a few degrees above the absolute zero, and at these low temperatures matter exhibits no properties which would lead us to conclude that the atoms are larger than at ordinary temperatures.

FLOW OF A COMPRESSIBLE FLUID.

The flow of a compressed gas from an orifice in the containing vessel.—It has been pointed out already (p. 406) that the flow of a compressible fluid, such as a gas, is much more difficult to determine than that of an incompressible fluid. The enhanced difficulty is due to the fact that an alteration of pressure generally produces an alteration in the density of a gas. Thus, as a gas travels along a tube of flow, if its pressure diminishes, its density also will diminish; further, as the gas expands, it gains energy at the expense of the heat which disappears.

Let it be supposed that gas, at a pressure p_1, is contained within a vessel, and escapes from a small orifice into the surrounding space where the pressure is lower. Within the vessel, tubes of flow will converge towards the orifice. At a distance from the orifice, a tube will be wide, say of cross-sectional area A, and the velocity V_1 of the gas will be vanishingly small; the pressure of the gas will be sensibly equal to p_1, the pressure that would exist everywhere within the vessel if the orifice were closed. Nearer to the orifice, let the cross-sectional area of the tube be a, the velocity of the gas being V, and its pressure p. Where the pressure of the gas is p_1, let the density of the gas be ρ_1; and where the pressure is p, let the density be ρ. Then, when the flow has become steady—

$$\rho a V = \rho_1 A V_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The mass of gas $\rho_1 A V_1$ which, in one second, crosses the section of area A, must expand so that its pressure falls from p_1 to p before it crosses the section of area a. Since gases are bad thermal conductors, it may be assumed that no appreciable quantity of heat flows across the lateral boundaries of a tube of flow, and in this case the expansion is approximately adiabatic. Let the absolute temperature of the gas fall from T_1 to T during the expansion; the heat lost by the gas must have been transformed into mechanical energy, so that the mechanical energy carried across the section of area a in one second, exceeds that carried across the section of area A in the same time, by the mechanical equivalent of the heat lost by the mass, $\rho_1 A V_1$ of gas in expanding adiabatically so that its pressure falls from p_1 to p.

If unit mass of gas is cooled, at constant volume, through $(T_1 - T)$ degrees, the heat lost is equal to $s_v (T_1 - T)$, where s_v is the specific heat of the gas at constant volume; in this case the heat is conducted away from the gas, and no work is done. If unit mass of the same gas expands adiabatically so that its temperature falls from T_1 to T, an equal quantity of heat is lost by the gas; for most gases are approximately perfect, and the thermal energy of a perfect gas is confined to the kinetic energy of its constituent molecules, so that a reduction in the mean molecular velocity of the gas, corresponding to a fall of temperature from T_1 to T, entails the same loss of thermal energy however the fall of temperature may be produced. But when a gas expands adiabatically, the thermal energy lost is converted into mechanical energy; thus, the mechanical energy E_1 gained by the adiabiatic expansion of unit mass of the gas so that its temperature falls from T_1 to T, is given by the equation

$$E_1 = Js_v(T_1 - T),$$

where J denotes the mechanical equivalent of unit quantity of heat.

If s_p denotes the specific heat of the gas at constant pressure, it can be proved[1] that—

$$s_p - s_v = R/J,$$

where R denotes the product of the pressure and volume of unit mass of the gas, divided by its absolute temperature. Multiplying the value of E_1 by (R/J), and dividing it by the equal quantity $(s_p - s_v)$, we find that—

$$\begin{aligned} E_1 &= \frac{s_v}{s_p - s_v} R(T_1 - T) \\ &= \frac{1}{\gamma - 1} R(T_1 - T), \end{aligned}$$

where $\gamma = \frac{s_p}{s_v}$.

The mechanical energy E, gained by the expansion of any mass m of the same gas under similar conditions, is given by the equation—

$$E = mE_1 = \frac{mR(T_1 - T)}{\gamma - 1}.$$

In this equation, $m R T_1$ is equal to the product of the pressure p_1 and the volume (say v_1) of the mass m of gas, before the expansion; similarly $m R T$ is equal to the product of the pressure p and the volume (say v) of the same mass of gas, after the expansion. Hence—

$$E = \frac{p_1 v_1 - pv}{\gamma - 1}.$$

The mass of gas $\rho_1 AV_1 = \rho aV$, has a volume AV_1 before the expansion, when the pressure is p_1; while its volume is equal to aV after the expansion, when the pressure has fallen to p. The mechanical energy E gained by the adiabatic expansion of this mass of gas, is given by the equation—

$$E = \frac{p_1 AV_1 - paV}{\gamma - 1}$$

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[1] See the Author's *Heat for Advanced Students* (Macmillan), p. 301.

From the fundamental equation given on p. 402, it follows that—

$$aV(p+\tfrac{1}{2}\rho V^2) - AV_1(p_1+\tfrac{1}{2}\rho_1 V_1^2) = \frac{p_1AV_1 - paV}{\gamma - 1};$$

$$\therefore\quad aV\left(p+\frac{p}{\gamma-1}+\tfrac{1}{2}\rho V^2\right) - AV_1\left(p_1+\frac{p_1}{\gamma-1}+0\right) = 0,$$

since V_1^2 is vanishingly small. Dividing the first term by ρaV, and the second term by the equal quantity $\rho_1 AV_1$, we obtain the equation—

$$\frac{\gamma}{\gamma-1}\frac{p}{\rho} + \tfrac{1}{2}V^2 - \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = 0.$$

$$\therefore\ V^2 = \frac{2\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1} - \frac{p}{\rho}\right).$$

When a mass of gas, which has a volume v_1 at a pressure p_1, expands adiabatically until the pressure falls to p and the volume increases to v, the following relation[1] holds—

$$pv^\gamma = p_1 v_1^\gamma.$$

Since the density is inversely proportional to the volume of the gas, it follows that—

$$p\left(\frac{1}{\rho}\right)^\gamma = p_1\left(\frac{1}{\rho_1}\right)^\gamma.$$

Thus—

$$\frac{1}{\rho} = \frac{1}{\rho_1}\left(\frac{p_1}{p}\right)^{\frac{1}{\gamma}},$$

so that—

$$V^2 = \frac{2\gamma}{\gamma-1}\left\{\frac{p_1}{\rho_1} - \frac{p}{\rho_1}\left(\frac{p_1}{p}\right)^{\frac{1}{\gamma}}\right\}$$

$$= \frac{2\gamma}{\gamma-1}\cdot\frac{p_1}{\rho_1}\left\{1 - \frac{p}{p_1}\cdot\left(\frac{p_1}{p}\right)^{\frac{1}{\gamma}}\right\}$$

$$= \frac{2\gamma}{\gamma-1}\frac{p_1}{\rho_1}\left\{1 - \left(\frac{p}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

Equation (2) would suffice to determine the velocity V of the escaping gas, if we might assume that the pressure p of the gas, as it leaves the orifice, is equal to the pressure outside the vessel at a distance from the orifice. This assumption, however, cannot be tenable in all cases. It would be tenable if the escaping gas always formed a jet similar to that formed by a liquid flowing from an orifice (p. 410). But if we suppose that the gas escapes into a vacuum, it is clear that the tubes of flow must expand as they leave the orifice, for the gas must flow towards all points in the external space ; and an alteration in the width of a tube of flow entails an alteration of pressure, so that in this case the pressure at the orifice must differ from that at distant external points.

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[1] See the Author's *Heat for Advanced Students* (Macmillan), p. 320.

Condition of flow of gas.—Let a gas flow along a tube of flow, subject to the condition that it continually proceeds to points at lower and lower pressures. Using the notation explained above, the velocity V, at any point in the tube where the pressure is equal to p, is given by equation (2). The mass of gas crossing any section of the tube in a second is independent of the position of the section selected, so that ρaV must be independent of p. Since—

$$\rho=\rho_1(p/p_1)^{\frac{1}{\gamma}},$$

it follows that

$$\rho a\mathrm{V} = \rho_1\left(\frac{p}{p_1}\right)^{\frac{1}{\gamma}} a \sqrt{\frac{2\gamma}{\gamma-1}\frac{p_1}{\rho_1}}\sqrt{\left\{1-\left(\frac{p}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right\}}$$

$$= a\sqrt{\frac{2\gamma p_1\rho_1}{\gamma-1}}\sqrt{\left\{1-\left(\frac{p}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right\}\left(\frac{p}{p_1}\right)^{\frac{2}{\gamma}}},$$

and the right-hand side of this equation must have the same value, whatever may be the value of the sectional area a and the pressure p at the point chosen.

Let

$$\left\{1-\left(\frac{p}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right\}\left(\frac{p}{p_1}\right)^{\frac{2}{\gamma}} = \mathrm{P};$$

then $a\sqrt{\mathrm{P}}$ must be constant, since the quantity $\sqrt{\{2\gamma p_1\rho_1/(\gamma-1)\}}$ does not involve p, and must therefore have the same value whatever section of the tube of flow is chosen.

Now, P is the product of two terms. The first is equal to zero when $p=p_1$, and its value increases as p decreases until it becomes equal to unity when $p=0$. The second term is equal to unity when $p=p_1$, and its value decreases as p decreases until it becomes equal to zero when $p=0$. Thus, as p decreases from the value p_1 to zero, the value of P at first increases, owing to the increase in the first term; ultimately, however, the value of P must diminish and become equal to zero when $p=0$, owing to the diminishing value of the second term. But $a\sqrt{P}$ must remain constant for all values of P; consequently the value of a must diminish so long as P increases, and then must increase to infinity as the value of P falls to zero.

Thus, **if a gas flows along a tube of flow, subject to the condition that the pressure falls from p_1 at the end where the gas enters with negligible velocity, to zero at the other end of the tube; then, as the gas flows to points at lower and lower pressures, the sectional area of the tube at first decreases, and attains a minimum value where the pressure p makes P a maximum; subsequently the tube expands laterally, and becomes infinitely wide where the pressure is equal to zero.**

To determine the value of p which makes P a maximum, let (p/p_1) be denoted by r; then—

$$P = \left(1 - r^{\frac{\gamma-1}{\gamma}}\right) r^{\frac{2}{\gamma}} = r^{\frac{2}{\gamma}} - r^{\frac{\gamma+1}{\gamma}}.$$

When P has a maximum value, a small increase in the value of p, which causes the value of r to increase to $(r+\delta)$, will produce no increase in the value of P. Now—

$$(r+\delta)^{\frac{2}{\gamma}} = r^{\frac{2}{\gamma}}\left(1+\frac{\delta}{r}\right)^{\frac{2}{\gamma}} = r^{\frac{2}{\gamma}}\left(1+\frac{2}{\gamma}\frac{\delta}{r}+\ldots\ldots\right).$$

Similarly—

$$(r+\delta)^{\frac{\gamma+1}{\gamma}} = r^{\frac{\gamma+1}{\gamma}}\left(1+\frac{\gamma+1}{\gamma}\frac{\delta}{r}+\ldots\ldots\right).$$

Therefore, the increase in the value of P, due to the increase in the value of r to $r+\delta$, is equal to—

$$r^{\frac{2}{\gamma}} \cdot \frac{2}{\gamma}\frac{\delta}{r} - r^{\frac{\gamma+1}{\gamma}} \cdot \frac{\gamma+1}{\gamma}\frac{\delta}{r}$$

and this must be equal to zero. Thus—

$$\frac{(\gamma+1)}{\gamma} r^{\frac{\gamma+1}{\gamma}} = \frac{2}{\gamma} r^{\frac{2}{\gamma}};$$

$$\therefore \quad r^{\frac{\gamma-1}{\gamma}} = \frac{2}{\gamma+1},$$

and the value of p (say p_0) which makes P a maximum, is given by the equation—

$$r = \frac{p_0}{p_1} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The following table gives the value of p_0/p_1 for a few typical gases—

Gas	γ	$\frac{p_0}{p_1}$
Air	1·408	0·525
Hydrogen	1·42	0·417
Carbon dioxide	1·26	0·537

Thus it appears that **a tube of flow will have a minimum sectional area at a point where the pressure is equal, roughly, to one half of the pressure at the end of the tube where the gas enters with negligible velocity.**

Velocity of efflux from an orifice, as influenced by the external pressure.—Let it now be supposed that a compressed gas escapes from an orifice in its containing vessel, into the external space where the pressure is equal to zero. Within the vessel, at a distance from the orifice, the pressure and density of the gas are respectively equal to p_1 and ρ_1. The tubes of flow within the vessel must become narrower as they approach the orifice; they must also attain a minimum sectional area at the point where the pressure has the critical value p_0 determined by equation (3), and must then expand laterally until they become infinitely wide. The only way in which these conditions can be complied with is, for the tubes of flow to contract laterally until a point in the immediate neighbourhood of the orifice is reached, and then to expand into the external space (Fig. 212). In this case the pressure at the orifice must be equal to p_0; and the velocity V_0 of the gas as it leaves the orifice is found by substituting the value of p_0 for p in equation (2). Thus—

$$V_0 = \sqrt{\frac{2\gamma}{\gamma - 1}\frac{p_1}{\rho_1}}\sqrt{\left\{1 - \frac{2}{\gamma+1}\right\}}$$

$$= \sqrt{\frac{2\gamma}{\gamma+1}\frac{p_1}{\rho_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This equation suffices to determine the velocity of efflux at the orifice.

Now let the external pressure be increased gradually. So long as it is less than the critical pressure p_0, the tubes of flow must have minimum sections in the immediate neighbourhood of the orifice, and therefore the pressure at the orifice must remain equal to p_0, and the velocity of efflux must remain unaltered. In the external space, the tubes of flow expand laterally, and the pressure within them falls until it becomes equal to the external pressure; subsequently, no further expansion of the tubes of flow can occur, for if it did, the pressure within them would fall below the external pressure. But the gas cannot flow away from the orifice in all directions in tubes of flow of constant area; hence, motion along stream-lines becomes impossible at points further from the orifice than those at which the tubes cease to expand, and the motion becomes turbulent.

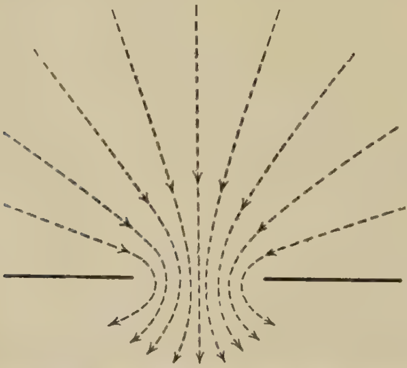

FIG. 212.—Stream-lines in a gas which escapes from a small orifice into a vacuum.

When the external pressure has been increased to p_0, the tubes of flow cannot expand into the external space, and turbulent motion commences at the orifice. The gas leaves the orifice in the form of a jet (Fig. 213), and the surrounding gas is thrown into eddies in order to avoid discontinuity of motion between it and the sides of the jet. **So long as the external pressure is not greater than the critical pressure p_0, the velocity of efflux at the orifice retains the value given by equation** (4).

When the external pressure is greater than p_0, the pressure cannot be as low as p_0 in any part of a tube of flow; in this case

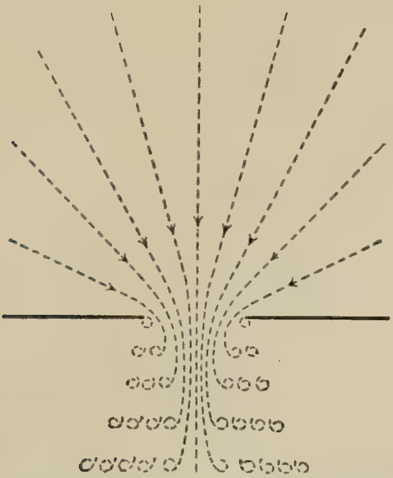

FIG. 213.—Stream-lines in a gas which escapes from a small orifice into a space where the pressure is greater than half the pressure within the vessel which the gas leaves.

the gas issues as a jet which sets the surrounding gas in eddying motion (Fig. 213), and the pressure at the orifice is equal to the external pressure, say p_2. Thus, the velocity of efflux is obtained by substituting p_2 for p in equation (2) (p. 440).

From the reasoning used above we may conclude, that if a vessel is divided into two compartments by a partition pierced with a small orifice, and compressed gas, contained in one compartment, escapes into the other through the orifice; then the velocity of escape is independent of the pressure in the receiving compartment, provided that this is less than about half the pressure in the discharging compartment. This result was first obtained in 1869 by R. D. Napier, from experiments on steam; somewhat later Wilde obtained the same result with relation to air. Its explanation was given some twenty years later by Prof. Osborne Reynolds, who also pointed out that much light is thrown on the subject by expressing the value of V_0, given in equation (4), in terms of the pressure p_0 and the density ρ_0 at the orifice.

From p. 440, it follows that—

$$p_0\left(\frac{1}{\rho_0}\right)^{\gamma} = p_1\left(\frac{1}{\rho_1}\right)^{\gamma}.$$

Using equation (3), p. 443, we find that—

$$\rho_1 = \rho_0\left(\frac{p_1}{p_0}\right)^{\frac{1}{\gamma}} = \rho_0\left(\frac{\gamma+1}{2}\right)\cdot{}^{\frac{1}{\gamma-1}}.$$

Now (p. 444)—

$$\begin{aligned} V_0 = \sqrt{\gamma\cdot\frac{2}{\gamma+1}\frac{p_1}{\rho_1}} &= \sqrt{\gamma\cdot\frac{2}{\gamma+1}\cdot\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\cdot\frac{p_1}{\rho_0}} \\ &= \sqrt{\gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}\cdot\frac{p_1}{\rho_0}} \\ &= \sqrt{\frac{\gamma p_0}{\rho_0}}, \end{aligned}$$

from equation (3) (p. 443); and this gives the velocity of sound in the gas, under the conditions which prevail at the orifice.[1]

The physical significance of the result obtained above, now becomes apparent. Any alteration in the flow within the discharging vessel must be accompanied by alterations of pressure within each tube of flow, and a change of pressure in the receiving vessel can affect the velocity of flow within the discharging vessel, only if it can affect the pressure in the tubes of flow within the discharging vessel. Now, a change of pressure in the receiving vessel will produce a wave of increased or diminished pressure, which travels through the gas issuing from the orifice with a velocity equal to that of sound; if the gas is issuing from the orifice with a velocity equal to that of sound, the wave cannot reach the interior of the discharging vessel, for it is carried outwards by the issuing gas just as fast as it travels inwards through the gas.

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[1] See the Author's *Heat for Advanced Students* (Macmillan), p. 329.

Hence it follows that **the maximum velocity with which a gas can flow through a small orifice, is equal to the velocity of sound in the gas under the conditions of temperature, density, and pressure which prevail at the orifice.**

QUESTIONS ON CHAPTER XIII.

1. Determine the value of the pressure on the surface of an isolated spherical source, of radius r and strength q, due to the flow of fluid of density ρ across the surface.

2. From the similarity between the values of the stresses across the transverse sections of tubes of flow and electric tubes of force, calculate the attraction exerted by a source of strength q_1 on another source of strength q_2, when the two are separated by a distance d.

3. Two spherical bodies, surrounded by a perfect incompressible fluid, are expanding and contracting rhythmically with a common period T, the phases of the oscillations of both bodies remaining equal. Calculate the maximum attraction exerted by one of the bodies on the other when the distance between the two is equal to d; the mean radius of either body being equal to r, and the linear amplitude of the oscillations of the surface of either being equal to a.

4. Let the flow in a perfect incompressible fluid be due entirely to sources and sinks, and let the component flow normal to each element of the boundary surface of the fluid be known; (the boundaries of the sources and sinks are considered to be part of the boundaries of the fluid). Prove that, if any chosen distribution of tubes of flow would produce the given flow normal to the boundary, this distribution is the only one that could be produced by the sources and sinks in the fluid.

5. A source of strength q is situated in an incompressible perfect fluid, at a distance d from a rigid plane which forms the only boundary of the fluid. Prove that the source is attracted by the plane, and calculate the value of the attraction. Assume that the fluid can slip freely along the plane.

6. A sphere is moving through an incompressible perfect fluid. Prove that the instantaneous flow in the fluid surrounding the sphere is similar to that which would be produced if the sphere were removed and a

doublet were placed at its centre. Determine the strength of the doublet in terms of the velocity V of the sphere.

7. A sphere is moving through an incompressible perfect fluid. Prove that the motion of the surrounding fluid increases the inertia of the sphere by an amount equal to half the mass of the fluid that would fill the space occupied by the sphere.

8. A very long cylindrical rod is moving, in a direction perpendicular to its length, through an incompressible perfect fluid. Prove that the motion of the surrounding fluid increases the inertia of the rod by an amount equal to the mass of the fluid that would fill the space occupied by the rod.

9. A straight vortex filament, of strength m, is situated in a perfect incompressible fluid bounded by a cylindrical vessel or radius R; the filament is parallel to the axis of the cylinder, the distance between the two being equal to d. Prove that the vortex filament will revolve about the axis of the cylinder with a uniform linear velocity equal to—

$$\frac{md}{2\pi(\mathrm{R}^2 - d^2)}$$

10. A straight vortex filament, of strength m, is situated with its axis parallel to a plane surface which forms the only boundary of the perfect incompressible fluid which surrounds the filament. Prove that, if the distance of the filament from the plane is equal to R, the filament must be moving in a direction perpendicular to its length and parallel to the plane, with a velocity equal to $m/4\pi\mathrm{R}$.

11. The air-tube of a bicycle tyre has an internal volume of 80 c. in. The tyre is inflated until the pressure of the contained air is equal to several atmospheres, when it is found that there is a slight leak at the valve; the mouth of the valve is wetted, and it is found that a bubble one-quarter of an inch in diameter is blown in six seconds. The tyre is then allowed to remain uninterfered with for 48 hours, when it is found that, on wetting the mouth of the valve, a bubble one-quarter of an inch in diameter is blown in twelve seconds. Estimate roughly the pressure inside the air tube immediately after its inflation.

12. In connection with question (11), estimate roughly the area of the orifice through which the air escapes from the tyre.

CHAPTER XIV

WAVES ON THE SURFACE OF A LIQUID

Wave length, periodic time, and velocity.—Everybody is familiar, to some extent, with the characteristics of waves on the surface of water. These can be observed, on a small scale, when a stone is thrown into a pond; on a large scale they may be studied to best advantage on the open sea, as, for instance, in crossing the Atlantic. The "choppy" waves seen in the English Channel are of a comparatively complicated character, due to conflicting tidal currents. In the simplest case, a train of waves consists of a series of parallel and equidistant ridges, separated one from another by depressions, the whole travelling over the surface of the water with a definite velocity. Fig. 214 represents the section of a wave-train by a vertical plane parallel to the direction of propagation; to a first approximation, the section of the surface takes the form of a series of simple harmonic curves.

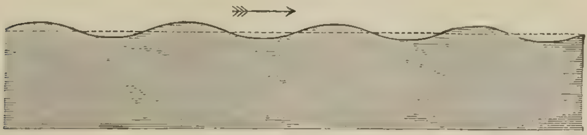

FIG. 214.—Waves on the surface of a liquid.

Since the waves obviously travel over the surface of the water, one is inclined to believe, at first sight, that the surface layers of the water travel with the waves. But if some small pieces of straw are scattered on the surface of a pond, and these are watched as a wave train travels past them, it will be seen that they are not carried forward appreciably; each piece of straw is seen to oscillate about an approximately fixed position, and therefore it may be concluded that each particle of the water oscillates in a similar manner. Each particle of water rises and falls, acquiring its highest position when it is beneath the crest of a wave, and its lowest position when it is beneath a trough. The distance from one crest to the next, or from one trough to the next, is called the **wave length**; this will be denoted by λ. If the waves are all similar, as represented in Fig. 214, all will have a common wave length.

Let a particle of water be on the crest of a wave at a given instant; as the crest moves forward, the particle sinks until it is at the bottom of a trough, and then rises until it once more attains its maximum elevation; during this process the first crest has moved forward through a distance λ, and another crest has travelled up to the particle through an equal distance from the rear. Thus, if T denotes the period of oscillation of the particle, and V denotes the velocity with which the waves travel over the surface of the water, it is evident that a crest, moving with the velocity V, traverses a distance λ in the time T, so that—

$$VT = \lambda \quad . \quad . \qquad . \quad . \quad . \quad . \quad . \quad (1)$$

Further consideration shows that each particle of water must do something more than merely oscillate in a vertical line; for no empty space is formed beneath the crest of a wave, and therefore the heaping up of water to form a crest must involve a horizontal flow, presumably from the space left empty in a neighbouring trough. We may therefore assume that a particle of water in the surface executes two linear oscillatory motions: a vertical oscillation of amplitude a, where a denotes the height of a crest above the mean level of the surfaces of the water; and a horizontal oscillation of amplitude β, a value which must be determined. If a particle once passes through any given position, it will repass the same position at regular intervals, each equal to T; hence it follows that the periods of the vertical and horizontal oscillatory motions must be equal, each having the value T. Further, when oscillations are small they invariably approximate to the simple harmonic type; thus when the amplitude of the waves is small in comparison with the wave length λ, we may assume that each particle of water in the surface executes a simple harmonic motion of amplitude a in a vertical direction, together with a simple harmonic motion of amplitude β in a horizontal direction, the periods of both oscillations being equal. The maximum velocity due to the vertical oscillation is equal to $(2\pi a/T)$ (p. 87), while the maximum velocity due to the horizontal oscillation is equal to $(2\pi\beta/T)$. In either case, the maximum velocity is attained at the middle point of the oscillation in question.

So far, attention has been concentrated on a particle in the free surface of the water. On referring to Fig. 214 it becomes obvious, without further argument, that motion cannot be confined to the particles in the surface. On the other hand, when the water is very deep (as, for instance, in the Atlantic Ocean) all particles, from the surface to the bottom of the water, cannot execute oscillations of equal amplitude; for if they did, a wave would possess a store of energy enormously in excess of that which experience warrants us in attributing to it. Hence we must conclude that a particle of water, at a distance below the free surface, executes vertical and horizontal oscillations similar to, but smaller than, those of a particle at the surface; the relation between the amplitude of oscillation, and the depth below the surface, must be determined.

In the ensuing investigations, waves on deep water will be studied. The relation between the amplitudes of the horizontal and vertical oscillations will be determined, the velocity of wave propagation will be deduced, and the relation between the amplitude and the depth will be found; then, expresssions for the potential and kinetic energies of a wave train will be obtained, and these will suffice to explain some interesting points in connection with the propagation of groups of waves.

The horizontal and vertical oscillations of a particle.—Let it be supposed that a series of imaginary horizontal planes are described in still water, each plane being at a distance δ below the one above it, the highest plane coinciding with the free surface of the water. The traces of these planes are indicated by the horizontal lines on the right-hand side of Fig. 215. Further, let it be supposed that all particles of water which lie in the imaginary planes, are marked in some manner so that the position of each plane becomes visible to the eye. When a train of waves travels from left to right over the water, all particles will be set in oscillation, and the particles which previously lay in a plane will now for an instant lie in a curved surface, of which the trace on the plane of the paper is a wave curve, as shown on the left hand side of Fig. 215. Particles in any vertical line will be in similar stages of their oscillations, that is, they will be in equal phases: the crests of all wave curves will lie in one vertical line, and the troughs of all wave curves will lie in another vertical line.

A particle at the crest of any wave curve will be at its maximum elevation, and therefore **at a crest, the vertical upward displacement of a particle has its maximum value, and the vertical velocity is equal to zero. A particle at the bottom of a trough has its maximum downward displacement, and its vertical velocity is equal to zero.** At the mid-point on the slope of a wave curve the vertical displacement is equal to zero, and the vertical velocity has its maximum value. The wave curves in Fig. 215 have been drawn so as to show that the amplitude of vertical oscillation becomes less, the greater the mean depth of the oscillating particle below the surface.

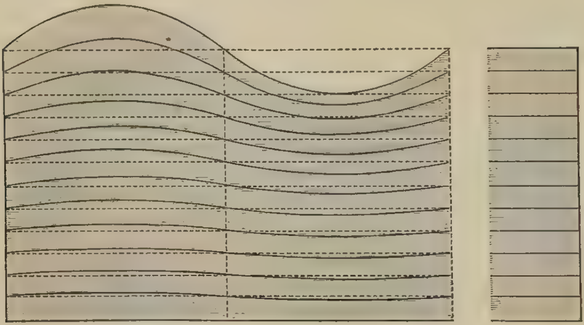

FIG. 215.—Wave motion near to the surface of a liquid.

The **mean position** of a particle is defined as the position of the point about which the particle oscillates; the distance, measured vertically downwards, from the mean position of the free surface to the mean position of any particle beneath the surface, is defined as the **mean**

depth of the particle below the surface. From Fig. 215 it will be seen that the vertical height of the water above a particle at the mid-point on the slope of a wave curve, is equal to the mean depth of all particles on that wave curve.

Let waves travel over water from left to right with a velocity V, and let it be supposed that a uniform velocity V, from right to left, is impressed on the whole of the water. Then the waves will remain stationary in space, and the water will stream past them ; a particle in the free surface of the water will travel over the free surface of the waves, and a particle in any wave curve will travel along that curve, so that the wave curves become stream-lines. If we imagine Fig. 215 to be displaced through unit distance perpendicular to the plane of the paper, then any two neighbouring wave curves will describe curved surfaces, which, together with the original and final planes of the diagram, form the lateral boundaries of a tube of flow. All tubes of flow are of unit thickness perpendicular to the plane of the paper, and therefore the cross-sectional area of a tube can be determined directly by inspection.

It will be noticed that the distance which separates neighbouring wave curves is greater under a crest than under a trough of the waves ; this is due to the fact that the amplitude of any wave curve is less than that of the curve immediately above it. Since the tubes of flow are horizontal under the crests and troughs of the waves, and the cross-sectional area of a tube is greater under a crest than under a trough, it follows that the horizontal velocity of the water has a minimum value under a crest, and a maximum value under a trough (p. 378). Now, the velocity of any particle of water in a tube of flow is the resultant of two components, viz., the horizontal velocity V, from right to left, impressed on all particles ; and the instantaneous velocity of the particle due to its oscillation. Since the horizontal velocity of a particle has a minimum value below the crest of a wave, and a maximum value below the trough, the velocity due to the horizontal oscillation of a particle must be opposed to the impressed velocity V under the crest, and concurrent with V under the trough ; and the numerical value of the velocity due to the horizontal oscillation of a particle must have a maximum value under the crests and the troughs. Hence, the character of the horizontal oscillation of a particle becomes evident; under the crest of a wave, all particles are moving, with maximum horizontal velocity, in the direction of wave propagation; and under the trough, all particles are moving, with maximum horizontal velocity, in an opposite direction. Since the maximum velocity of an oscillating particle is acquired at the instant when the corresponding displacement is zero, it follows that **the horizontal displacement is equal to zero under the crests and troughs of the waves, and therefore the horizontal displacement has its maximum value under the mid-points on the slopes of the waves.** Under the crest of a wave, a particle has its maximum upward displacement, and zero horizontal displacement; while under a trough, a particle has its maximum downward displacement and zero horizontal displacement. Under the mid-point on the slope of a wave, a particle has zero vertical displacement and maximum horizontal displacement. From this it becomes evident that **the horizontal and vertical oscillations of a particle differ in phase by $\pi/2$.**

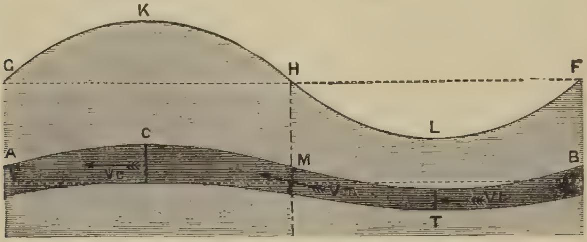

FIG. 216.—Tube of flow beneath the free surface of a liquid.

Let ACTB, Fig. 216, represent a tube of flow beneath the free surface of the water, the vertical thickness of the tube being so small that all particles within it oscillate with approximately equal amplitudes. Let particles within the tube oscillate with a vertical amplitude α, and a horizontal amplitude β; then the maximum horizontal velocity of oscillation is equal to $(2\pi\beta/T)$, where T is the period of oscillation (p. 451); and since $VT=\lambda$ (p. 450) it follows that the maximum horizontal velocity of oscillation is equal to $(2\pi V\beta/\lambda)$. Similarly, the maximum vertical velocity of oscillation is equal to $(2\pi V\alpha/\lambda)$.

When the uniform velocity V has been impressed on the whole of the water, let the resultant velocity of a particle at the point T (Fig. 216), directly beneath the trough of the wave, be equal to V_t. Then—

$$V_t = V + 2\pi V\beta/\lambda = V\{1 + (2\pi\beta/\lambda)\} \quad . \quad . \quad . \quad . \quad (2)$$

Let the resultant velocity of a particle at C, directly beneath the crest of the wave, be equal to V_c; then—

$$V_c = V\{1 - (2\pi\beta/\lambda)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let the velocity of a particle at M, directly beneath the mid-point on the slope of the wave, be denoted by V_m; then V_m is the resultant of the impressed horizontal velocity V, and the maximum vertical velocity of oscillation $(2\pi V\alpha/\lambda)$, so that—

$$V_m^2 = V^2\{1 + (2\pi\alpha/\lambda)^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Relation between α and β.—It now becomes necessary to utilise the fundamental equation (p. 402) expressing the condition of transfer of energy through a fluid. Values of the velocity at three different points in the tube of flow ACTB have been obtained, and any two of these may be selected for substitution in the fundamental equation. Let the points C and T be selected. At first sight, it would appear that the pressure at C must be greater than that at T, since the vertical height of the water is greater above C than above T. It must be clearly understood, however, that the pressure at a point below the free surface of a liquid is not necessarily proportional to the distance of that point below the surface, unless the liquid is stationary. For instance, when a liquid issues from an orifice in a cistern, the pressure inside the issuing jet is equal to the atmospheric pressure, whatever may be the distance of the orifice below the free surface of the liquid in the cistern. In a stationary liquid the weight of each particle is borne by the liquid beneath it (p. 35); the resultant force acting on each particle is equal to zero, and the particle has no acceleration. When the liquid is in motion, the product of the mass and the acceleration of any particle is equal to the resultant force acting upon it (p. 19). Therefore, if a particle is accelerated in a downward direction, it must be acted upon by a resultant downward force; that is, its weight cannot be borne entirely by the underlying liquid, and the difference between the pressures just below and just above the particle must be less than if the particle were stationary or moving with a constant velocity. If a vessel containing a liquid is allowed to fall freely under the action of gravity, the vessel and its contents fall with the constant acceleration g, and therefore the acceleration of each particle of the liquid is equal to g; the resultant downward force acting on any particle of mass m is equal to mg, and therefore no part of the weight of the particle is borne by the underlying liquid, so that the pressures just below and just above the particle are equal, and at any point in the liquid the pressure is equal to that at the free surface of the liquid. When a liquid is not falling freely under the action of gravity, part of the weight of each particle must be borne by the underlying liquid; but the greater the downward acceleration of the particle, the smaller will be the fraction of the weight of the particle borne by the underlying liquid, and the less will the particle contribute to the pressure of the liquid beneath it.

Now consider a particle traversing the tube of flow BTCA, (Fig. 216) from right to left. At M the particle acquires its maximum upward velocity, and at A the particle acquires its maximum downward velocity; therefore, as the particle moves from M through C to A, its acceleration is always in a downward direction, being greatest at C and equal to zero at M and A (compare p. 92). At M and A the weight of the particle is sustained entirely by the underlying liquid, since the vertical acceleration of the particle is equal to zero; similar reasoning applies to all tubes of force above ACTB, and therefore the pressure at M or A is equal to $g\rho h$, where h is the height of the vertical column of liquid above M or A. But at points between M and A, the weight of a particle is not borne entirely by the underlying liquid; thus the pressure at C must be less than that due to the vertical column of liquid above C. Similar reasoning shows that at T the pressure is greater than that due to the vertical column of liquid above T; for all particles between B and M are accelerated in an upward direction, the acceleration having its maximum value at T, and therefore the liquid beneath a particle not only sustains the weight of the particle, but exerts on it a resultant upward force equal to the product of the mass and the upward acceleration of the particle.

Let the amplitude of the vertical oscillation of a particle on the free surface of the liquid be equal to a_0; then the crest K is at a height a_0 above the mid-point H on the slope of the wave, and C is at a height a above M; thus, the height CK of the column of liquid above C is equal to $(h+a_0-a)$, where $MH=h$. Similarly, the height TL of the column of liquid above T is equal to $(h-a_0+a)$. The pressure P at M is equal to $g\rho h$. The pressure at C will be determined later; for the present it will be assumed to be equal to $(P+p)$, where the excess of pressure p is due partly to the additional height $(a-a_0)$ of liquid above C, due allowance being made for the maximum downward accelerations of all particles between C and K. The column of liquid above T is shorter by $(a-a_0)$ than that above M, and allowance must be made for the maximum upward accelerations of all particles between T and L; and as the maximum upward acceleration of a particle between T and L is numerically equal to the maximum downward acceleration of a corresponding particle between C and K, we may assume that the pressure at T is equal to $(P-p)$.

At the points C, M, and T in the tube of flow ACTB, let the cross-sectional areas of the tube be equal to A_c, A_m, and A_t respectively. Applying the fundamental equation (p. 402) to the flow of the liquid from T to C, and remembering that each unit mass of liquid gains $(g\times 2a)$ units of gravitational energy in passing from T to C, we have—

$$A_cV_c(P+p+\tfrac{1}{2}\rho V_c^2)-A_tV_t(P-p+\tfrac{1}{2}\rho V_t^2)=-\rho A_cV_c\,.\,2ga.$$

Applying the fundamental equation to the flow from M to C, we have—

$$A_cV_c(P+p+\tfrac{1}{2}\rho V_c^2)-A_mV_m(P+\tfrac{1}{2}\rho V_m^2)=-\rho A_cV_c\,.\,ga.$$

Dividing these equations through by—

$$A_cV_c=A_tV_t=A_mV_m,$$

and simplifying, we have—

$$\tfrac{1}{2}\rho V_c^2-\tfrac{1}{2}\rho V_t^2=-2p-2g\rho a \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\tfrac{1}{2}\rho V_c^2-\tfrac{1}{2}\rho V_m^2=-\ p-\ g\rho a \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Multiplying equation (6) by two, and subtracting the result from (5), we obtain—

$$-\tfrac{1}{2}\rho V_c^2-\tfrac{1}{2}\rho V_t^2+\rho V_m^2=0;$$

$$\therefore\quad V_c^2+V_t^2=2V_m^2.$$

Substituting the values of V_c, V_t, and V_m, given by equations (2), (3) and (4) (p. 455) we obtain—

$$V^2\left(1 - \frac{2\pi\beta}{\lambda}\right)^2 + V^2\left(1 + \frac{2\pi\beta}{\lambda}\right)^2 = 2V^2\left\{1 + \left(\frac{2\pi\alpha}{\lambda}\right)^2\right\};$$

$$\therefore \quad 1 + \left(\frac{2\pi\beta}{\lambda}\right)^2 = 1 + \left(\frac{2\pi\alpha}{\lambda}\right)^2,$$

and therefore $\beta = \alpha$.

Thus, the **amplitudes of the vertical and horizontal oscillations are equal**, and this result holds for all particles of the liquid, that is, for all values of the mean depth h. It has been proved already (p. 454) that the phases of the vertical and horizontal oscillations differ by $\pi/2$, and it is known that two mutually perpendicular oscillations, equal in amplitude and period, but differing in phase by $\pi/2$, are equivalent to a uniform circular motion (p. 87). Hence, we conclude that **each particle of liquid revolves with uniform velocity in a circular orbit**, the radius of the circle being equal to a, the maximum vertical displacement of the particle. The instantaneous displacements of a number of particles in a wave train, together with the circular orbits of the particles, are represented in Fig. 217.

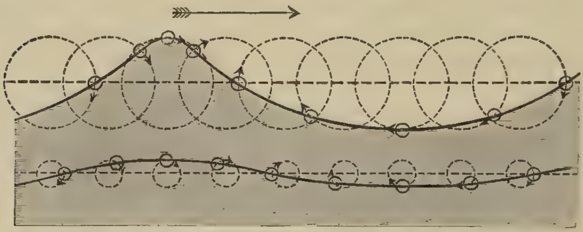

FIG. 217. Circular orbits of particles of a liquid, during the passage of a wave train over the surface of the liquid.

The upper series of circles in Fig. 217 represents the orbits of nine particles in the surface layer of the liquid. If the liquid were untroubled by waves, these nine particles would lie at equidistant points on the plane surface of the liquid, the distance between any two neighbouring particles being equal to $\lambda/8$. During the passage of the waves each particle revolves in a circular orbit about a fixed point; the radius of each orbit is equal to the amplitude of the wave disturbance at the surface. The direction in which the orbits are traversed is determined by the condition that a particle at the crest of a wave must move forwards, that is, in the direction in which the waves are travelling; while a particle at the bottom of a trough must move backwards. Further, the phases of any two neighbouring particles must differ by $2\pi/8$. It becomes evident that, when the amplitude is large, the section of the surface differs considerably from a simple harmonic curve; in fact, the section could be described in the following manner. **Let a circle, of radius $\lambda/2\pi$, roll along the underside of a horizontal line at a height $\lambda/2\pi$ above the mean level of the surface; then the section of the waves will be described by a point at a distance a from the centre of the rolling circle**, where a denotes the amplitude of the waves. A curve described in this manner is called a **trochoid.**[1] When the amplitude a is small, the trochoid approximates to a simple harmonic curve (p. 332). The lower series of circles in Fig. 217 represents the orbits of nine particles at a mean depth of about a fifth of the wave length. The relative amplitudes, at the surface and beneath it, are drawn to scale.

Velocity of wave transmission.—In equation (6) (p. 457), substitute the values of V_c and V_m given by equations (3) and (4) (p. 455), together with the condition that $\beta = a$. Then—

$$\tfrac{1}{2}\rho V^2\left\{1 - \frac{2\pi a}{\lambda}\right\}^2 - \tfrac{1}{2}\rho V^2\left\{1 + \left(\frac{2\pi a}{\lambda}\right)^2\right\} = -p - g\rho a;$$

$$\therefore \quad -\rho V^2 \cdot \frac{2\pi a}{\lambda} = -p - g\rho a,$$

and

$$V^2 = \frac{p + g\rho a}{\rho \cdot \dfrac{2\pi a}{\lambda}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now, V denotes the velocity of transmission of the wave disturbance at a mean depth h below the free surface of the liquid, the amplitude of oscillation at this depth being equal to a. It is essential that V shall have the same value at all depths in the liquid, since the crests of all the wave curves within the liquid must retain their relative positions one below another (p. 452). Hence, V **must be independent of** h, and must therefore be independent of a; in other words, the right-hand side of (7) must be independent of the value of a. This condition can be complied with only if p is proportional to a, that is, if $p = ka$, where k is a constant.

---

[1] Greek *trochos*, a wheel. A point on one of the spokes of a wheel describes a trochoid when the wheel rolls along a horizontal straight line.

If the wave-length is great, the curvature of the free surface of the wave will be small at all points, and in these circumstances the effects of surface tension may be neglected (p. 349), and the pressure just beneath the free surface will be equal to the atmospheric pressure. Hence, considering a tube of flow just beneath the free surface, the pressure P at the point H (Fig. 216) is equal to the atmospheric pressure, and the pressure $(P+p)$ at K has the same value; therefore at the surface, where $a = a_0$, we have—

$$p = ka_0 = 0; \quad \therefore \; k = 0.$$

Thus, **when the effects of surface tension may be neglected, the pressure is uniform at all points in a tube of flow,** and therefore $p = 0$. In this case

$$V^2 = \frac{g\rho a}{\rho \cdot \frac{2\pi a}{\lambda}} = \frac{g\lambda}{2\pi}$$

and

$$V = \sqrt{\left(\frac{g\lambda}{2\pi}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence, **when the waves are due entirely to gravity, the velocity is proportional to the square root of the wave length.**

When the effects of surface tension are taken into account, the pressure just beneath the free surface of the liquid is greater than the atmospheric pressure (p. 318) by (S/R), where S is the surface tension of the liquid, and R is the radius of curvature of the free surface at the point in question. The trace of the wave surface approximates to a simple harmonic curve, and therefore at the mid-point H (Fig. 216) on the slope of the curve the value of (1/R) is equal to zero, while at the crest K the value of (1/R) is equal to $a_0(2\pi/\lambda)^2$ (p. 336). Thus at H, the pressure P just beneath the free surface is equal to the atmospheric pressure, while just beneath the crest K the pressure $P+p$ is greater than that of the atmosphere by $a_0 S(2\pi/\lambda)^2$, and therefore—

$$p = ka_0 = a_0 S(2\pi/\lambda)^2.$$

Thus—

$$k = S(2\pi/\lambda)^2,$$

and in general—

$$p = ka = aS(2\pi/\lambda)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Substituting the result in (7) (p. 495), we obtain—

$$V^2 = \frac{\alpha S\left(\frac{2\pi}{\lambda}\right)^2 + g\rho\alpha}{\rho\left(\frac{2\pi\alpha}{\lambda}\right)}$$

$$\therefore \quad V = \sqrt{\frac{\lambda}{2\pi}\left\{g + \frac{S}{\rho}\left(\frac{2\pi}{\lambda}\right)^2\right\}} \quad . \quad . \quad . \quad . \quad (10)$$

The results to be deduced from the equation have been discussed fully in a previous chapter (p. 350).

Relation between the amplitude and the mean depth below the surface.—Let it be supposed that, while the surface of a liquid is untroubled by waves, imaginary horizontal planes are described in it, the distance between any plane and the one above it being very small and equal to δ. Now, let a wave train of amplitude a_0 travel up from the left with velocity V; the planes become deformed into curved surfaces of which the traces on the plane on the diagram are wave curves (Fig. 215). While the liquid to the right is still undisturbed, let a velocity V from right to left be impressed on the whole of the liquid. The waves remain stationary, and the liquid streams past them. Let Fig. 215 represent the wave curves corresponding to various mean depths below the free surface; if the diagram is displaced through unit distance perpendicular to the plane of the paper, any two neighbouring wave curves will describe curved surfaces which, together with the original and final planes of the diagram, form the boundaries of a tube of flow.

A particle at the crest of the wave curve which forms the trace of the upper boundary of any tube of flow, is displaced upwards through a distance α' from its mean position, and a particle at the crest of the next lower curve is displaced upwards through a distance α'', where α' and α'' are the amplitudes corresponding to the two wave curves. Thus, the vertical distance between the particles at the crest of the two wave curves is equal to $(\delta + \alpha' - \alpha'')$; and the cross sectional area A_c of the tube, at a point beneath the crest of the wave, is equal to $(\delta + \alpha' - \alpha'') \times 1$. Similar reasoning shows that the sectional area A_t of the same tube of flow, at a point beneath the trough of the wave, is equal to $(\delta - \alpha' + \alpha'') \times 1$. At a point to the right, where there are no waves, the velocity of the liquid is uniform and equal to V, and the sectional area of the tube of flow is equal to $\delta \times 1$; and since the product of the velocity and the sectional area of the tube is constant (p. 378), it follows that—

$$V_cA_c = V_c(\delta + \alpha' - \alpha'') = V\delta\,; \quad \therefore\ V_c = V\,\frac{\delta}{\delta + \alpha' - \alpha''}$$

Similarly—

$$V_tA_t = V_t(\delta - \alpha' + \alpha'') = V\delta\,; \quad \therefore\ V_t = V\,.\,\frac{\delta}{\delta - \alpha' + \alpha''}\,;$$

$$\therefore\quad V_c^2 - V_t^2 = V^2\left\{\frac{1}{\left(1 + \frac{\alpha' - \alpha''}{\delta}\right)^2} - \frac{1}{\left(1 - \frac{\alpha' - \alpha''}{\delta}\right)^2}\right\}$$

$$= V^2\,.\,\frac{-4\,.\,\frac{\alpha' - \alpha''}{\delta}}{\left\{1 - \left(\frac{\alpha' - \alpha''}{\delta}\right)^2\right\}^2}.$$

In this equation, $(\alpha' - \alpha'')$ denotes the falling off in the amplitude due to an increase δ in the mean depth beneath the free surface of the liquid. If the amplitude at the free surface is small, both α' and α'' must be smaller still, and thefore $(\alpha' - \alpha'')$ must be very small; let it be assumed that $(\alpha' - \alpha'')$ is small in comparison with δ, so that $\{(\alpha' - \alpha'')/\delta\}^2$ is small in comparison with unity. In this case—

$$V_c^2 - V_t^2 = -V^2\left(4\frac{\alpha' - \alpha''}{\delta}\right).$$

Substitute this result in equation (5) (p. 457). On the right-hand side of that equation, α' may be substituted for α, since the amplitude varies to a very small extent in the tube of flow. Then, substituting the value of p obtained from equation (9) (p. 460)—

$$\tfrac{1}{2}\rho(V_c^2 - V_t^2) = -\tfrac{1}{2}\rho V^2\left(4\frac{\alpha' - \alpha''}{\delta}\right) = -2\alpha'\left(S\left(\frac{2\pi}{\lambda}\right)^2 + g\rho\right);$$

$$\therefore\quad V^2(\alpha' - \alpha'') = \alpha'\delta\left(\frac{S}{\rho}\left(\frac{2\pi}{\lambda}\right)^2 + g\right)$$

$$= \alpha'\delta\frac{2\pi}{\lambda}V^2, \text{ from equation (10) (p. 461);}$$

$$\therefore\quad \alpha' - \alpha'' = \frac{2\pi}{\lambda}\alpha'\delta,$$

and

$$\alpha'' = \alpha'\left(1 - \frac{2\pi}{\lambda}\delta\right).$$

This result shows that if we know the amplitude α' of a particle at a mean depth h below the surface, the amplitude α'' of a particle at a mean depth $(h+\delta)$ can be obtained by multiplying α' by $\{1-(2\pi\delta/\lambda)\}$.

Let the amplitude of a particle in the free surface of the liquid be α_0, and let the amplitudes of particles at mean depths δ, 2δ, 3δ, . . . $n\delta$ be equal to α_1, α_2, α_3, . . . α_n.

Then

$$\alpha_1 = \alpha_0\left(1 - \frac{2\pi}{\lambda}\delta\right).$$

$$\alpha_2 = \alpha_1\left(1 - \frac{2\pi}{\lambda}\delta\right) = \alpha_0\left(1 - \frac{2\pi}{\lambda}\delta\right)^2.$$

$$\alpha_3 = \alpha_0\left(1 - \frac{2\pi}{\lambda}\delta\right)^3.$$

$$\ldots = \ldots\ldots\ldots$$

$$\alpha_n = \alpha_0\left(1 - \frac{2\pi}{\lambda}\delta\right)^n.$$

Let $n\delta$, the mean depth below the surface of the particle whose amplitude is equal to α_n, be denoted by h; and let $(2\pi\delta/\lambda)=(1/x)$. When δ is infinitely small, x must be infinitely great. Further—

$$n = h/\delta = 2\pi x h/\lambda,$$

and therefore—

$$\alpha_n = \alpha_0\left(1 - \frac{1}{x}\right)^{\frac{2\pi x h}{\lambda}}$$

$$= \alpha_0\left\{\left(1 - \frac{1}{x}\right)^{-x}\right\}^{-\frac{2\pi h}{\lambda}},$$

and if $\left(1 - \frac{1}{x}\right)^{-x}$ be expanded by the binomial theorem, and an infinitely large value be substituted for x we obtain a series which is equal to the base ϵ of the Napierian logarithms (compare p. 82). Thus—

$$\alpha_n = \alpha_0\epsilon^{-\frac{2\pi h}{\lambda}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This equation suffices to determine α_n, the amplitude of oscillation of a particle at a mean depth h beneath the free surface of the liquid. It shows that the amplitude falls off very rapidly as the mean depth increases; the extent of this falling off can be judged from Fig. 215, which is drawn to scale. At a mean depth below the surface equal to one wave length, the amplitude of oscillation is only 0·2 per cent. of the amplitude at the free surface of the liquid. When ripples (p. 351) travel over the surface of a liquid, the motion is confined almost entirely to a very thin superficial layer. The disturbance of the Atlantic due to waves of 30 ft. length, and 10 ft. amplitude, falls off to an oscillation of 0·24 inch amplitude at a mean depth of 30 ft.

Potential energy of a wave train.—When a liquid is at rest, its surface is plane and horizontal, and its potential energy has its smallest possible value. When waves travel over the liquid, they entail an increase of potential energy due to two causes. Firstly, liquid is removed from certain places, thus forming troughs, and is heaped up at other places, thus forming crests; the work done in this process represents the gravitational energy of the wave train. Secondly, the free surface of the liquid is stretched, and this entails an increase in the surface energy of the liquid; part of the energy is due to heat absorbed (p. 295), but this part is not transmitted with the waves, so that the only part of the surface energy which need be considered is that represented by the work done in stretching the free surface. The gravitational and surface energies of the wave train must now be determined.

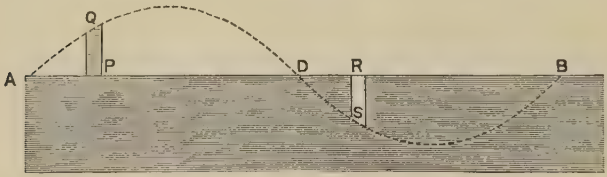

FIG. 218.—Potential energy of a wave.

Let it be assumed that the trace of the waves on the plane of the diagram (Fig. 218) is represented by the equation—

$$y = a_0 \sin\frac{2\pi x}{\lambda},$$

where x denotes the distance from A to any point P on the line AB, and $y = PQ$ denotes the height above AB of a point Q on the wave curve. As before, a_0 denotes the small amplitude of the wave curve.

Starting with the liquid untroubled by waves, let it be supposed that the liquid is removed from the trough DSB (Fig. 218), and is heaped up so as to form the crest AQD of the wave. Let E_g denote the gravitational energy of a slab of the wave, of unit thickness perpendicular to the plane of the paper, and of length λ from left to right; then E_g is equal to the work done in removing a slab of unit thickness from the trough DSB, and replacing it so as to form the crest AQD.

Select n equidistant points on the line AB, and through these points draw planes perpendicular to AB and to the plane of the paper; by this means the liquid to be removed from the trough is divided into $(n/2)$ elements similar to RS, each having unit thickness perpendicular to the plane of the paper, and a breadth equal to (λ/n) parallel to AB. If DR = AP, the element RS may be removed from the trough and replaced at PQ. If $AP = x$, the length PQ = RS of the element is equal to $\alpha_0 \sin(2\pi x/\lambda)$, and the centre of gravity of the element is raised through a distance $(PQ + RS)/2 = \alpha_0 \sin(2\pi x/\lambda)$, so that the work done is equal to—

$$g\rho \cdot \frac{\lambda}{n} \cdot \alpha_0^2 \sin^2\left(\frac{2\pi x}{\lambda}\right).$$

Now for the various elements removed from the trough, the value of $(2\pi x/\lambda)$ will vary from 0 to π. The average value of the square of the sine of an angle, between the limits 0 and $\pi/2$, is equal to 1/2 (p. 53); and between the limits 0 and π the average value of the square of the sine must also be equal to 1/2. Hence, E_g, **the work done in removing the $n/2$ elements from the trough DSB, and replacing them so as to form the crest AQD of the wave, is given by the equation—**

$$E_g = \frac{n}{2} \times g\rho \cdot \frac{\lambda}{n} \alpha_0^2 \cdot \times \{\textit{average value of } \sin^2(2\pi x/\lambda) \textit{ between } 0 \textit{ and } \pi\}$$

$$= \tfrac{1}{4} g\rho\lambda\alpha_0^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Let E_s denote the increase in the surface energy of that part of the liquid selected already, due to the stretching of the surface. The value of E_s will be equal to twice the work done on a strip of the surface of length APD = λ/2 (Fig. 218), and of unit breadth, while it is being stretched so that its length becomes equal to that of the curve AQD. It may be supposed that the strip is pushed upwards from below, so that its trace changes from APD to AQD, retaining the form of a simple harmonic curve of which the amplitude varies from zero to α_0.

Meanwhile, at a point above P, the pressure due to the surface tension increases uniformly from zero to $S(2\pi/\lambda)^2 PQ = S(2\pi/\lambda)^2 \alpha_o \sin\theta$, and the average value of the pressure is equal to half this value; this average pressure represents the average force per unit area overcome in pushing the surface around P upwards to Q, and therefore the work done per unit area in the neighbourhood of P is equal to—

$$\frac{S}{2}\left(\frac{2\pi}{\lambda}\right)^2 \alpha_o^2 \sin^2\left(\frac{2\pi x}{\lambda}\right).$$

The work done in pushing the whole of the strip APD upwards to AQD, is equal to—

$$\frac{\lambda}{2} \times \frac{S}{2}\left(\frac{2\pi}{\lambda}\right)^2 \alpha_o^2 \times \textit{average value of } \sin^2\left(\frac{2\pi x}{\lambda}\right),$$

where x varies from 0 to $\lambda/2$, and $2\pi x/\lambda$ varies from 0 to π. The average value of $\sin^2(2\pi x/\lambda)$ between these limits is equal to $\frac{1}{2}$; therefore, finally—

$$E_s = 2 \times \frac{\lambda}{2} \times \frac{S}{2}\left(\frac{2\pi}{\lambda}\right)^2 \alpha_o^2 \times \tfrac{1}{2} = \frac{\lambda S}{4}\left(\frac{2\pi}{\lambda}\right)^2 \alpha_o^2;$$

and the total potential energy, $E_p = (E_g + E_s)$ is given by the equation—

$$E_p = \frac{\lambda}{4}\alpha_o^2\left\{g\rho + S\left(\frac{2\pi}{\lambda}\right)^2\right\} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Kinetic energy of a wave train.—All particles in the wave train, at a common mean depth below the surface of the liquid, revolve with uniform velocity in circular orbits of equal radii.

When the liquid is at rest, let imaginary horizontal planes be described at distances h and $(h+\delta)$ below the free surface; then these planes, at a small distance δ apart, form the upper and lower boundaries of a tube of length λ, unit breadth, and thickness δ; and the particles of liquid contained in this tube are caused by the waves to revolve in circular orbits of radii equal to a_n. The velocity of each particle is equal to $(2\pi a_n/T) = (2\pi a_n V/\lambda)$, and therefore the kinetic enegy of all the particles within the tube is equal to—

$$\tfrac{1}{2} \cdot \rho\delta\lambda \cdot \left(\frac{2\pi\alpha_n V}{\lambda}\right)^2$$

Now (p. 463))— $\alpha_n = \alpha_o\left(1 - \dfrac{2\pi\delta}{\lambda}\right)^n = \alpha_o K^n$,

if $\qquad K = \{1 - (2\pi\delta/\lambda)\}$.

All particles of the liquid, from the free surface downwards, which are comprised in a slab of unit thickness and of length λ parallel to the direction of wave propagation, together possess an amount of kinetic energy E_k given by the equation—

$$E_k = \tfrac{1}{2}\rho\delta\lambda\left(\frac{2\pi V}{\lambda}\right)^2\{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 \; . \; . \; . \; . \; . \; .\}$$

$$= \tfrac{1}{2}\rho\delta\lambda\left(\frac{2\pi V}{\lambda}\right)^2\alpha_0^2\{1 + K^2 + K^4 + K^6 + \; . \; . \; . \; . \; . \; . \; .\}.$$

The geometrical series within the brackets, summed to n terms, is equal to—

$$\frac{1 - K^{2n}}{1 - K^2};$$

and since K is less than unity, K^{2n} will be infinitely small when n is infinitely great. Also—

$$1 - K^2 = 1 - \left\{1 - \frac{2\pi\delta}{\lambda}\right\}^2$$

$$= 1 - \left\{1 - \frac{4\pi\delta}{\lambda} + \left(\frac{2\pi\delta}{\lambda}\right)^2\right\}$$

$$= \frac{4\pi\delta}{\lambda},$$

when δ is so small that terms involving δ^2 may be neglected. Thus, the series $\{1 + K^2 + K^4 + \; . \; . \; . \; .\}$ is equal to $(\lambda/4\pi\delta)$, and

$$E_k = \tfrac{1}{2}\rho\delta\lambda\left(\frac{2\pi V}{\lambda}\right)^2\alpha_0^2 \times \frac{\lambda}{4\pi\delta}$$

$$= \tfrac{1}{4}\rho\lambda \, . \, \frac{2\pi}{\lambda} \, . \, V^2\alpha_0^2 \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (13)$$

From equation (10) (p. 461)—

$$\frac{2\pi}{\lambda}V^2 = g + \frac{S}{\rho}\left(\frac{2\pi}{\lambda}\right)^2,$$

and therefore $E_k = \dfrac{\lambda}{4}\alpha_0^2\left\{g\rho + S\left(\dfrac{2\pi}{\lambda}\right)^2\right\} = E_p$, from equation (12).

Thus, **when a wave train travels over the surface of a liquid, one half of the energy of the waves is potential, and the other half is kinetic.**

Transmission of energy by a train of gravitational waves.—When a train of similar waves is travelling over the surface of a liquid, each particle of the liquid revolves with constant velocity in an orbit of constant radius. Hence, the kinetic energy of a particle suffers no change as the wave train progresses ; in other words, **the kinetic energy of a wave train is not transmitted from particle to particle.** The potential energy of the wave-train is obviously transmitted, for each particle moves forward when it is above the level of its mean position (Fig. 217), that is, when its gravitational energy is greater than if the liquid were still ; and it moves backwards when it is below the level of its mean position, that is, when its gravitational energy is less than if the liquid were still. Since the potential energy of a wave train is equal to half the total energy of the waves, and only the potential energy is transmitted by the waves, it follows that only one half of the energy of the wave train is transmitted. Let an imaginary plane be drawn perpendicular to the direction in which the waves are travelling ; then while a complete wave, comprising a crest and a trough, travels across this plane, only one half of the energy of one wave length is transmitted across it ; and for the energy possessed by one wave length of the liquid to be transmitted across the plane, two complete waves must have travelled across it. This result may be expressed by the statement that **a train of gravitational waves transmits energy at half the rate at which the individual waves travel.**

A group, comprising a limited number of gravitational waves, may be looked upon as a vehicle for the transmission of energy over the surface of a liquid. The individual waves of the group travel with a speed determined by equation (8) (p. 460) ; but energy is transmitted with half this speed, and therefore **the group as a whole must travel with half the speed of the individual waves.** As each wave approaches the anterior boundary of the group, its amplitude diminishes, owing to the fact that only one half of the energy of the wave is transmitted to the still water in front of the group. The last wave of the group leaves half of its energy behind as it travels forwards, and therefore a new wave of diminished amplitude appears just behind it. Thus, the waves dwindle as they approach the front of the group, and fresh waves continually arise in the rear ; and the group as a whole travels forward with half the velocity of the individual waves. These phenomena can be observed quite plainly when a stone is cast into a pond. A somewhat similar phenomenon can be observed at sea; if a wave which is larger than its neighbours is followed by the eye, it will be noticed that it soon loses its extra size, which is acquired by the next following wave.

Wave velocity and group velocity.—Let a train of waves, of unit amplitude and wave length λ, travel with a velocity V. Each particle in the surface of the liquid oscillates about its mean position; if t denotes the time which has elapsed since the instant when a selected particle passed upwards through its mean position, then its instantaneous displacement above that position is equal to $\sin(2\pi Vt/\lambda)$. Let the waves be travelling from left to right; then the displacement of a particle in the surface, at a distance x to the right of the particle first selected, will be the same as that of the first particle at a time t' seconds earlier, where $Vt'=x$; for each displacement travels along the surface with a velocity V. Hence, at the instant when the first particle has the displacement $\sin(2\pi Vt/\lambda)$ the second particle will have the displacement $\sin\{2\pi V(t-t')/\lambda\}$, and this is equal to $\sin\{(2\pi/\lambda)(Vt-x)\}$. Thus, if y denotes the instantaneous displacement at a distance x from the first particle, measured in the direction in which the waves are travelling, we have—

$$y = \sin\frac{2\pi}{\lambda}(Vt-x).$$

Let $p=(2\pi/\lambda)$; then—

$$y = \sin p\,(Vt-x).$$

Now let another train of waves, also of unit amplitude, but of slightly different wave length λ', travel with a correspondingly different velocity V' in the same direction as the first train. Then, writing $p'=(2\pi/\lambda')$, the displacement y' of the particle at x, due to the second wave train, is given by the equation—

$$y' = \sin p'\,(V't-x).$$

The resultant displacement Y of the particle, due to both wave trains, is given by the equation—

$$Y=y+y' = \sin p\,(Vt-x)+\sin p'\,(V't-x).$$

Now—

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B,$$

where A and B denote any angles. Let—

$$pVt - px = A + B,$$
$$p'V't - p'x = A - B;$$

then $$2A = (pV + p'V')t - (p + p')x,$$

and $$2B = (pV - p'V')t - (p - p')x,$$

and—

$$Y = 2\cos\tfrac{1}{2}\{(pV - p'V')t - (p - p')x\}.\ \sin\tfrac{1}{2}\{(pV + p\ V')t - (p + p')x\} \ldots (14)$$

To understand the meaning of this equation, let a graph be drawn showing the connection between Y′ and x for any selected value of t, where—

$$Y' = 2\cos\tfrac{1}{2}\{(pV - p'V')t - (p - p')x\} \quad . \quad . \quad . \quad (15)$$

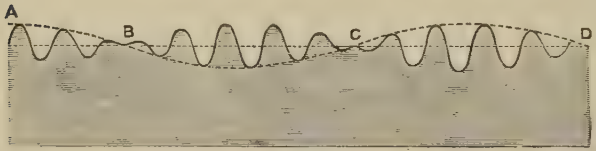

Fig. 219.—Groups of waves.

This graph has the form of the dotted curve ABCD (Fig. 219). From this it will be seen that, for any selected value of t, equation (15) represents a simple harmonic curve. Also, if—

$$\frac{1}{2(p - p')} = \frac{2\pi}{L},$$

and $$\frac{pV - p'V'}{p - p'} = v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

equation (15) may be written in the form—

$$Y' = 2\cos\frac{2\pi}{L}(vt - x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

It at once becomes evident that this equation represents a simple harmonic curve, of wave length L, which travels with a velocity v along the axis of x.

Now from (14),

$$Y = Y'\sin\tfrac{1}{2}\{(pV + p'V')t - (p + p')x\},$$

and if p' and V' are very nearly equal to p and V, the expression $(1/2)(pV+p'V')$ represents the mean of the values pV and pV', and this, for our present purpose, may be assumed to be equal to pV, to a sufficiently close approximation. Also, $(1/2)(p+p')=p$, to the same degree of approximation. Thus—

$$Y = Y' \sin p(Vt-x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If Y' were constant, this equation would represent an ordinary train of waves travelling with velocity V. Since Y' varies both with t and x, it follows that each of the individual waves represented by (18) travels along the axis of x with a velocity V, its amplitude changing in such a manner that the wave always touches the simple harmonic curve represented by (17), which travels along the axis of x with a velocity v. Thus the simple harmonic curve represented by (17) (the dotted curve ABCD in Fig. 219), envelops the individual waves, and the waves between B and C constitute a group which, as a whole, travels forward with a velocity v, while each individual wave of the group travels forward with a velocity V.

The precise meaning to be attached to the reasoning employed above, may be made clearer by the aid of Fig. 220. The curves in the lower part of the diagram represent the instantaneous sections of two trains of waves, of equal amplitude but of different lengths. The resultant displacement at any point on the surface of the liquid, is equal to the algebraical sum of the displacements at that point, due to the two trains of waves; hence, the section of the surface takes the form represented in the upper part of the diagram. At A and C, crests of the two trains of waves coincide, so that at these points the amplitude of the resultant disturbance has a maximum value. At B and D, a crest of one train of waves coincides with a trough of the other train, and at these points there is no resultant displacement of the surface. The space between the points B

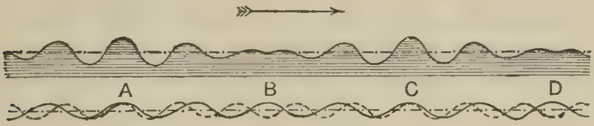

FIG. 220.—Groups of waves.

and D comprises a group of waves ; the group length is equal to the distance between B and D, or that between A and C.

If both trains of waves travel with equal velocities, the groups will travel with the same velocity as the waves ; for we may suppose that Fig. 220, as a whole, travels from left to right with a velocity equal to that of the waves. If, however, the shorter waves travel the faster, they will continually overtake the longer waves ; in this case we may suppose that the dotted curve in Fig. 220 travels from left to right with a velocity greater than that with which the full line curve travels in the same direction. As a consequence, the maximum displacement travels forward faster than either train of waves. Similarly, if the shorter waves travel more slowly than the longer waves, the maximum displacement travels forward more slowly than either train of waves.

Fig. 220 shows that the number of the shorter waves between A and C must be greater by unity than the number of the longer waves in the same space ; that is, if a group comprises n of the longer waves, it must comprise $(n+1)$ of the shorter waves. Similarly, two groups must comprise $2n$ of the longer waves, and $(2n+2)$ of the shorter waves. Hence, in general, **the number of groups comprised in a given space is equal to the difference between the number of the shorter and the number of the longer waves comprised in that space.** Now, if the length of the longer waves is equal to λ', the number of these waves comprised in unit length is equal to $(1/\lambda')$; similarly, the number of the shorter waves, of wave length λ, comprised in unit length, is equal to $(1/\lambda)$. Hence, **the number of groups per unit length is equal to** $\{(1/\lambda)-(1/\lambda'\}$.

Now let attention be concentrated on a given point of the surface, say A, Fig. 220. There will be a maximum displacement at A at the instant when two crests pass that point simultaneously, and A will be the middle of a group. As the two trains of waves travel forward, the displacement at A diminishes, reaches a minimum, and then increases ; it attains its maximum value once more when n of the longer and $(n+1)$ of the shorter waves have passed. In this time a group of waves has passed A. Hence, **the number of groups that pass a point in any given time, is equal to the difference between the number of the shorter and the number of the longer waves that pass in that time.**

Let the longer waves travel with a velocity V' ; then in one second a length V' of the train of longer waves will pass any point, and since each of these waves has a length λ', it follows that the number of the longer waves that pass a point in a second is equal to (V'/λ'). Similarly, the number of the shorter waves that pass the point in a second is equal to (V/λ), where V is the velocity of the shorter waves. Hence, the difference between the numbers of shorter and longer waves that pass a point in a second is equal to $\{(V/\lambda) - V'/\lambda'\}$, and this gives the number of groups that pass the point in a second.

Let v be the velocity of a group; then the maximum displacement that passes through a point at a given instant will travel forward through a distance equal to v units of length in one second, and each unit length of this distance will comprise $\{(1/\lambda) - (1/\lambda')\}$ separate groups, all of which have passed through the given point in a second; hence, the number of groups that pass a point in a second is also equal to $v\{(1/\lambda) - (1/\lambda')\}$. Thus

$$v\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{V}{\lambda} - \frac{V'}{\lambda'};$$

$$\therefore \quad v = \frac{\dfrac{V}{\lambda} - \dfrac{V'}{\lambda'}}{\dfrac{1}{\lambda} - \dfrac{1}{\lambda'}}.$$

This value of v becomes identical with that given by equation (16), p. 470, when the numerator and denominator are both multiplied by 2π.

Let it be assumed that the wave velocity V varies as the nth power of the wave length λ; that is—

$$V = K\lambda^n,$$

where K is a constant for the particular type of waves in question. Substituting in (16) the values of p and p' in terms of λ and λ', we obtain—

$$v = \frac{\lambda' V - \lambda V'}{\lambda' - \lambda}.$$

Let $\lambda' = \lambda + \delta$, where δ is very small; then—

$$v = \frac{(\lambda + \delta)K\lambda^n - \lambda K(\lambda + \delta)^n}{\delta};$$

and $$(\lambda + \delta)^n = \lambda^n\left(1 + \frac{\delta}{\lambda}\right)^n = \lambda^n\left(1 + n\frac{\delta}{\lambda} + \ldots\right),$$

when powers of δ/λ greater than the first are neglected. Thus—

$$v = K\frac{\left(\lambda + \delta\right)\lambda^n - \lambda\,.\,\lambda^n\left(1 + n\dfrac{\delta}{\lambda}\right)}{\delta} = K\lambda^n(1 - n);$$

$$\therefore \quad v = V(1 - n) \quad . \quad . \quad . \quad . \qquad (19)$$

For gravitational waves on a deep liquid, $n = 1/2$ (equation (8), p. 460). Therefore for these waves—

$$v = (1/2)\,\mathrm{V},$$

that is, **the group velocity has half the value of the wave velocity.**

When the length of the waves which travel over a liquid is very small, the value of $(S/\rho)(2\pi/\lambda)^2$ becomes great in comparison with g, and the velocity V of these waves is given by the equation—

$$\mathrm{V} = \sqrt{\frac{2\pi S}{\lambda\rho}}, \text{ (equation (10), p. 461).}$$

Very short waves are sometimes called ripples, and sometimes capillary waves; for these $n = -(1/2)$, and—

$$v = (3/2)\,\mathrm{V},$$

so that **the group velocity is one-and-a-half times as great as the wave velocity.**

A few other types of waves may be briefly referred to.

Sound waves and light waves are propagated with a velocity which is independent of the wave length; that is, $\mathrm{V} = \mathrm{K}\lambda^{0}$, and $n = 0$, so that $v = \mathrm{V}$

A flexural wave of length λ is propagated[1] along an elastic rod with a velocity which varies inversely as λ; that is, $\mathrm{V} \propto \lambda^{-1}$, and $n = -1$. In this case, $v = 2\mathrm{V}$; that is, **the group velocity is twice as great as the wave velocity.**

If a number of pendulums are hung side by side in a vertical plane, and a finger be drawn along them, the pendulums will be set oscillating in such a manner that the phase of any one is intermediate between those of the preceding and succeeding pendulums. A wave disturbance travels along the line of pendulums, and the distance between the two nearest pendulums which are in the same phase of oscillation is the wave length λ of the disturbance. A displacement, equal to the amplitude of swing, travels along the line through a distance λ during the period of oscillation T common to all the pendulums, and the velocity of wave transmission is equal to λ/T. Since T is determined by the length of the pendulums, and is therefore constant, it follows that $\mathrm{V} = \mathrm{K}\lambda$, and $n = 1$. In this case the group velocity v is equal to zero; it will be noticed that there is no transmission of energy.

The propagation of groups of waves.—Two instances of the propagation of groups of waves must now be noticed.

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[1] See question 8, p. 284. The answer to this question is given at the end of the book.

In the first instance, let it be supposed that some disturbance has caused the water of a lake to be heaped up in the neighbourhood of a particular point (Fig. 221, *a*). It may be supposed that for an instant the heaped up water is stationary; immediately afterwards the middle of the heap commences to subside, and the sides of the heap move outwards (Fig. 221, *b*). This entails the formation of two wave-crests which travel away from each other. The momentum of the subsiding water between the two crests carries it downwards below the general surface of the lake, but after a time the downward motion ceases, and then an upward motion takes its place (Fig. 221, *c*). Thus, a new heap of water shoots upwards between the two receding crests, and in its turn attains its maximum altitude; then the middle of the heap begins to subside (Fig. 221, *d*), and two new crests move outwards from it. As this process is repeated, a number of waves travel away from the point at which the original disturbance occurred (Fig. 221, *e* and *f*); the long waves move most quickly (if the wave length is greater than the critical value given on p. 351), and the waves in the front of the group continually dwindle, while new waves appear in the rear of the group. Waves of this type can be produced by throwing a stone into a pond; the form of the surface, just after the stone has sunk beneath the surface, is similar to that represented in Fig. 221, *b*.

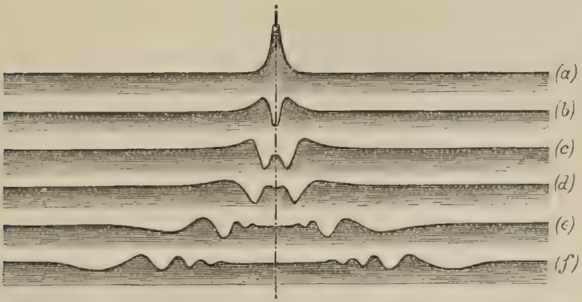


FIG. 221.—Waves due to an arbitrary disturbance. (After Lord Kelvin.)

The question naturally arises, why is it that the crest of a wave can advance without being continually divided and re-divided like the heaped-up water from which the waves originate? The answer to this question becomes obvious, when it is remembered that the heaped-up water is stationary at the instant when it commences to divide; while there is a forward motion at the crest of an advancing wave, and this motion determines that energy shall be transferred only in the forward direction.

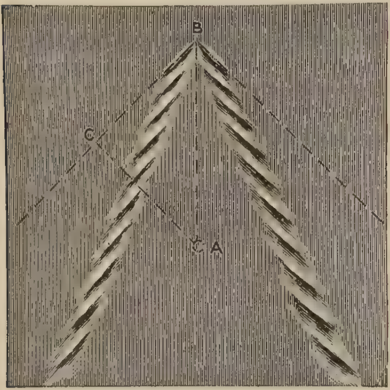


FIG. 222.—Waves produced by a partly immersed body which is moving along the surface of a liquid.

In the second instance, let a partly immersed body be moving over still water; the water is heaped up just in front of the body, and this causes waves to be formed which travel away over the surrounding surface. If a single wave could be propagated without change of form, the disturbance originated at A (Fig. 222) would travel outwards to C while the body moved onwards to B, and two long ridges of water would slant backwards from the position occupied by the body at any instant. Let V be the wave velocity in this case, and let u be the velocity of the body. Then the ridges of water would make an angle $ABC=\theta$ with the direction AB in which the body is moving, where—

$$\sin\theta = AC/AB = V/u.$$

But a single wave cannot be propagated without change of form; the wave continually dwindles as it advances, and new waves appear in its rear, the group thus formed being propagated with half the wave velocity V. As a consequence, no wave will have reached C when the body arrives at B, but midway along the line AC there will be a group of waves; and instead of a single wave slanting backwards from A, there will be a number of waves arranged in steps (Fig. 222). That end of a step which is farthest from B represents the dwindling crest of the foremost wave of one group; and the other end of the same step represents the wave just formed in the rear of another group. As each step-like wave advances, it appears to drift sideways along the surface of the water; this is due to the decay of the end of the wave remote from B, and the growth of the other end. Waves of this type can be observed when a boat moves rapidly over a lake, or when a duck swims across a pond.

Retarding force due to the waves produced by a ship.—When a floating body travels over the surface of a liquid, its progress is retarded by forces due to two distinct causes. In the first place, the wetted surface of the body is acted upon by a retarding force somewhat akin to friction; this force depends on the area of the wetted surface, the nature of the liquid, and the velocity with which the surface is moving through the liquid, and can be determined from the results of direct experiments. In the second place, the body produces waves which continually carry energy away from it, and this energy must be supplied by the agent which propels the body; hence, work must be done in propelling the body, and the motion of the body must be opposed by a force due to the waves produced. In connection with large ships, the retarding force due to wave-making is of great importance.

Let the retarding force, due to the production of waves by the body, be denoted by f. Then f may in some way depend on g, the acceleration due to gravity; for the only waves of importance, when the body is large, are of the gravitational type: let f vary as the nth power of g. The value of f will also depend on the mass m of the liquid displaced by the body, that is, on the mass of the body; let f vary as the xth power of m. Further, f will vary with some power (say the yth) of the linear magnitude l of the body, and also with some power (say the zth) of the velocity V with which the body is moving. These are the only factors which could influence the wave-making properties of a floating body, and therefore

$$f = kg^n m^x l^y \mathrm{V}^z,$$

where k is a constant depending on the shape but not on the magnitude of the body.

The dimensions (p. 19) of f are $(\mathrm{ML/T^2})$; those of g are $(\mathrm{L/T^2})$, those of m are M, while those of l and V are respectively L and L/T. Since the dimensions of both sides of the equation must be equal, and k has no dimensions, it follows that

$$\frac{\mathrm{ML}}{\mathrm{T}^2} = \left(\frac{\mathrm{L}}{\mathrm{T}^2}\right)^n \mathrm{M}^x\mathrm{L}^y\left(\frac{\mathrm{L}}{\mathrm{T}}\right)^z.$$

M occurs to the first power on the left-hand side of this equation, and to the power x on the right-hand side: therefore $x=1$. Equating powers of L on the two sides of the equations:

$$1=n+y+z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equating powers of T,

$$-2=-2n-z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \quad z=2(1-n),$$

and from (1),

$$y=1-n-z=1-n-2+2n=n-1.$$

Thus

$$f=kmg^n l^{(n-1)}\mathrm{V}^{2(1-n)},$$

and

$$\frac{f}{m}=kg^n\left(\frac{\mathrm{V}^2}{l}\right)^{1-n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The method of dimensions does not suffice to determine the value of n, but it is obvious that n cannot be equal to unity, since, if it were, f/m would be independent of V and l, which is absurd. The value of f/m gives the retarding force per unit mass of the liquid displaced by the body; and from (3) it appears that f/m is constant so long as V^2/l is constant. Hence, if a model of length l requires the application of a force f_1 per unit mass, in order to tow it, at a velocity V, against the retarding force due to the waves it produces; a ship similar to the model, but of N times its linear magnitude, will need the application of the same force per unit mass when it is propelled at a velocity $\mathrm{V}\sqrt{\mathrm{N}}$ Thus, experiments on a model suffice to determine the power needed by the engines of a full-sized ship; and the form of ship which will entail the smallest loss of energy due to the production of waves can also be determined. The law deduced from equation (3) above is known as *Froude's Law of Comparison*, in honour of its discoverer.

Problem.—*A model* 10 *feet long needs the expenditure of* 0·5 *horse-power in order to tow it at a velocity of* 3 *miles per hour against the opposing force due to the waves produced. What horse-power would be required to propel a ship, similar to the model but* 250 *feet long, at the "corresponding" velocity* $3\sqrt{(250/10)}=15$ *miles per hour, against the opposing force due to the waves produced?*

Let p be the power required by the model, and P that required by the ship; and let m be the mass of the model, and M that of the ship, while v and V denote the "corresponding" velocities of the model and the ship.

The power expended is proportional to the product of the opposing force and the velocity, and therefore the power divided by the velocity is proportional to the opposing force. At "corresponding" speeds, the opposing force per unit mass is constant, so that—

$$\therefore \quad \frac{p}{mv} = \frac{P}{MV};$$

$$\therefore \quad P = p\frac{MV}{mv} = 0{\cdot}5 \times (25)^3 \times 5 = 1{,}562 \text{ horse-power,}$$

since the mass of the displaced water is proportional to the cube of the linear magnitude of the ship.

The foregoing investigation is of a general character, no particular shape of the floating body having been assumed. Some advantage may be gained, however, by working out the problem for a body of specific shape. Let the body have the form of a rectangular box of length l and breadth b; and let it be immersed to a depth d in the liquid. When the body is moving forward with a velocity V, the liquid immediately in front of it must also be moving with a velocity V; for this to be possible, the liquid must be heaped up in front of the body (Fig. 223). To determine the height h to which the liquid must be heaped up, let it be supposed that a velocity, equal and opposite to that with which the body is actually moving, is impressed on the body and the whole of the liquid. Then, if A is a point on the surface of the liquid to which the disturbance created by the body has not extended, the velocity of a particle of the liquid at A is equal to V; and if the particle has unit mass, its kinetic energy is equal to $V^2/2$. When this particle reaches

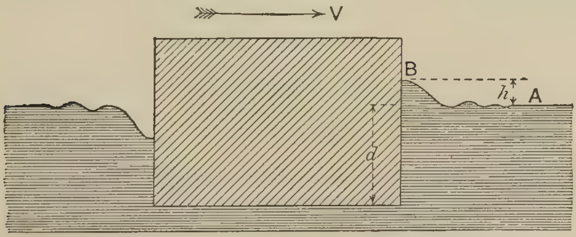


FIG. 223.—Partly immersed body moving along the surface of a liquid.

the point B at the summit of the ridge immediately in front of the body, its velocity in the plane of the diagram (Fig. 223) has decreased to zero, and the particle has risen to a height h above the undisturbed surface of the liquid; thus, the increase in its potential energy is equal to gh, and this increase has been gained at the expense of the kinetic energy which has disappeared. Thus

$$\frac{V^2}{2} = gh, \text{ and } V^2 = 2gh.$$

In this calculation, the lateral velocity imparted to each particle as it approaches the body has been neglected; this lateral velocity must exist in order that the liquid may flow round the body. Now, if h is small in comparison with the depth of immersion d, the force acting on the body, due to the heaping up of the water in front of it, is approximately equal to $g\rho hbd$. In order that the liquid may follow the body, there must be a depression of depth h immediately behind it; and this entails a force $g\rho hbd$ which also opposes the motion of the body. Then, the total opposing force f is given by the equation

$$f = \rho bd \,.\, 2gh.$$

The mass m of the displaced liquid is equal to ρlbd. Thus—

$$f = m\frac{2gh}{l} = m\frac{V^2}{l}.$$

This result agrees with equation (3) (p. 478) if $n = 0$ and $k = 1$. But whereas the result just obtained is only an approximate solution relating to a body of a particular form, equation (3) is absolutely true, no matter what may be the form of the body.

The calming of waves by a shower of rain.—Sailors are familiar with the fact that a heavy shower of rain tends to calm the waves on the sea. The popular explanation of this phenomenon is that the rain "beats the waves down"; but it is obvious that this explanation is insufficient, since the rain falls not only on the crests, but also in the troughs, and the act of pressing both the crests and the troughs downwards would have no tendency to reduce the waves. The true explanation can be inferred from the result of a simple experiment. If ink, or any coloured liquid, is allowed to fall drop by drop into water, it will be observed that each drop forms a vortex ring which travels downwards through the water, expanding as its velocity diminishes (p. 434). Each vortex ring carries a considerable volume of the surrounding water with it (p. 437). Thus, each drop of rain that falls on the sea forms a vortex ring by which water from the surface of the sea is carried downwards to a considerable depth. When waves are travelling over the surface of the sea, the particles of water are moving in circular orbits, and anything which destroys the regularity of their motion will destroy the waves. The vortex rings formed from the raindrops obviously destroy the regularity of the wave-motion, by carrying particles from the surface, where the orbital velocity of the particles is greatest, down into the depths of the sea where the orbital velocity is insignificant; with the result that internal eddies are formed, and the energy of the waves is ultimately dissipated in the form of heat.

The calming of waves by the spreading of a layer of oil over the surface of the sea has been discussed already (p. 298); in this case, the variation in the surface tension, due to the unequal stretching of the superficial layer of oil, destroys the regular orbital motion of the water particles near to the surface of the sea, and so destroys the waves.

The characteristics of waves on the surface of a shallow liquid.—When waves travel over the surface of a liquid, the motion of each particle of the liquid must comply with the geometrical and mechanical conditions which apply to any moving liquid. When the liquid is deep, the only other condition to be complied with is, that the motion of each particle must be consistent with the motion at the free surface of the liquid. In this case each particle describes a circular orbit, the radius of the orbit depending on the mean depth of the particle below the free surface of the liquid (p. 458). When the mean depth of the particle is great in comparison with the wave-length, the orbital motion is insignificant; but when the mean depth of the particle is small, the radius of the circular orbit described by the particle has a finite value.

If the total depth of the liquid is small in comparison with the length of a wave, additional conditions must be complied with by each particle of the liquid. If we suppose that the liquid is bounded below by a horizontal rigid plane, it follows that a particle of liquid in contact with this plane cannot execute vertical oscillations; for to do so it would have to descend through the plane at one instant, and rise vertically, leaving an empty space between it and the plane, at another instant. On the other hand, it might be possible for a particle in contact with the plane to oscillate in a horizontal direction. Hence, we see that the particles of the liquid cannot describe simple circular orbits as they would if the liquid were deep. The vertical oscillations of the particles must decrease rapidly with the depth, so that there is no vertical oscillation at the lower boundary of the liquid; while the horizontal oscillations of the particles may be practically equal at all depths.

Velocity of wave propagation.—Let it be supposed that the maximum vertical displacement at the free surface of a liquid is small in comparison with H, the total depth of the liquid; and

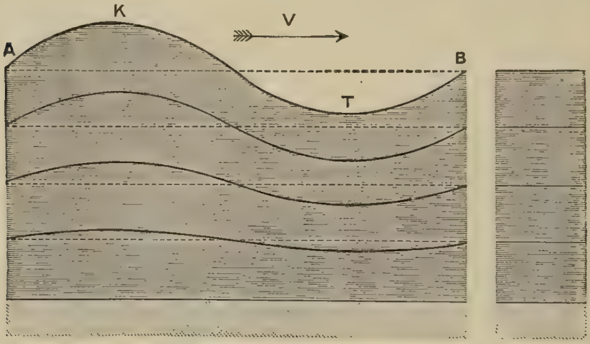


FIG. 224.—Wave on the surface of a shallow liquid.

also that H is small in comparison with the length of the waves which travel along the free surface. The maximum horizontal displacement is unrestricted and will have the same value at all depths; therefore the motion of each particle will approximate closely to a horizontal rectilinear oscillation.

Let A K T B (Fig. 224) represent a wave disturbance travelling from left to right over a shallow liquid; the form of the wave need not approximate to a simple harmonic curve, but may be of any shape. To the right of the point B let the water be undisturbed. Let it be supposed that, before the liquid was disturbed by the waves, it was divided into *n* layers of equal thickness bv imaginary horizontal planes, the distance between any one of these planes and the plane immediately above it being equal to H/n. The waves distort these planes into curved surfaces, of which the traces are represented in Fig. 224. Let V denote the velocity of the waves, from left to right; and let an equal velocity be impressed on the whole of the liquid, from right to left. Then the waves become stationary in space, and the liquid streams past them; each of the undulating curves in Fig. 224 becomes a stream line, and if we suppose that the diagram is displaced through unit distance perpendicular to the plane of the paper, we obtain tubes of flow similar to those described on p. 453. Now, the horizontal oscillations of the particles at all depths are equal, and therefore the maximum velocity due to the horizontal oscillation is the same at all depths; and as the velocity of flow at any point beneath the crest K of the wave is the resultant of the uniform velocity V impressed on the whole of the liquid, and the maximum velocity due to the horizontal oscillation at that point, it follows that the velocity of flow is identical at all points beneath K. From this it follows at once that, beneath K, all tubes of flow must have equal cross-sectional areas; and therefore the crests of the undulatory curves must be equidistant one from another.

If the crest K of the wave is at a distance y above the free surface of the untroubled water, the total depth of the water below K is equal to $(H+y)$; and since this depth comprises n tubes of flow of equal vertical thickness, it follows that the vertical distance between any undulatory curve and the curve immediately above it, is equal to $(H+y)/n$, and this also gives the cross-sectional area of any tube of flow below K. Let V_c be the velocity of flow at any point below K; then the product of the velocity and the cross-sectional area is constant for all points in a tube of flow (p. 378), and in the untroubled water to the right of B, where the velocity of flow is equal to V, the cross-sectional area of each tube is equal to H/n. Thus—

$$V_c \cdot \frac{H+y}{n} = V \cdot \frac{H}{n};$$

$$\therefore \quad V_c = V \cdot \frac{H}{H+y}$$

To determine the value of V, consider the flow in the tube immediately beneath the free surface. The liquid flows across any cross-section to the right of B with a velocity V; and before reaching the cross-section immediately beneath K, the liquid has risen through a height y against the downward force of gravity. The pressure in the tube is everywhere equal to the atmospheric pressure P. Thus, by reasoning precisely similar to that employed on p. 402, we obtain—

$$(P+\tfrac{1}{2}\rho V_c^2)-(P+\tfrac{1}{2}\rho V^2)=-g\rho y\,;$$

$$\therefore\quad \tfrac{1}{2}(V_c^2-V^2)=-gy.$$

Substituting the value of V_c obtained above—

$$V_c^2-V^2=V^2\left\{\left(\frac{H}{H+y}\right)^2-1\right\}=-2gy.$$

$$V^2\left\{\frac{1}{\left(1+\dfrac{y}{H}\right)^2}-1\right\}=-2gy\,;$$

$$\therefore\quad -V^2.\,\frac{\dfrac{2y}{H}+\left(\dfrac{y}{H}\right)^2}{\left(1+\dfrac{y}{H}\right)^2}=-\,2gy.$$

Since y is small in comparison with H, the value of $(y/H)^2$ will be negligibly small in comparison with that of $(2y/H)$; also, y/H will be negligibly small in comparison with unity. Thus—

$$V^2\,.\,\frac{2y}{H}=2gy,$$

so that $V^2=gH$, and $V=\sqrt{(gH)}$.

Thus, **when waves, subject to the conditions explained above, travel over the surface of a shallow liquid, the velocity of wave propagation is proportional to the square-root of the depth of the liquid, and is independent of the wave length.**—A disturbance of any shape can travel unchanged over the surface of a shallow liquid, without breaking up into separate waves. For instance, an isolated crest can travel over the surface without change of shape.

Scott Russel studied these waves on canals. When a barge is towed along a canal, the water becomes heaped up in front of the barge, and thus produces waves which travel forwards with the velocity $V=\sqrt{(gH)}$, where H is the depth of the canal. If the speed of the barge is less than V, the waves continually carry energy away ahead of the barge, and a corresponding amount of work must be performed by the agent by which the barge is towed. If, however, the speed of the barge is equal to V, only one wave is formed, and this accompanies the barge in its progress, so that no progressive loss of energy is entailed. Hence, the work expended in towing the barge with the velocity $V = \sqrt{(g\,H)}$, is less than if the speed of the barge were smaller.

On a canal 8 ft. deep, the critical velocity V is given by the equation—

$$V = \sqrt{(32 \times 8)} = 16 \text{ ft. per second, or nearly } 11 \text{ miles per hour.}$$

The above investigation gives a clue to the reason why sea waves curl over and break as they approach a shelving beach; as the waves travel toward the beach, each crest is always in deeper water than the trough immediately in front of it, and thus the crest travels the faster, and ultimately topples over into the trough.

When waves, extending in parallel straight lines along the surface of the sea, approach a shelving beach obliquely, the end of a wave which is nearest to the beach travels more slowly than the other end of the wave; as a consequence, the waves wheel round and finally break parallel to the beach.

It may be left as an exercise for the student, to prove that if ripples (that is, waves of very short wave length) travel over a liquid of which the depth is small in comparison with λ, then the velocity V with which the ripples travel is given by the equation—

$$V = \sqrt{H\left\{g + \left(\frac{2\pi}{\lambda}\right)^2 \frac{S}{\rho}\right\}}.$$

Boundary conditions.—A point of importance yet remains to be discussed. It has been assumed that particles in contact with the lower boundary of the liquid can move freely along that boundary—that is, that there is free slip between a liquid and its solid boundary. In the next chapter it will be explained that this assumption is inadmissible. Nevertheless, the laws derived above, with regard to the wave-motion of a shallow liquid, have been amply verified; hence we may conclude that there is some peculiarity about the motion of the liquid in contact with its solid boundary which permits the superincumbent liquid to move in the manner described. Mrs. Ayrton has found that this is, in fact, the case; horizontal vortex filaments are formed at the boundary, and these serve as rollers on which the superincumbent liquid moves to and fro. The ridges and furrows frequently to be seen on a sandy beach, when the tide has ebbed, are due to these vortices. The narrow striations of the dust-figures formed in Kundt's tube are probably due to the same cause.

QUESTIONS ON CHAPTER XIV

1. Two wave trains, equal in wave length and amplitude, are travelling in opposite directions over the surface of deep water. Prove that at certain points (called nodes) on the surface, separated by distances equal to half the wave length of the waves, there will be no vertical motion; while at points midway between the nodes (called antinodes) there will be maximum vertical motion, of amplitude equal to twice the amplitude a of either of the wave trains. Also, prove that, at the nodes, there is a horizontal oscillatory motion of amplitude $2a$; while at a point between a node and an antinode, there is a linear oscillatory motion of amplitude $2a$ in a direction inclined to the vertical.

2. Waves on the surface of deep water travel up to a vertical wall on which they are incident normally. Prove that a series of "standing waves" (see question 1) will be formed, and that there will be an antinode at the surface of the wall.

3. Draw a curve showing the velocity of waves on water, for wave lengths between zero and a very large magnitude (surface tension of water $= 70$ dyne/cm.).

4. Ripples of wave length 1 cm. are travelling over the surface of a liquid; at what depth will the amplitude of the oscillatory disturbance be equal to 1 per cent. of the amplitude at the surface?

5. A canal is five feet deep; at what speed must a barge be towed along this canal in order that the expenditure of energy shall be as small as possible?

6. A straight canal, of uniform depth H, is bounded laterally by vertical banks; at one place a small ridge extends transversely across the bed of the canal. The water in the canal is flowing with a uniform velocity V. Prove that there will be a transverse ridge on the surface of the water above the ridge on the bed of the canal, if $V > \sqrt{gH}$; and that there will be a transverse depression on the surface of the water if $V < \sqrt{gH}$. What will happen if $V = \sqrt{gH}$?

CHAPTER XV

THE FLOW OF A VISCOUS FLUID

Coefficient of viscosity.—When contiguous layers of a material fluid are moving relatively to each other, forces are called into play which tend to destroy their relative motion; the layer which is moving more quickly is acted upon by a retarding force, and the layer which is moving more slowly is acted upon by an accelerating force. That property of a material fluid which opposes the relative motion of its parts is called *viscosity*.

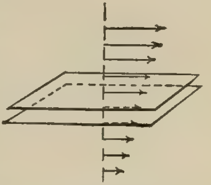


FIG. 225.—Velocity gradient in a fluid.

Let Fig. 225 represent two imaginary planes described in a fluid, and let all particles in the upper plane be moving from left to right with a velocity v_1, while all particles in the lower plane are moving from left to right with a smaller velocity v_2. In passing from the upper to the lower plane, there is no abrupt change of velocity; if the distance between the planes is equal to d, the velocity changes from v_1 to v_2 in the distance d, and the change of velocity per unit distance is equal to $(v_1-v_2)/d$. This quantity is called the **velocity gradient**, and will be denoted, in the ensuing investigations, by the symbol v'.

The fluid above the upper plane is moving more quickly than the fluid below it, and therefore the fluid beneath the upper plane is acted upon by an accelerating force, and the fluid above the upper plane is acted upon by an equal retarding force. Let the value of either of these forces be denoted by f;

then it is self-evident that f is proportional to the area a of the plane.

Since the force f is called into play by the relative motion of the parts of the fluid above and below the plane, it cannot depend on the actual velocity of either of these parts. The simplest assumption that can be made is that **the force f is proportional to the velocity gradient v' in the immediate neighbourhood of the plane.** This assumption cannot be justified by direct experiments, but predictions based on it have been tested experimentally, and the close agreement between the phenomena predicted and those observed affords an indirect proof of the validity of the assumption.

Since the force f is proportional to the area a and the velocity gradient v', it follows that—

$$f = \eta a v',$$

where η is a constant depending on the nature and physical conditions of the fluid: this constant is called the **coefficient of viscosity** of the fluid under the given conditions. When f, a, and v' can be measured, the value of η is obtained directly from the equation—

$$\eta = \frac{f}{av'}.$$

Thus, **the coefficient of viscosity may be defined as the tangential force per unit area per unit velocity gradient.**

The dimensions of η are those of a force divided by an area and a velocity gradient, and the dimensions of a velocity gradient are $(L/T) \div L = 1/T$. Hence the dimensions of η are—

$$\frac{ML}{T^2} \div \left(L^2 \times \frac{1}{T}\right) = \frac{M}{LT}.$$

Orderly and turbulent motion: the critical velocity.—The two characteristic types of motion of a fluid have been referred to already (p. 403). In one type of motion, all particles of the liquid travel along stream-lines, and this type of motion may be called orderly; in the other type, no definite stream lines are formed, and the resulting motion may be called turbulent.

Experiments show that when a fluid flows through a narrow tube at a constant rate, there is a difference of pressure between the ends of the tube. Energy is being transformed into heat within the tube, and therefore the pressure where the fluid enters is greater than where it leaves the tube (p. 404). When the rate of flow is increased, it is found that the difference of pressure is proportional to the rate of flow, so long as this rate is small; but for high rates of flow the difference of pressure is proportional approximately to the square of the rate. Thus, if the average velocity across any cross-section of the tube is denoted by V, and the difference of pressure between the ends of the tube is denoted by P, it appears that P is proportional to V when V is less than a certain critical value, and P becomes approximately proportional to V^2 when V is greater than the critical value.

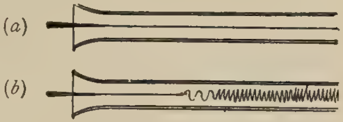


FIG. 226.—Orderly motion, and turbulent motion.

Prof. Osborne Reynolds found that, below the critical velocity, the motion of a fluid is orderly; while above the critical velocity the motion is turbulent. Water was allowed to flow along a glass tube, and the nature of its motion was rendered evident by introducing a little coloured water, which flowed from a small jet near to the axis of the tube. When the velocity of the water was small, the coloured water flowed in a straight line along the axis of the tube (Fig. 226, *a*); but at a certain velocity the band of coloured water became sinuous (Fig. 226, *b*), and at a slightly higher velocity the colouring matter became uniformly distributed throughout the tube, thus showing that orderly motion along stream-lines had ceased.

The full investigation of the conditions which determine whether the motion of a fluid shall be orderly or turbulent is a matter of considerable difficulty; but the way in which the critical velocity V_c depends on the properties of the fluid, and the radius of the tube through which the fluid flows, can be determined easily by the method of dimensions. Let

$$V_c = k\rho^x \eta^y r^z,$$

where ρ denotes the density and η the coefficient of viscosity of the fluid, while r denotes the radius of the tube, and k is a numerical constant. The dimensions of both sides of the equation must be equal, so that

$$\frac{L}{T} = \left(\frac{M}{L^3}\right)^x \left(\frac{M}{LT}\right)^y L^z.$$

Equating the coefficients of M,

$$x + y = 0.$$

Equating the coefficients of T,

$$-1 = -y \quad \therefore\ y = 1, \text{ and } x = -1.$$

Equating the coefficients of L,

$$1 = -3x - y + z = 3 - 1 + z.$$

$$\therefore \quad z = -1.$$

Hence, $$V_c = k\frac{\eta}{\rho r}.$$

Thus, **the critical velocity of flow of a given fluid, under given physical conditions, is inversely proportional to the radius of the tube.** If different liquids are caused to flow through a given tube, the critical velocity of each is directly proportional to the coefficient of viscosity of the liquid, and inversely proportional to the density of the liquid. Thus, a high viscosity tends to promote orderly motion, and a high density tends to promote turbulent motion. From this we may conclude that an inviscid material fluid (if such could be found) would be incapable of orderly motion at all velocities ; in other words, the motion of an inviscid fluid is essentially unstable. It is only the viscosity of material fluids which renders it possible for their flow to approximate to that of a perfect fluid (p. 397).

Prof. Osborne Reynolds found that for water at 5°C. flowing through a composition tube of 0·615 cm. diameter, the critical velocity V_c is equal to 44·3 cms. per sec. At 5°C. the viscosity of water is equal to 0·015 gr./cm. sec. Hence, under these conditions—

$$k = \frac{V_c \rho r}{\eta} = \frac{44{\cdot}3 \times 1 \times 0{\cdot}307}{0{\cdot}015} = 907.$$

It must be remembered, however, that the viscosity of water varies rapidly with the temperature, so that it is difficult to determine k with great exactitude. In round numbers, k may be taken as equal to 1,000. Thus, for liquids in general

$$V_c = 1{,}000\frac{\eta}{\rho r}.$$

Thrupp has shown that Osborne Reynolds's formula holds only for small pipes; in large pipes and channels, the critical velocity is much greater—sometimes thousands of times greater.

Prof. Osborne Reynolds found that, for velocities above the critical velocity, the difference of pressure between the ends of a tube varies as $V^{1 \cdot 723}$, where V denotes the average velocity across any cross section of the tube; previous investigators had supposed that the pressure varied as V^2.

When the critical velocity is exceeded, the pressure required to drive a liquid through a tube is independent of the viscosity, and depends mainly on the density of the liquid. Thus, if treacle is driven through a tube at a velocity exceeding the critical velocity for treacle, the difference of pressure between the ends of the tube is almost exactly equal to that which would be required to drive water through the tube at the same velocity. Since the critical velocity is inversely proportional to the radius of the tube, it follows that, in a very large tube, treacle would flow as readily as water. If the Mississippi were a river of treacle, it would flow at the same rate as that at which the water actually flows. An instance of the ready flow of a large quantity of highly viscous liquid, is afforded by the lava which sometimes flows down the sides of volcanoes, the rate of flow being comparable with that of water.

Flow of a liquid through a narrow tube.—Let it be supposed that a liquid is flowing through a narrow tube of circular section, at an average velocity less than the critical velocity. In this case the flow of the liquid is orderly, and each particle travels in a straight line parallel to the axis of the tube. There is weighty evidence in favour of the assumption that the layer of liquid immediately in contact with the walls of the tube is practically stationary; this condition can be expressed by saying that **there is no slip between the walls of the tube and the liquid immediately in contact with them.** The condition of no slip at the walls of the tube, together with the law connecting the velocity gradient at any point with the coefficient of viscosity, can be used to determine the exact nature of the flow in the tube.

The lines of flow are parallel to the axis of the tube; therefore there can be no radial pressure gradient within the tube. A radial pressure gradient would imply, either a radial flow opposed by a frictional force, or a radial acceleration. Since there is no radial flow, there can be no radial acceleration, and it follows that **the pressure must be constant over any cross-section of the tube.** This law applies also to points just outside the ends of the tube where the stream lines are parallel; but if the tube is long in comparison with its internal diameter, the distance between the points near to its ends, where the stream lines begin to diverge, will be practically equal to the length of the tube.

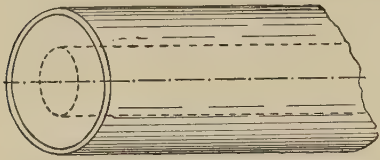


FIG. 227.—Flow of a viscous liquid through a tube.

Let it be supposed that the difference of pressure between the ends of the tube is equal to P. Let Fig. 227 represent one end of the tube; and let an imaginary cylinder, of radius r, be described about the axis of the tube. Then it is obvious that the velocity gradient will be constant over the surface of the cylinder; for the flow across any cross-section of the tube must be precisely similar to that across any other cross-section, and therefore there can be no variation in the flow along any generating line of the cylinder; also, the flow across any cross-section of the tube must be symmetrical with respect to the axis, and thus the velocity gradient can depend only on the distance from the axis of the tube. Let v' denote the velocity gradient common to all points on the surface of the imaginary cylinder; then the force exerted on the liquid inside the cylinder, by the liquid outside of it, is given by the equation—

$$f = \eta a v' = \eta \, . \, 2\pi r l \, . \, v',$$

where l denotes the length of the tube. This result is derived directly from the fundamental formula obtained on p. 488, by supposing that the cylindrical surface is made up of a great number of plane strips.

The liquid inside the imaginary cylinder is moving with greater velocity than that outside, and therefore the force f

opposes the motion of the liquid inside the cylinder. The difference between the pressures acting on the plane ends of the cylinder produces a force $\pi r^2 P$, which tends to accelerate the motion of the liquid within the cylinder. Since the flow is steady, the forces acting on the liquid within the cylinder must be in equilibrium; therefore—

$$\eta \,.\, 2\pi r l v' = \pi r^2 P,$$

$$\therefore \quad v' = r\frac{P}{2\eta l}.$$

Thus **the velocity gradient is proportional to the distance r from the axis of the tube.** The velocity gradient is equal to zero at the axis, and has its greatest value at points in contact with the walls of the tube.

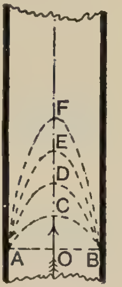


FIG. 228. Flow of a viscous liquid through a tube.

In passing inwards from the walls of the tube to a point at a distance r from the axis, the velocity gradient (rate of change of velocity per unit distance) decreases regularly: and the liquid immediately in contact with the walls of the tube is stationary. If R denotes the internal radius of the tube, the velocity gradient at the walls is equal to $(RP/2\eta l)$; at a distance r from the axis, the velocity gradient is equal to $(rP/2\eta l)$. Thus, over the distance $(R-r)$, the velocity gradient has the *average* value $\{P(R+r)/4\eta l\}$; therefore in this distance the velocity changes by—

$$P\frac{R+r}{4\eta l}\,.\,(R-r) = P\frac{R^2-r^2}{4\eta l}.$$

Since the velocity is equal to zero at the walls, the value just obtained gives the velocity at a distance r from the axis. At the axis, where $r=0$, the velocity is equal to $PR^2/4\eta l$.

A clear idea of the way in which the velocity varies over the cross-section of the tube can be obtained from Fig. 228. The particles which simultaneously cross the diameter AOB of the tube at a given instant, will be distributed along the parabola ACB at the end of a short time t; while at the ends of the times $2t$, $3t$, $4t$, they will be distributed along the parabolas ADB, AEB, and AFB, where OD = 2 × OC, OE = 3 × OC, etc.

It now becomes necessary to determine the volume V of liquid which crosses any cross-section of the tube in a second; this quantity, of course, gives the volume of liquid which enters one end of the tube and leaves the other end in a second.

Let two cylinders, of nearly equal radii r_a and r_b, be described coaxially with the tube. These cylinders cut off an annular strip, of area $\pi(r_b^2 - r_a^2)$, from any cross-section of the tube; and the velocity of the liquid which crosses this strip is approximately equal to

$$\frac{P}{4\eta l}\left(R^2 - \frac{r_b^2 + r_a^2}{2}\right).$$

Hence (p. 378), the volume of liquid which crosses the annular strip in a second is equal to

$$\frac{P}{4\eta l}\left(R^2 - \frac{r_b^2 + r_a^2}{2}\right)\pi(r_b^2 - r_a^2)$$
$$= \frac{P\pi}{4\eta l}\left\{R^2(r_b^2 - r_a^2) - \frac{r_b^4 - r_a^4}{2}\right\}$$

Now describe a large number of coaxial cylinders of radii r_0, r_1, r_2. . . . . r_n, where $r_0 = 0$, and $r_n = R$. These cylinders divide any cross-section of the tube into annular strips, and the volume of liquid which crosses each of these strips in a second can be written down from the result just obtained. Hence, the total volume V of liquid which crosses the complete cross-section in a second is the sum of the following quantities :—

$$\frac{P\pi}{4\eta l}\left\{R^2(r_1^2 - r_0^2) - \frac{r_1^4 - r_0^4}{2}\right\};$$
$$\frac{P\pi}{4\eta l}\left\{R^2(r_2^2 - r_1^2) - \frac{r_2^4 - r_1^4}{2}\right\};$$
$$\frac{P\pi}{4\eta l}\left\{R^2(r_3^2 - r_2^2) - \frac{r_3^4 - r_2^4}{2}\right\};$$
$$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$
$$\frac{P\pi}{4\eta l}\left\{R^2(r_n^2 - r_{n-1}^2) - \frac{r_n^4 - r_{n-1}^4}{2}\right\};$$

The sum of these quantities is equal to—

$$\frac{P\pi}{4\eta l}\left\{R^2(r_n^2 - r_0^2) - \frac{r_n^4 - r_0^4}{2}\right\}$$
$$= \frac{P\pi}{4\eta l} \cdot \frac{R^4}{2}, \text{ since } r_n = R, \text{ and } r_0 = 0.$$
$$\therefore \quad V = \frac{P\pi R^4}{8\eta l}.$$

This result shows that the volume V of liquid which enters one end and leaves the other end of the tube in a second, is directly proportional to the difference of pressure P between the ends, and to the fourth power of the radius R of the tube; while it is inversely proportional to the coefficient of viscosity η of the liquid. The dependence of the flow on the fourth power of the radius of the tube was discovered originally by Poiseuille.

Determination of the coefficient of viscosity of a liquid.—The coefficient of viscosity of a liquid may be determined accurately by the aid of the apparatus represented diagrammatically in Fig. 229. The glass vessel AB has a narrow neck at C, and another at E; each of these necks is marked with a circular scratch. The neck C can be closed by means of a stopper attached to a glass rod D. Below the neck E the vessel is sealed on to a wide glass tube EF which is bent at right-angles at F. Into the open end of the tube EF a capillary tube GK is fitted.

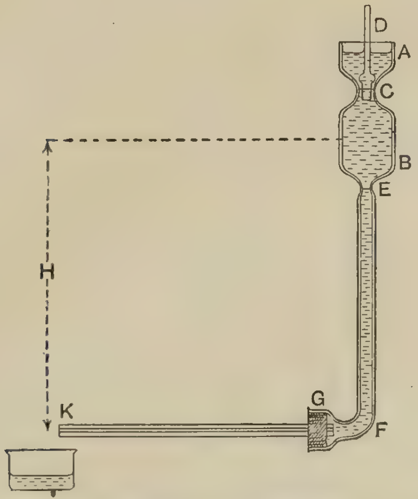


FIG. 229.—Experimental arrangement for the determination of the coefficient of viscosity of a liquid.

In carrying out an experiment, the liquid to be tested is poured into A and allowed to flow through the apparatus until the vessel AB, the wide tube EF, and the capillary tube GK are filled. The neck C is then closed by the stopper, and the apparatus is placed in a thermostat and left until the whole of the liquid has attained a constant temperature. The stopper is then removed, without opening the thermostat; and the time that elapses between the instant when the surface of the liquid passes the scratch in the neck C, and that when it passes the neck E, is observed. The volume between the scratches in the necks E and C has been determined experimentally, and this volume, divided by the time observed, gives the average volume V of liquid that flows in at one end and out at the other end of the capillary tube GK during one second. The average pressure during the experiment is equal to $g\rho$H, where ρ is the density of the liquid, and H is the mean of the heights of the scratches in the necks E and C above the end K of the capillary tube.

The capillary tube should be as uniform as possible in section; its uniformity can be tested by measuring the length of a short thread of mercury in various positions in the tube. Since the fourth power of the radius of the tube occurs in the expression for the coefficient of viscosity, care must be exercised in determining the value of the radius. The tube should be nearly filled with mercury, and the length of the thread of mercury should be measured accurately; the mercury should then be weighed, and its volume determined from the known density of mercury. The volume of the mercury divided by the length of the thread gives the average cross-sectional area πR^2 of the tube; and from this result the value of R^4 can be calculated.

Since the viscosity of a liquid varies rapidly with the temperature, the temperature of the thermostat must be observed with care. In the neighbourhood of 15° C., the coefficient of viscosity of water decreases by about 3 per cent. for each degree rise of temperature; hence, where accuracy is aimed at, a sensitive thermometer, which has been calibrated carefully, must be used.

Correction for loss of pressure due to gain of kinetic energy.—The difference of pressure between the ends of the capillary tube is not exactly equal to $g\rho$H, where H denotes the height of the free surface of the liquid in the vessel B (Fig. 229) above the horizontal plane in which the capillary tube lies. This is due to the circumstance that the velocity of the liquid increases as it approaches the entrance of the capillary tube, and therefore the pressure suffers a corresponding diminution.

Consider a tube of flow extending from the free surface of the liquid in the vessel B (Fig. 229) to the centre of the orifice of the capillary tube. At the free surface of the liquid the velocity is negligibly small: let it be denoted by v_s, and let a_s denote the area of the tube of flow at the surface. In the neighbourhood of the orifice, the tube of flow will taper rapidly; where it crosses the orifice, let its area be equal to a_c, and let the corresponding velocity be equal to v_c. At all points in this tube of flow, the velocity will be small, except in the immediate neighbourhood of the orifice; and since the tube extends to the centre of the orifice, the lateral velocity gradient at its boundaries will be negligibly small, even where the velocity is finite, just as at the centre of the tube. Hence, between the free surface and the centre of the orifice, there will be no appreciable loss of pressure due to viscosity. If p_s and p_c denote the respective pressures at the free surface and at the centre of the orifice of the capillary tube, it follows from the fundamental law of the flow of a liquid (p. 402) that—

$$a_c v_c(p_c + \tfrac{1}{2}\rho v_c^2) - a_s v_s(p_s + \tfrac{1}{2}\rho v_s^2) = g\mathrm{H} \,.\, \rho a_c v_c.$$

Divide through by $a_c v_c = a_s v_s$ (p. 378), and write $v_s = 0$. Then—

$$p_c - p_s = g\rho\mathrm{H} - \tfrac{1}{2}\rho v_c^2.$$

The quantity $(p_s + g\rho\mathrm{H})$ gives the pressure at the centre of the orifice if the flow were to cease; and we see that, owing to the flow, the pressure is diminished by $\rho v_c^2/2$.

Now, although the velocity has different values at different points of the orifice, the pressure must be constant at all points in the plane of the orifice (p. 492); that is, at all points in the plane of the orifice the pressure must be equal to p_c. At points near to the edge of the orifice, the velocity of the liquid is small, and therefore the loss of pressure due to gain of kinetic energy is small; but at these points the total loss of pressure must be equal to that at the centre of the orifice, and therefore an additional loss of pressure must be produced somehow, presumably by the viscosity of the liquid. Thus, at all points of the orifice the lower limit to the loss of pressure is equal to $(\rho v_c^2/2)$, where v_c is the velocity at the centre of the orifice.

From p. 493, it follows that—

$$v_c = \frac{\mathrm{PR}^2}{4\eta l},$$

and the average velocity over any cross-section of the tube is equal to—

$$\frac{V}{\pi R^2} = \frac{PR^2}{8\eta l} = \frac{v_c}{2};$$

$$\therefore \quad v_c = 2\frac{V}{\pi R^2};$$

and the corrected difference of pressure between the ends of the capillary tube is equal to—

$$g\rho H - \tfrac{1}{2}\rho v_c^2 = g\rho\left(H - \frac{2V^2}{g\pi^2 R^4}\right).$$

The generally accepted correction for the loss of head due to gain of kinetic energy is $V^2/g\pi^2R^4$; this correction is obtained by a train of reasoning which leads to the erroneous conclusion that the pressure varies over the transverse section of the capillary tube.

Simple apparatus for the determination of the coefficient of viscosity of a liquid.—The experimental determination of the coefficient of viscosity of a liquid, in the manner discussed above, is complicated by the necessity of using a thermostat. The arrangement now to be described suffices to give accurate results, while all avoidable complications are dispensed with.

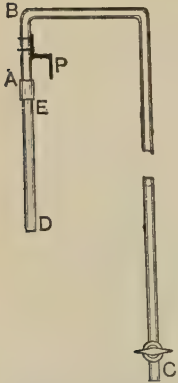


FIG. 230.—Detail of apparatus for the determination of the coefficient of viscosity of a liquid.

Expt. 55—To determine the coefficient of viscosity of a liquid at various temperatures.

A piece of glass tube, about 70 cm. in length and of about 0·5 cm. internal diameter, is bent into the form of a U (Fig. 230) of which the shorter limb AB is about 10 cm. long. A stop-cock with a wide passage is sealed on to the end C of the long limb of the U tube. DE represents a piece of uniform thermometer tube, about 10 cm. long and of about 0·5 mm. internal diameter, connected to AB by means of a piece of rubber tube. P represents a pin, bent twice at right-angles, and fixed to AB by means of wire: the use of this pin will be explained subsequently.

In carrying out an experiment, the U tube is held with the stop-cock at the top, and is filled with some of the liquid to be tested; this can be done by inserting the drawn-out end of a piece of wide tube through the opened stop-cock, and using the tube as a funnel. When the U tube is full and the liquid commences to flow through the capillary tube DE, close the stop-cock and invert the U tube. Immerse the thermometer tube in some more of the liquid to be tested, contained in a boiling tube, the boiling tube being surrounded by water contained in a beaker, and the beaker being supported on a sand-bath (Fig. 231). A thermometer dips into the liquid contained in the boiling tube and indicates its temperature. By adjusting the height of a Bunsen flame beneath the sand bath, the temperature of the liquid can be raised to any desired value, and kept constant at that value. The stop-cock is then opened, and the liquid begins to syphon over from the boiling tube. At the instant when the surface of the liquid in the boiling tube passes the point of the bent pin P, a measuring glass or flask is placed under the stop-cock, and the time is observed. The boiling tube is raised as the liquid flows out of it, so that the surface of the contained liquid always touches the point of the bent pin. When a sufficient quantity of liquid has syphoned over, the time is again observed; the volume of the liquid that has syphoned over may be observed directly if a measuring glass has been used to catch it, or a more accurate determination may be made by weighing the liquid. Thus we obtain the volume of liquid that has flowed through the capillary tube in an observed time. The difference of pressure between the ends of the capillary tube is equal to $g\rho H$, where H is the vertical distance between the point of the bent pin and the plane which passes through the outlet of the stop-cock, and ρ is the density of the liquid.

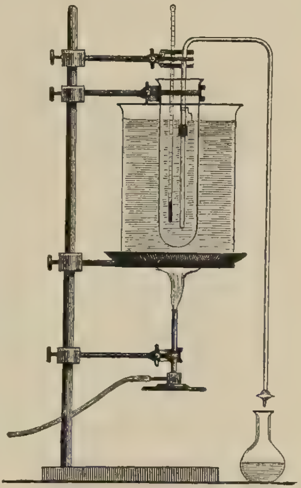


FIG. 231.—Experimental arrangement for the determination of the coefficient of viscosity of a liquid. (Due to Prof. Osborne Reynolds.)

After closing the stop-cock, and pouring the liquid that has syphoned over back into the boiling tube, the temperature of the liquid can be raised, and another determination made; and so on.

Viscosity of liquids at various temperatures.—The curve in Fig. 232 shows the value of the viscosity of water for various temperatures between 0°C. and 100°C. ; it has been plotted from the experimental results obtained by Thorpe and Rodger. It appears that the viscosity at any temperature t°C. can be represented by the empirical formula—

$$\eta = \frac{0{\cdot}017944}{(1 + 0{\cdot}023121t)^{1{\cdot}5423}}.$$

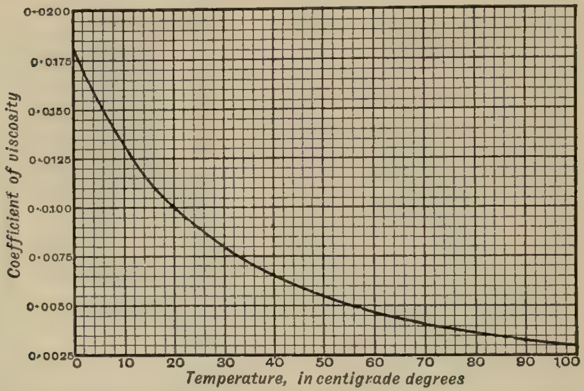


Fig. 232.—Coefficient of viscosity of water at temperatures between 0°C. and 100°C.

Thorpe and Rodger found that the coefficient of viscosity of any liquid can be represented by a formula of the type—

$$\eta = C/(1 + bt)^n.$$

The values of the constants C, b and n, for a number of liquids are given in the following table—

Liquid	C	b	n
Bromine	0·012535	0·008935	1·4077
Chloroform	0·007006	0·006316	1·8196
Carbon tetrachloride	0·013466	0·010521	1·7121
Carbon bisulphide	0·004294	0·005021	1·6328
Ethyl ether	0·002864	0·007332	1·4644
Benzene	0·009055	0·011963	1·5554
Methyl alcohol	0·008083	0·006100	2·6793
Ethyl alcohol	0·017753	0·004770	4·3731

According to Umani, the coefficient of viscosity of mercury is equal to 0·01577 at 10°C.

When a substance is dissolved in a liquid, the viscosity of the solution may be either greater or less than that of the pure liquid; no general laws have been discovered that would enable us to predict the value of the viscosity of a solution or a mixture of liquids.

Determination of the viscosity of a fluid by the "revolving cylinder" method.—Let ABC and DEF (Fig. 233) represent the transverse sections of two hollow cylinders, the common axis of both being a line through O perpendicular to the plane of the paper. Let the space between the cylinders be filled with a viscous fluid, and let the outer cylinder ABC be rotated about its axis with a constant angular velocity ω, while the inner cylinder DEF is prevented from rotating. Then, since there is no slip between the outer cylinder and the fluid immediately in contact with it, the latter will be revolving about O with the angular velocity ω; and since there is no abrupt change of velocity between contiguous particles of fluid, each particle of the fluid must be revolving about O, except the particles immediately in contact with the stationary cylinder DEF. The angular velocity of rotation of the fluid must therefore fall off from ω at the outer cylinder to zero at the inner cylinder.

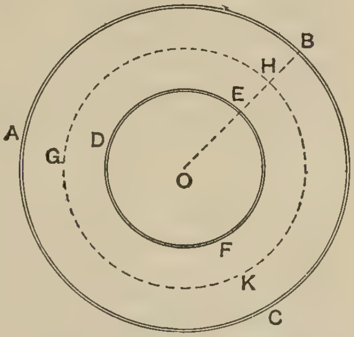


FIG. 233.—The "revolving cylinder" method of determining the coefficient of viscosity of a fluid. (The space between the two cylinders is exaggerated considerably.)

Let an imaginary cylinder, of which the transverse section is GHK, be described about the axis through O; then the velocities of particles on opposite sides of this surface will differ, and there will be a velocity gradient perpendicular to the surface; let this velocity gradient be denoted by v'. Then, if r is the radius, and l the length of the cylindrical surface, its area is equal to $2\pi rl$, and each unit of area will be acted upon by a tangential stress equal to $\eta v'$, where η denotes the coefficient of viscosity of the fluid. The tangential stresses tend to accelerate the velocity of rotation of the fluid between DEF and GHK, and the torque exerted about the axis through O is equal to $\eta v' \times 2\pi r l \times r = 2\pi r^2 l \eta v'$. Since the fluid between DEF and GHK is in steady motion, the forces acting on it must be in equilibrium; hence, the cylinder DEF must exert a torque $\tau = 2\pi r^2 l \eta v'$ tending to retard the rotation of the fluid in contact with it; and an equal but opposite torque must be exerted on the stationary cylinder. If the cylinder DEF is supported by a wire, the torque will twist the wire until the restoring torque is equal to τ. The value of τ_1, the torque per unit twist of the wire, can be determined in the manner explained on p. 112; if the observed twist is denoted by θ, the value of τ is determined from the equation—

$$\tau = \tau_1 \theta.$$

In determining the value of the velocity gradient v', it must be remembered that there would be no relative displacement of neighbouring particles of the fluid, if the whole of the fluid were rotating like a solid, that is, with constant angular velocity. Draw the radius OB intersecting the circle GHK in H; and on this radius choose two adjacent points equidistant from and on opposite sides of the point H. Let these points be at distances r_a and r_b from O, and let the corresponding angular velocities of rotation be equal to ω_a and ω_b. The actual difference between the velocities at these points is equal to $\omega_b r_b - \omega_a r_a$.

Now—

$$\omega_b r_b - \omega_a r_a = \tfrac{1}{2}(\omega_b + \omega_a)(r_b - r_a) + \tfrac{1}{2}(\omega_b - \omega_a)(r_b + r_a).$$

The first term on the right-hand side of this equation gives the difference in the velocities at the two points, due to a uniform rotation about O with the angular velocity $(\omega_b + \omega_a)/2$; this difference of velocities would produce no viscous stress in the fluid. The second term gives the difference in the velocities which produces the viscous stress; and since $(r_b + r_a)/2 = r$, it follows that

$$v' = r(\omega_b - \omega_a)/(r_b - r_a),$$

and

$$\tau = 2\pi r^3 l \eta \,.\, \frac{(\omega_b - \omega_a)}{r_b - r_a};$$

$$\therefore \quad \frac{2\pi l \eta}{\tau}(\omega_b - \omega_a) = \frac{r_b - r_a}{r^3}.$$

Multiply the numerator of the fraction on the right-hand side of this equation by $(r_b+r_a)/2$, and multiply the denominator by the equal quantity r; then

$$\frac{2\pi l\eta}{\tau}\cdot(\omega_b-\omega_a)=\frac{1}{2}\,\frac{r_b^2-r_a^2}{r^4}.$$

Now let r_b differ only infinitesimally from r_a. In this case we may write $r^2=r_br_a$ (pp. 48 and 193), and—

$$\frac{4\pi l\eta}{\tau}\cdot(\omega_b-\omega_a)=\frac{r_b^2-r_a^2}{r_b^2r_a^2}=\frac{1}{r_a^2}-\frac{1}{r_b^2}.$$

Now let the distance EB (Fig. 233) be divided into n infinitesimal elements of length by means of points at distances $r_\circ$, r_1, r_2, r_3 . . . r_n from O, where $r_\circ=\mathrm{OE}=\mathrm{R}_1$, and $r_n=\mathrm{OB}=\mathrm{R}_2$; and let the angular velocities at these points be equal to $\omega_\circ$, ω_1, ω_2, ω_3 . . . ω_n; where $\omega_\circ=0$, and $\omega_n=\omega$, the angular velocity of rotation of the cylinder ABC. Then we have—

$$\frac{4\pi l\eta}{\tau}(\omega_1-\omega_\circ)=\frac{1}{r_\circ^2}-\frac{1}{r_1^2}$$

$$\frac{4\pi l\eta}{\tau}(\omega_2-\omega_1)=\frac{1}{r_1^2}-\frac{1}{r_2^2}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad=\quad\cdot\quad\cdot\quad\cdot$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad=\quad\cdot\quad\cdot\quad\cdot$$

$$\frac{4\pi l\eta}{\tau}(\omega_n-\omega_{n-1})=\frac{1}{r^2_{n-1}}-\frac{1}{r_n^2}.$$

Adding these equations together, we obtain—

$$\frac{4\pi\eta l}{\tau}(\omega_n-\omega_\circ)=\frac{1}{r_\circ^2}-\frac{1}{r_n^2};$$

$$\therefore\quad 4\pi l\eta\omega=\tau\left(\frac{1}{\mathrm{R}_1^2}-\frac{1}{\mathrm{R}_2^2}\right).$$

Determination of the coefficient of viscosity of a fluid by the "revolving disc" method.—Let a circular disc of large diameter be mounted so that it can be rotated at a constant speed about a vertical axis through its centre and perpendicular to its plane; just above this, let another disc be supported by a fine wire, in such a manner that the two discs are parallel and the wire coincides with the axis of the lower disc (Fig. 234). If the space between the discs is filled with a fluid, the rotation of the lower disc tends to drag the fluid round, and thus to rotate the upper disc; as a consequence, the upper disc is twisted about its axis, until the torque due to the viscous drag of the fluid is just equal to the restoring torque τ called into play by the twist of the wire suspension. If θ denotes the angle through which the upper disc has been twisted, and τ_1 denotes the torque per unit twist of the wire (p. 108), then $\tau=\tau_1\theta$.

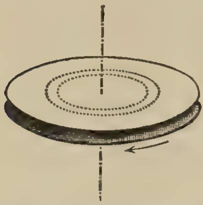


FIG. 234.—The "revolving disc" method of determining the coefficient of viscosity of a fluid.

Let the lower disc rotate with an angular velocity ω; then a point on it, at a distance r from the axis of rotation, is moving with a velocity equal to $r\omega$, and the fluid immediately in contact with the disc at this point must be moving with an equal velocity. The upper disc and the fluid immediately in contact with it are stationary. Thus, at a distance r from the axis of rotation, the velocity of the fluid varies from $r\omega$ at the surface of the lower disc to zero at the upper disc, and the velocity gradient is equal to $r\omega/d$, if the distance between the discs is equal to d. Hence, the tangential force acting on a small area a of the upper disc, at a mean distance r from the axis, is equal to $\eta a.r\omega/d$, and the corresponding torque about the axis is equal to $\eta ar^2\omega/d$.

Divide the surface of the upper disc into concentric annular strips by circles of radii $r_0, r_1, r_2, \ldots . r_n$. Then the area of the strip which lies between the circles of radii r_1 and r_0 is equal to $\pi(r_1^2-r_0^2)$, and for this strip the value of r^2 may be taken as equal to $(r_1^2+r_0^2)/2$, (p. 48), so that the torque about the axis, due to the viscous drag on the strip, is equal to—

$$\frac{\eta\omega}{d}\cdot\pi(r_1^2-r_0^2)\cdot\frac{(r_1^2+r_0^2)}{2}=\frac{\eta\omega\pi}{2d}(r_1^4-r_0^4).$$

The torques due to the viscous drag on the other strips can be written down in a similar form. The sum of the torques acting on all the strips is equal to—

$$\frac{\eta\omega\pi}{2d}\{(r_1^4-r_0^4)+(r_2^4-r_1^4)+\ldots\ldots+(r_n^4-r^4_{n-1})\}$$

$$=\frac{\eta\omega\pi}{2d}(r_n^4-r_0^4).$$

Let $r_n = \mathrm{R}$, the radius of the upper disc, and let $r_0 = 0$. Then—

$$\tau = \frac{\eta\omega\pi\mathrm{R}^4}{2d};$$

and since τ, R, and d can be measured, it follows that η can be determined.

In carrying out the above investigation, it has been assumed tacitly that the velocity gradient is equal to $r\omega/d$, right up to the edge of the disc; and that there is no viscous drag on the upper surface of the top disc. These requirements can be fulfilled by using the arrangement represented in Fig. 235. AB represents the lower disc; the upper disc CD nearly fills a circular hole in a larger plate EF, so that the flow of the fluid is almost identical with that which would be produced if CD and EF were one continuous plate; hence the velocity gradient is equal to $r\omega/d$, right up to the edge of CD. A shallow case GH rests on EF, and protects the upper surface of CD from disturbance.

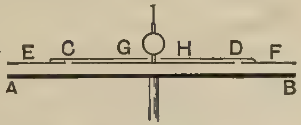


FIG. 235. — The "revolving disc" method of determining the coefficient of viscosity of a fluid.

The viscosity of air can be determined accurately by the arrangement represented diagrammatically in Fig. 235, if the suspended disc is about 40 cm. in diameter, and the distance d between the discs is equal to about 3 mm. The lower disc should be rotated at a rate of one turn in about two or three seconds.

Lubrication.—It is well known that the friction between surfaces in contact can be diminished greatly by lubricating them with oil; indeed, the fact is observed so frequently that it is liable to escape consideration, and to be classed among those phenomena which are reputed to need no explanation. Nevertheless, the way in which friction is diminished by lubrication presents many aspects which are of great scientific interest; and an attentive study of these reveals important principles which might otherwise escape recognition.

Before attention is devoted to the effects of lubrication, it may be well to consider briefly the laws of friction between dry surfaces. Experience shows that when two dry surfaces are in contact, the application of a force is needed in order to make one slide over the other; and when one surface is sliding over the other, a force must be applied in order to maintain the motion. In either case, the magnitude of the force that must be applied is proportional to the force which presses the surfaces against each other; but the force necessary to maintain the sliding motion is, in general, somewhat smaller than that required to initiate it. The magnitude of the applied force is independent of the area of contact; and when the sliding motion has commenced, the force required to maintain it is independent of the velocity, unless this is so great that the surfaces become heated. The **coefficient of friction** is the ratio of the force that must be applied in order to initiate or to maintain the motion, to the force that presses the surfaces against each other. The value of the coefficient of friction depends on the substances whose surfaces are in contact.

When the surfaces are well lubricated, the force required to maintain the sliding motion is approximately independent of the force that presses the surfaces against each other; it is independent of the substances whose surfaces are sliding one over the other; it varies directly as the velocity of relative motion of the surfaces; and it depends on the nature of the lubricant, and on the temperature. Thus we are led to infer that the lubricated surfaces do not come into contact, but that they are separated from each other by a layer of the lubricant; and that the necessity of applying a force in order to keep the surfaces in relative motion is due to the viscosity of the lubricant. Let a be the area of either of the surfaces, while d is the distance between the two, and v is the velocity of relative motion; then the force required to keep one sliding over the other is equal to $a(v/d)\eta$, where η is the coefficient of viscosity of the lubricant.

At this point two difficulties arise. In the first place, why is it that the surfaces are not in contact, even when they are pressed together by a considerable force? The surfaces come into contact when they are stationary, however well they may be lubricated; why is it, then, that the surfaces become separated when they are in relative motion? Again, experience shows that the lubricant must have a high coefficient of viscosity if the friction between the surfaces is to be reduced as much as possible; on the other hand, it appears, from the formula obtained above, as if the frictional force could be made small by using a lubricant with a small coefficient of viscosity.

These difficulties will be found to vanish when the action of the lubricant is studied; at present, it is enough to notice that the friction depends not only on η, but also on d: hence, we infer that the high value of η is necessary in order to prevent d from becoming infinitely small, that is, to prevent the surfaces from coming into contact.

Fugitive elasticity of a viscous fluid.—Let AB and CD (Fig. 236) represent two plane and parallel surfaces, AB being stationary while CD is moving with constant velocity from right to left; and let the space between the surfaces be filled with a viscous fluid. Let the dotted lines represent the traces of planes perpendicular to the surfaces and to the plane of the paper. At a given instant, let the particles of the viscous fluid which are passing through these planes be marked in some way. After a short interval of time, the marked particles will be passing through planes of which the traces are represented by the unbroken straight lines. Let the lower surface be moving with the constant velocity v, and let d denote the distances between the surfaces; then the viscous drag on unit area of either surface is equal to $\eta v/d$.

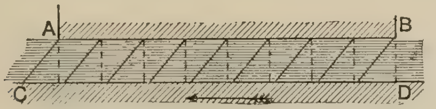
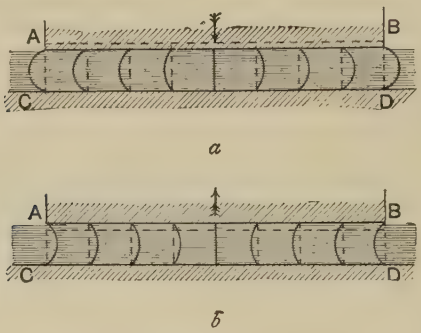


FIG. 236.—Fugitive elasticity of a viscous fluid.

Now, if the space between the surfaces had been occupied by a medium endowed with shear elasticity, and the lower surface had been displaced tangentially so as to produce in the medium a shearing strain equal to ϕ, the tangential stress acting on unit area of either surface would have been equal to $n\phi$, where n denotes the coefficient of shear elasticity of the medium (p. 228). If the shearing strain increased from zero to ϕ in the time τ, and the elasticity then broke down, the average shearing stress during the time τ would have been equal to $n\phi/2$; and if the medium then recovered its shear elasticity, the shearing stress during the next interval of time τ would have had an equal value. Hence, if the lower surface were moving with a uniform velocity v, and the space between the surfaces were filled with a medium endowed with a fugitive shear elasticity which broke down at intervals of time each equal to τ, the average tangential stress which would act on either surface would be equal to $n\phi/2$, where $\phi = v\tau/d$. Thus the average tangential stress would be equal to $nv\tau/2d$. On comparing this value with that of the drag exerted by the viscous medium on unit area of either surface, it becomes apparent that a medium endowed with fugitive shear elasticity would possess the properties of a viscous medium, if $n\tau/2 = \eta$. Hence, it may be stated that **the properties of a viscous medium may be accounted for, by supposing the medium to possess a fugitive shear elasticity which breaks down at very short intervals of time.**

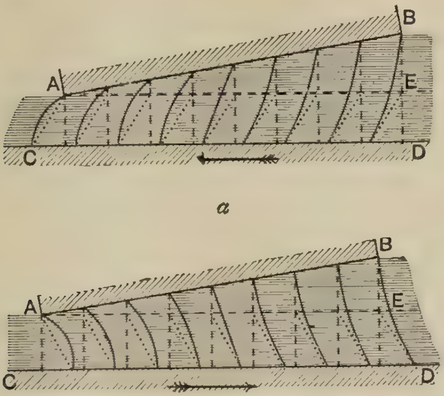


a

b

FIG. 237.—Fugitive elasticity of a viscous fluid.

Now let two plane and parallel surfaces be separated by an incompressible medium endowed with shear elasticity. If one surface approaches the other through a small distance, some of the medium must be extruded from the intervening space ; and if there is no slip between the medium and the surfaces, the flow of the medium will be greatest midway between the surfaces. If planes, perpendicular to the plane of the paper, are drawn through the dotted lines in Fig. 237, *a*, then the particles of the medium that lay in these planes before the flow commenced, will be carried by the flow of the medium to the curved surfaces of which the sections are indicated by the continuous curved lines. Thus, a filament of the medium that originally lay along one of the straight dotted lines, will be stretched by the flow so that it lies along the corresponding continuous line ; and this stretching of each filament will produce a tensile stress[1] which tends to make the filament recover its original position and length. The medium on the concave side of each curved surface will be at a higher pressure than the medium on the convex side, since each curved surface may be considered to represent the section of a stretched elastic membrane ; and the pressure will be greatest midway between the lateral edges of the space between the surfaces. This pressure will tend to stop the relative approach of the surfaces.



a

b

FIG. 238.—Fugitive elasticity of a viscous fluid.

If the medium between the plane surfaces (Fig. 237, *a*) is merely viscous, and one surface continuously approaches the other, the continual extrusion of the medium from the space between the surfaces will evoke the fugitive shear elasticity of the medium, and a pressure tending to stop the relative approach of the surfaces will be produced.

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[1] From the investigation carried out on p. 232 it appears that, in an incompressible medium ($k=\infty$), the value of Young's modulus E is equal to $3n$, where n is the coefficient of shear elasticity.

If one of the plane surfaces, instead of approaching, recedes from the other, the medium must flow into the space between the surfaces, and the particles that lay in the straight dotted lines (Fig. 237, *b*) before the flow commenced will be carried by the flow to the continuous curved lines, which are concave outwards. The tension along these curved lines produces a negative pressure between the surfaces; that is, a stress which tends to stop the relative recession of the surfaces.

Now let AB and CD (Fig. 238, *a*) represent two plane surfaces inclined to each other at a small angle, and let the space between the surfaces be filled with an incompressible medium endowed with shear elasticity. Let planes perpendicular to the plane of the paper be drawn through the dotted lines normal to CD; and let it be required to determine the displacement of the particles in these planes, due to a small tangental motion, towards the left, of the lower surface CD.

Through the edge A of the upper surface, describe an imaginary plane AE parallel to CD; and let it be supposed that the medium between AE and AB is perfectly rigid while the surface CD is being displaced. Then the medium between AE and CD is subjected to a pure shear by the tangential displacement of CD, and the particles of the medium which previously lay in the dotted lines which extend perpendicularly from AE to CD, are carried to the dotted lines that slope downwards from right to left. Now let the medium between AE and AB recover its elasticity; the tangential force exerted on the plane AE, from right to left, causes a displacement of that plane from right to left, and the straight dotted lines, which meet at an angle on the line AE, give place to the continuous curved lines which join AB and CD. It is obvious that towards the right-hand side of the diagram, the curved lines must be convex towards the right, and must lie to the left of the straight dotted lines which meet at an angle on AE. From this it follows that, on the right-hand side of the diagram, the medium flows into the space between the surfaces; and, since the volume of the space between the surfaces does not change, the medium must flow outwards at the left-hand side of the diagram. Therefore the dotted lines towards the left-hand side of the diagram give place to continuous curved lines which are convex towards the left. Hence, proceeding from the right to the left-hand side of the diagram, the curved lines are at first convex towards the right, but the convexity diminishes as we proceed, and one particular line is straight; on proceeding further towards the left-hand side, the continuous lines are convex towards the left. The continuous curved lines represent filaments of the medium in a state of tension, and we infer that the pressure of the medium increases from the edges inwards, until the straight continuous line is reached; and this pressure tends to force the surfaces apart.

If the space between AB and CD is filled with a medium which is merely viscous, then a continuous displacement of the surface CD, from right to left, will evoke the fugitive shear elasticity of the medium, and a pressure tending to separate the surfaces will be produced. It is this pressure which prevents lubricated surfaces from coming into contact, so long as the surfaces are in relative motion. It must be noticed, however, that this pressure can be produced only when the relative motion of the surfaces tends to carry the viscous medium from the wide to the narrow part of the intervening space. Thus, a motion of CD from right to left, or a motion of AB from left to right, will produce the pressure. On the other hand, the motion of CD from left to right (Fig. 238, *b*) causes the displacement-lines of the medium to be concave outwards, so that a negative pressure is produced in the medium, and the surfaces AB and CD are drawn towards each other. The way in which the displacement-lines are obtained in Fig. 238, *b*, will be evident without further explanation.

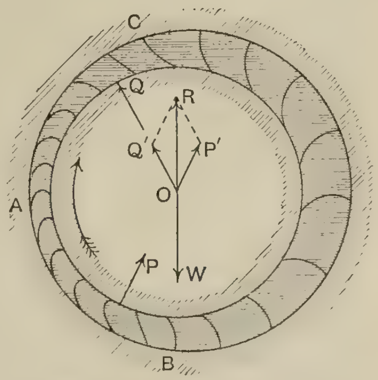

FIG. 239.—Shaft rotating in a well lubricated bearing.

The effect of lubrication, on a shaft which rotates in a cylindrical bearing, can now be understood. Let the shaft rotate in the direction indicated by the arrow (Fig. 239). At first sight it would appear probable that the space between the shaft and the bearing would be narrowest immediately beneath the shaft ; but it will appear presently that the narrowest space must lie at some point near to A, where A is on that side of the shaft to which the lubrication is carried from below. The displacement-lines in the oil are shown in Fig. 239, and it becomes evident that the oil between B and A exerts a pressure P on the shaft, while the oil between A and C exerts a negative pressure Q. From the centre of the shaft, draw OP′ parallel and equal to the positive pressure P, and OQ′ parallel and equal to the negative pressure Q ; then the resultant OR of OP′ and OQ must be equal to the weight OW of the shaft. If the narrowest section were not somewhere near to A, it would be impossible for the weight of the shaft to be supported by the oil.

The magnitude of the pressure which may be produced in the oil between the points A and B is worthy of note. Mr. Beauchamp Tower measured this pressure in a particular instance, and found that it was equal to 625 lb. per sq. in., or about 40 atmospheres. Of course, the pressure depends on the speed at which the shaft is driven ; in machinery driven at high speeds, the oil must be forced into the bearings under high pressure. Beauchamp Tower found that when a shaft was pressed downwards by a considerable force f, and the speed of rotation was slow, the frictional force that had to be overcome in driving the shaft was as large as $f/3$; in this case the shaft came into contact with the bearing. On increasing the speed of rotation to a sufficient extent, it was found that the frictional force fell to the much smaller value $f/400$, showing that fluid friction had been substituted for the friction between solid surfaces.

"Ducks and drakes."—It is well known that a stone will rebound from the surface of water, if its component velocity parallel to the surface of the water is great in comparison with its component velocity normal to the surface. A projectile fired from a cannon will rebound from the surface of water under similar conditions. These interesting phenomena can be explained[1] in terms of the fugitive elasticity of a viscous liquid.

In the first place, a body never rebounds when it falls normally on the surface of water. It will be seen from Fig. 237 that the fugitive elasticity of the water will produce an upward force when the body is sinking into the water, but this force will vanish when the body ceases to sink, and would be replaced by a downward force if the body commenced to rise. When the body possesses a considerable velocity tangential to the surface of the water, the case is different. Let AB (Fig. 236, p. 507) represent a flat body moving from left to right along the surface of water. The surface layer of the water suffers a shearing strain, while the lower layers of the water (represented by CD) remain at rest. In this case, the body is subjected merely to a viscous drag which tends to stop its horizontal motion. If, however, the front edge of the body tilts upwards, the strain in the surface layer of the water will resemble that represented in Fig. 238, *a*, (p. 509), and an upward force will be evoked which may suffice to project the body upwards. If the rear edge of the body were to tilt upwards, the strain in the surface layer of the water would resemble that represented in Fig. 238, *b*, and a force would be evoked which would pull the rear edge of the body downwards, and so destroy the tilt. Thus it appears that the tangential motion of the body, with its flat surface parallel to the surface of the water, is unstable; a small upward tilt of the front edge of the body suffices to produce a force which projects the body upwards out of the water.

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[1] An explanation of "ducks and drakes," bearing a general resemblance to that given in the text, was published by Prof. W. Sutherland in the *Philosophical Magazine* for 1896, pp. 111–115.

Orderly flow rendered possible by viscosity.—Some idea may now be formed as to the way in which viscosity tends to prevent the turbulent motion of a fluid. Let the horizontal straight lines in Fig. 240 represent lines of flow in a fluid, and let the corresponding tubes of flow have square sections perpendicular to the plane of the diagram. Consider the motions of the three cubical masses of liquid represented by A, B, and C. As A moves from right to left, let some accidental circumstance cause it to expand laterally so as to acquire the shape indicated by A′. Then B and C must contract laterally so as to acquire the shapes indicated by B′ and C′, and it is clear that the centre of gravity of B′ or C′ has advanced farther than that of A′. In fact, if A′ has expanded laterally its forward velocity must have diminished, and if B′ and C′ have contracted laterally, their forward velocities must have increased. Since the pressure diminishes as the velocity increases, it follows that the pressure within A′ is greater than that in B′ and C′, and the difference

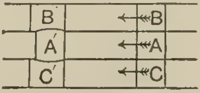

FIG. 240.—Orderly flow rendered possible by viscosity.

of pressure between A′ and B′ or C′ tends to make A′ expand still further. In the absence of viscosity, no force is called into play which tends to stop the lateral expansion of A′, and therefore the expansion continues and the motion along stream-lines breaks up. But if the fluid is viscous, the first tendency to expand is opposed by the fugitive elasticity of the fluid ; and the lateral expansion of A′ is opposed, just as if its lateral walls were elastic membranes.

Flow of a gas through a narrow tube.—When a liquid flows through a tube, the product of the velocity at any point and the cross-sectional area of the tube at that point must be constant (p. 378), since a liquid is practically incompressible. This law does not hold for a gas ; when a gas flows through a tube, the condition to be satisfied is that the mass of gas crossing a section of the tube is constant for all sections, and therefore (p. 379)—

$$\rho a \mathrm{V} = \text{a constant},$$

where ρ denotes the density at the section of area a, and V is the velocity of flow across that section. If the tube is uniform throughout its length, a is constant, and therefore $\rho\mathrm{V}$ must be constant.

The flow of a liquid through a tube has been investigated already (p. 494); the conditions under which a gas flows through a uniform narrow tube must now be determined.

Let it be assumed that the flow of the gas is isothermal ; that is, that the heat generated by the flow of the gas is conducted away immediately through the walls of the tube. At the ends of a very small element of length of the tube, let the pressures be equal to p_a and p_b, where p_a and p_b differ only infinitesimally ; and let the ends of the element be at distances x_a and x_b from the end of the tube at which the gas enters. Then the density of the gas may be considered to be constant within the element, and if a volume V_a of gas flows through the element in a second, the formula obtained on p. 494, for the flow of a fluid of constant density, may be applied. Thus—

$$\mathrm{V}_a = \frac{(p_a - p_b)\pi r^4}{8\eta(x_b - x_a)}.$$

Experiment shows that **the coefficient of viscosity of a gas is independent of the pressure** ; this very remarkable result will be discussed subsequently.

Let ρ denote the density of the gas under unit pressure; and let it be assumed that the gas obeys Boyle's law that the density is proportional to the pressure. Then, within the element under consideration, the average pressure is equal to $(p_a+p_b)/2$, and the density of the gas is equal to $\rho(p_a+p_b)/2$; therefore the mass of gas that flows through the element in a second is equal to $V_a\rho(p_a+p_b)/2$. Let the pressure at the end of the tube where the gas enters be equal to P, and let a volume V of gas enter the tube per second; then the mass of gas entering per second is equal to ρPV, and an equal mass of gas must cross each section of the tube per second. Thus—

$$\rho PV = \rho \cdot \frac{p_a+p_b}{2} \cdot V_a = \rho \frac{(p_a^2-p_b^2)\pi r^4}{16\eta(x_b-x_a)};$$

$$\therefore \quad PV = \frac{(p_a^2-p_b^2)\pi r^4}{16\eta(x_b-x_a)}.$$

Let the tube be divided into elements by cross-sections at distances $x_0, x_1, x_2, \ldots . x_n$ from the end of the tube where the gas enters, and let the pressures at these sections be equal to $p_0, p_1, p_2, \ldots . p_n$. Then

$$PV(x_1-x_0) = \frac{\pi r^4}{16\eta}(p_0^2-p_1^2)$$

$$PV(x_2-x_1) = \frac{\pi r^4}{16\eta}(p_1^2-p_2^2)$$

$$\ldots\ldots = \ldots\ldots$$

$$PV(x_n-x_{n-1}) = \frac{\pi r^4}{16\eta}(p_{n-1}^2-p_n^2).$$

Adding these equations together, we obtain—

$$PV(x_n-x_0) = \frac{\pi r^4}{16\eta}(p_0^2-p_n^2).$$

Now, let $x_0=0$; then $p_0=P$. Let $x_n=l$, where l denotes the length of the tube; and let p_n, the pressure of the gas when it leaves the tube, be denoted by p. Then

$$PV = \frac{(P^2-p^2)\pi r^4}{16\eta l}.$$

This equation suffices to determine η in terms of P, p, V, r, and l.

Simple arrangement for the determination of η for hydrogen or oxygen.

Expt. 56.—To determine the coefficient of viscosity of hydrogen or oxygen.

Let hydrogen, generated in an ordinary water voltameter (Fig. 241) be led through a drying tube D containing pumice soaked in strong sulphuric acid, then through a glass tube to which a manometer M containing Fleuss oil is fused, and finally into the atmosphere through a piece of thermometer tube AB, of which the internal diameter is equal to about 0·4 mm. The hydrogen can be evolved by sending a current through the voltameter from a few small secondary cells; the current is measured by an ammeter or a tangent galvanometer G. The oxygen must be allowed to escape freely into the atmosphere. The dilute sulphuric acid in the limb of the voltameter in which the hydrogen is generated will be below the level of the dilute acid in the open limbs of the tube; thus the hydrogen is generated under a pressure which suffices to force it through the capillary tube AB.

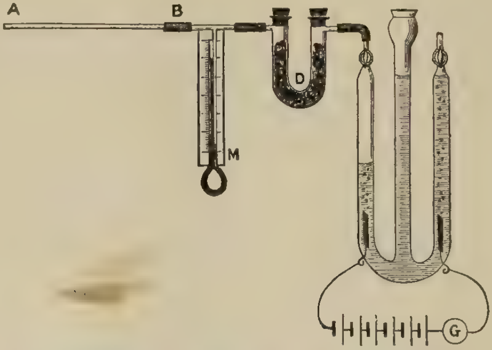

FIG. 241.—Simple arrangement for the determination of the coefficient of viscosity of hydrogen. (Due to Dr. Lehfeldt.)

To determine the pressure P at which the hydrogen enters the capillary tube, the pressure indicated by the manometer M must be added to the barometric pressure, both expressed in dynes per sq. cm.; the pressure p at which the hydrogen leaves the capillary tube is the atmospheric pressure. The value of the electric current being known, the mass of hydrogen generated in a second can be calculated; one ampere liberates 0·00001036 gram of hydrogen per second, and one gram of hydrogen at 0°C., and under a pressure equal to 76 cm. of mercury, occupies a volume of 11,120 c.c. Thus, if the electric current is steady and equal to C amperes, and the absolute temperature of the gas as it flows through the capillary tube is equal to T, the volume V of hydrogen that enters the capillary tube in a second is given by the equation

$$V=(P'/P)C \times 0{\cdot}00001036 \times 11{,}120 \times (T/273),$$

where P′ denotes the standard atmospheric pressure (equivalent to 76 cm. of mercury), expressed in dynes per sq. cm. Therefore—

$$PV=P'C \times 0{\cdot}00001036 \times 11{,}120 \times (T/273).$$

The value of P′ may be taken as 10^6 dynes per sq. cm.

Simple arrangement for the determination of η for atmospheric air.

EXPT. 57.—To determine the coefficient of viscosity of atmospheric air.

A piece of glass tubing ABC (Fig. 242) is bent into the form of a U, of which the longer limb AB is from 30 to 40 cm. in length, while the shorter limb is from 10 to 15 cm. in length. The internal diameter of this tube must be about 3·5 mm. A piece of thermometer tubing DE, about 20 cm. in length and 0·15 mm. in internal diameter, is connected to the shorter limb of the U tube by means of a piece of rubber tubing CD; a pellet of mercury M, about 5 or 6 cm. in length is introduced into the limb of AB. The U tube is then placed with its limbs vertical and an observation is made of the time at which the lower surface of the pellet of mercury passes a mark near to the top of the limb AB; subsequently, the time is observed at which the lower surface of the pellet passes another mark about 20 cm. lower down on the limb AB. Some of the mercury is then removed from the limb AB and weighed, and the procedure outlined above is repeated with the smaller pellet which still remains in the limb AB. Observations should be made with five or six pellets of mercury of varying masses; it is best to diminish the mass of the pellet by removing some of the mercury, and not to introduce fresh mercury into the tube.

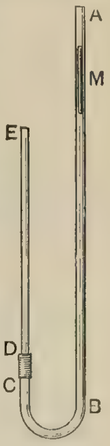

FIG. 242.—Simple arrangement for the determination of the coefficient of viscosity of atmospheric air. (Due to Dr. Rankine.)

Let the atmospheric pressure be equal to **P** gm. per sq. cm., and let the pressure due to the pellet of mercury M be equal to **p** gm. per sq. cm. The mercury sticks to some extent to the walls of the fall tube; let this diminish the pressure due to the pellet by δ gm. per sq. cm. Then, in the formula given on p. 515, $P=(\mathbf{P}+\mathbf{p}-\delta)g$ dynes per sq. cm. and $p=\mathbf{P}g$ dynes per sq. cm. Let the volume between the two marks in the fall tube be equal to Q c.c.; and let t be the time required for the lower surface of the pellet to pass from the upper to the lower of these marks; then $V = Q/t$ c.c. per sec. Thus—

$$(\mathbf{P} + \mathbf{p} - \delta)g\frac{Q}{t} = \frac{\{(\mathbf{P} + \mathbf{p} - \delta)^2 g^2 - \mathbf{P}^2 g^2\}\pi r^4}{16\eta l}$$

$$\therefore \quad Q = \frac{\left\{(\mathbf{P} + \mathbf{p} - \delta)\,g - \mathbf{P}\left(\dfrac{\mathbf{P}}{\mathbf{P} + \mathbf{p} - \delta}\right)g\right\}\pi r^4 t}{16\eta l}$$

Now $$\frac{\mathbf{P}}{\mathbf{P}+\mathbf{p}-\delta} = \frac{1}{1 + \dfrac{\mathbf{p}-\delta}{\mathbf{P}}} = 1 - \frac{\mathbf{p}-\delta}{\mathbf{P}} + \frac{(\mathbf{p}-\delta)^2}{\mathbf{P}^2} - \dots$$

where powers of the small quantity $(\mathbf{p} - \delta)/\mathbf{P}$ higher than the second are neglected. Thus, if k is written for $\pi r^4/8l$, we have

$$Q = \left\{(\mathbf{P}+\mathbf{p}-\delta)g - \left[\mathbf{P} - (\mathbf{p}-\delta) + \frac{(\mathbf{p}-\delta)^2}{\mathbf{P}}\right]g\right\}\frac{kt}{2\eta}$$

$$= \left\{2(\mathbf{p}-\delta) + \frac{(\mathbf{p}-\delta)^2}{\mathbf{P}}\right\}\frac{gkt}{2\eta}.$$

The value of δ is fairly small in comparison with $\mathbf{p}$, and therefore $\mathbf{p}^2/\mathbf{P}$ may be substituted for $(\mathbf{p} - \delta)^2/\mathbf{P}$, since this term as a whole is small. Thus, finally—

$$\eta = \left(\mathbf{p} - \frac{\mathbf{p}^2}{2\mathbf{P}} - \delta\right)t\,.\,\frac{gk}{Q}\,. \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Since η is constant, and gk/Q is also constant, it follows that $\left(\mathbf{p} - \dfrac{\mathbf{p}^2}{2\mathbf{P}} - \delta\right)t$ is constant. This enables us to determine the value of δ, from the results of observations of the times of descent of two pellets of mercury of different masses.

The following are the details of a determination of the value of η for atmospheric air by this method.

Room temperature during experiment = 17° C.

Volume Q between the two marks on the fall tube = 2·42 c.c. Sectional area of fall tube = 0·0990 sq. cm. The value of $\mathbf{p}$ is obtained by dividing the mass of the pellet of mercury by this area.

Barometric pressure $\mathbf{P} = 1050$ gm. per. sq. cm.

Value of $r^4 = 4{\cdot}56 \times 10^{-9}$ (cm.)^4. Length l of capillary tube = 20·15 cm.

The first column in the following table gives the observed values of the time of fall t for the pellets of mass m given in the second column. The third column gives the value of $\mathbf{p}$, obtained by dividing the mass m by the area (0·0990 sq. cm.) of the fall tube. The fourth column is obtained by calculation.

t (sec.)	m (gm.)	p gm./(cm.)^2	$\left(p - \frac{p^2}{2P}\right)$	$\left(p - \frac{p^2}{P} - \delta\right)t$
78·5	7·369	74·43	71·79	5475
89·2	6·474	65·39	63·36	5470
98·4	5·856	59·15	57·49	5456
123·4	4·701	47·48	46·41	5475
141·0	4·130	41·71	40·88	5476

The value of δ is obtained from the first and fourth horizontal rows of figures. Since $\left(p - \frac{p^2}{P} - \delta\right) \times t$ is constant, we have—

$$(71{\cdot}79 - \delta)\ 78{\cdot}5 = (46{\cdot}41 - \delta)\ 123{\cdot}4$$

$$\therefore \quad \delta = 2{\cdot}04 \text{ gm./(cm.)}^2.$$

The fifth column is obtained by calculation, using the value of δ just determined. The average value of $\left(p - \frac{p^2}{P} - \delta\right)t$ is found to be 5470. Then from (1), remembering that $k = \pi r^4/8l$,

$$\eta = \frac{5470 \times 981 \times 3{\cdot}14 \times 4{\cdot}56 \times 10^{-9}}{2{\cdot}42 \times 8 \times 20{\cdot}15} = 1{\cdot}97 \times 10^{-4} \text{ gm./cm. sec.}$$

Accurate determination of the coefficient of viscosity of a gas.—The accurate determination of the coefficient of viscosity is generally much more difficult in the case of a gas than in the case of a liquid. When only a small quantity of gas can be experimented with, the difficulty is still further increased. A simple arrangement, devised by Dr. A. O. Rankine, eliminates most of the difficulties that heretofore have confronted investigators; by its aid the coefficient of viscosity of a gas can be determined with comparative ease, and only a few cubic centimetres of gas need be used.

Rankine's apparatus is represented in Fig. 243. AB is a capillary tube, of which the internal diameter is equal to about 0·2 mm. A glass tube ADEB, with an internal diameter of about 3·5 mm., is bent into the shape of an elongated C, and the ends of this tube are fused to the ends of the capillary tube AB. The total length of the apparatus is about 70 cm. Stopcocks F and G are fused on to the wider tube, so that the interior of the tubes can be evacuated, or filled with any gas. A pellet of mercury M is introduced into the wide tube; when the apparatus is placed with AB and DE vertical, the mercury falls under the action of gravity, and the gas below M is forced through the capillary tube.

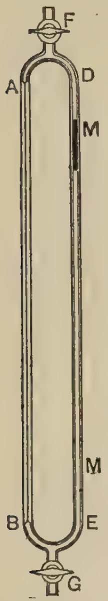

FIG. 243.—Rankine's apparatus for the determination of the coefficient of viscosity of a gas.

Let it be supposed that the apparatus, filled with the gas to be examined, is laid flat on a table. The pressure of the enclosed gas is now uniform throughout: let it be equal to p. Let the total volume of the enclosed gas be equal to v, and let ρ denote the value of the density of the gas if its pressure were equal to unity. Then the mass of the enclosed gas is equal to $\rho p v$, and this quantity must remain constant throughout the experiment.

Now let the apparatus be placed with the tubes AB and DE vertical; at a given instant let the volume of the gas, between the end A of the capillary tube and the upper surface of the mercury, be equal to v_1. The volume of the gas between the end B of the capillary tube and the lower surface of the mercury is equal to $(v-v_1)$, since the volume of gas within the capillary tube may be neglected. Let w denote the weight (in dynes) of the pellet of mercury, and let a be the cross-sectional area of the tube DE; then the pressure of the gas below the mercury exceeds that of the gas above the mercury by w/a. Let p_1 denote the pressure of the gas above the mercury. Then since the mass of gas is constant,

$$\rho p v = \rho p_1 v_1 + \rho\left(p_1 + \frac{w}{a}\right)(v-v_1)$$

$$= \rho\left(p_1 v + \frac{w}{a}(v-v_1)\right);$$

$$\therefore \quad p_1 v = v\left(p - \frac{w}{a}\right) + \frac{w}{a}v_1,$$

and

$$p_1 = p - \frac{w}{a} + \frac{wv_1}{av} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When it is remembered that p, w, a, and v are constants, while v_1 increases as the pellet of mercury descends, it becomes obvious that the pressure of the gas above the mercury increases as the pellet of mercury descends. Let P_1 denote the pressure of the gas below the mercury; then—

$$P_1 = p_1 + \frac{w}{a} = p + \frac{w}{a}\frac{v_1}{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

P_1 also increases as the pellet of mercury descends; but $P_1 - p_1$ remains constant.

Now let it be supposed that the pellet of mercury has descended to M′, (Fig. 243), the volume of gas between its upper surface and the end A of the capillary tube having increased to v_2; and let the pressure of this gas now be equal to p_2. Then—

$$p_2 = p - \frac{w}{a} + \frac{w}{a}\frac{v_2}{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If P_2 is now the pressure of the gas below the pellet of mercury,

$$P_2 = p + \frac{wv_2}{av} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When the pellet of mercury is at M, the mass of gas between its lower surface and the end B of the capillary tube is equal to $\rho P_1(v - v_1)$, and the corresponding mass when the pellet is at M′ is equal to $\rho P_2(v - v_2)$. Thus, the mass of gas which has been forced through the capillary tube, while the pellet descends from M to M_1, is equal to—

$$\begin{aligned} \rho\{P_1(v - v_1) - P_2(v - v_2)\} &= \rho\{(P_1 - P_2)v - (P_1v_1 - P_2v_2)\} \\ &= \rho\left\{\frac{w}{a}(v_1 - v_2) - p(v_1 - v_2) - \frac{w}{av}(v_1^2 - v_2^2)\right\}. \end{aligned}$$

Let the volumes v_1 and v_2 be chosen so that $v_1 + v_2 = v$; this condition will be complied with, if the two positions M and M′ of the pellet of mercury are symmetrical with respect to the middle of the tube ADEB, and the volume of gas between A and the upper surface of the pellet at M, is equal to the volume between B and the lower surface of the pellet at M′. In this case

$$\frac{w}{a}\frac{(v_1^2 - v_2^2)}{v} = \frac{w}{a}(v_1 - v_2) \, . \, \frac{v_1 + v_2}{v} = \frac{w}{a}(v_1 - v_2),$$

and the mass of gas forced through the capillary tube is equal to—

$$-\rho p(v_1 - v_2) = \rho p(v_2 - v_1).$$

If this mass of gas is forced through the capillary tube in the time T, the average rate at which the mass is forced through the tube is equal to $\rho p(v_2 - v_1)/\mathrm{T}$.

The formula obtained on p. 515 gives the value of the product of the pressure and volume of the gas forced through a capillary tube in a second under a steady difference of pressure between the ends of the tube. Let q denote the volume of gas forced through the tube in a small time t, when the gas enters the tube at a pressure P_1 and leaves it at a pressure p_1; then—

$$\mathrm{P}_1 \frac{q}{t} = \frac{(\mathrm{P}_1{}^2 - p_1{}^2)\pi r^4}{16\eta l}.$$

On multiplying this equation through by ρ, the density of the gas if its pressure were equal to unity, we obtain an expression for the mass of gas forced through the tube per second. Now—

$$\mathrm{P}_1{}^2 - p_1{}^2 = (\mathrm{P}_1 - p_1)(\mathrm{P}_1 + p_1).$$

And from (2) and (1),

$$\mathrm{P}_1 - p_1 = \frac{w}{a}$$

$$\mathrm{P}_1 + p_1 = 2p - \frac{w}{a} + 2\frac{w}{a}\frac{v_1}{v};$$

$$\therefore \quad (\mathrm{P}_1{}^2 - p_1{}^2) = \frac{w}{a}\left(2p - \frac{w}{a} + 2\frac{w}{a}\frac{v_1}{v}\right).$$

Similarly—

$$(\mathrm{P}_2{}^2 - p_2{}^2) = \frac{w}{a}\left(2p - \frac{w}{a} + 2\frac{w}{a}\frac{v_2}{v}\right).$$

These equations indicate that the mass of gas is forced through the capillary tube at a rate which increases uniformly with the distance through which the pellet of mercury descends. While the pellet of mercury descends from M to M′, the average rate at which the mass of gas is forced through the capillary tube is proportional to—

$$\{(\mathrm{P}_1{}^2 - p_1{}^2) + (\mathrm{P}_2{}^2 - p_2{}^2)\}/2;$$

that is, to—

$$\frac{w}{a}\left(2p - \frac{w}{a} + \frac{w}{a}\frac{v_1 + v_2}{v}\right)$$

$$= \frac{2wp}{a},$$

since $v_1 + v_2 = v$.

Thus, the average mass of gas forced through the capillary tube per second is equal to—

$$\rho \cdot \frac{\frac{2wp}{a} \cdot \pi r^4}{16\eta l} = \rho \cdot \frac{wp \cdot \pi r^4}{8a\eta l};$$

$$\therefore \quad \rho p \frac{v_2 - v_1}{\mathrm{T}} = \rho p \frac{w\pi \rho^4}{8a\eta l};$$

$$\eta = \frac{w\pi r^4 \mathrm{T}}{8al(v_2 - v_1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It may be noticed that the same formula would apply if the apparatus were filled with a liquid instead of a gas.

In carrying out an experiment with Rankine's apparatus, two marks are made on the tube DE (Fig. 243), in such positions that the volume between the end A of the capillary tube and one mark, is equal to the volume between the end B of the capillary tube and the other mark. The time is observed when the *upper* surface of the pellet of mercury passes the mark nearer to A, and again when the *lower* surface of the pellet passes the mark nearer to B; T is the interval of time that elapses between these observations. The quantity $(v_2 - v_1)$ is obtained by subtracting the volume of the pellet of mercury from the volume between the marks on the tube DE. The radius r of the capillary tube is measured in the manner described on p. 496.

The pellet of mercury sticks, to some extent, to the walls of the tube DE, and thus the pressure that it exerts is diminished; in other words, the effective weight of the mercury, instead of being w, is $(w - \delta)$. To determine the value of the correction δ, two experiments must be performed with pellets of mercury of different weights, w_1 and w_2. Let T_1 and T_2 be the times required for these pellets to traverse the distance between the same two marks, and let $(v_2 - v_1)$ and $(v_2' - v_1')$ denote the volumes swept out by the pellets in the two cases. Then from equation (5), since η, r, a, and l have identical values in the two experiments—

$$\frac{(w_1 - \delta)\mathrm{T}_1}{(v_2 - v_1)} = \frac{(w_2 - \delta)\mathrm{T}_2}{v_2' - v_1'}.$$

From this equation the value of δ can be calculated.

There is some evidence that the layer of gas immediately in contact with the walls of the tube is not absolutely at rest; in other words, there is a small amount of slip between the gas and the walls of the tube. To correct for this slip, the value of r used in (5) above should exceed the true radius of the capillary tube by the mean free path of the gas under the conditions that prevail during the experiment. The meaning of the mean free path will be explained in the next chapter.

Rankine's apparatus can be used to prove, in a very conclusive manner, that the coefficient of viscosity is independent of the density of a gas. It will be found that the value of T is not affected by pumping one-half or two-thirds of the gas out of the apparatus, provided that the original pressure was comparable with the atmospheric pressure. Formula (5) above cannot be used, however, when the gas enclosed by the apparatus is very rarefied; *i.e.* when the mean pressure is comparable with that due to the pellet of mercury.

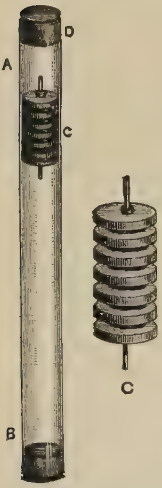

Fig. 244.—Apparatus for the determination of the coefficient of viscosity of a gas.

Modified form of Rankine's apparatus, for use by students.—Rankine's apparatus is not well adapted to the use of students, since the tubes must be cleaned and calibrated, once for all, before they are fused together. The apparatus now to be described has many of the advantages of that due to Rankine, and in addition it can be taken to pieces readily, so that the capillary tube may be cleaned and calibrated.

AB (Fig. 244) is a piece of glass tube, about 12 to 14 inches in length, and about 1·25 inches in diameter; this tube must be selected carefully, since it must be straight, circular in section, and uniform in diameter throughout its length. C is a piston made of ebonite, which slides freely in the tube AB; it is about two inches long, and has five or six deep circular grooves turned in its curved surface. The grooves are filled with mercury, introduced through a small hole D near to one end of the glass tube; the rings of mercury prevent air from passing between the piston and the walls of the tube AB, while the piston can slide along AB without friction.

A piece of capillary tube about three inches long passes axially through the piston, and is held in position by stuffing boxes. The ends of the tube AB are closed by rubber stoppers, one of which also closes the small hole D. On placing the tube AB with its axis vertical, the piston slowly sinks, the gas contained in the tube being forced from the lower to the upper side of the piston through the capillary tube. The formula for obtaining η is identical with that obtained above in connection with Rankine's apparatus. Two marks must be made on the tube AB, the distance between one and the lower surface of the stopper that closes the end A of the tube being equal to the distance between the other and the upper surface of the stopper that closes the end B of the tube; T is the time that elapses between the passage of the upper surface of the piston across one mark and the passage of the lower surface of the piston across the other mark.

If the lower surface of the piston meanwhile descends through a distance d, and R is the internal radius of the tube AB, the value of (v_2-v_1) is equal to $\pi R^2 d$. To obtain the value of w, the tube AB is weighed without the piston, and then with the piston inserted with its grooves filled with mercury. The value of a is equal to πR^2.

If stop-cocks are inserted through the rubber stoppers, the apparatus can be evacuated, or filled with any gas. There is a slight amount of friction between the piston and the walls of the tube: the correction for this can be obtained by performing two experiments; one with two metal cylinders attached to the ends of the piston, so as to increase its weight; and the other with the piston unweighted. The correction is small, amounting to about 3 per cent. of w, the weight of the piston.

The viscosity of gases.—The following table gives the values of the coefficient of viscosity for various gases, at 0°C.: the values are the means of results obtained by various observers.

Gas	Coefficient of Viscosity
Hydrogen	0·0000864
Aqueous vapour	0·0000093
Carbon monoxide...	0·0001628
Ethylene	0·0000944
Nitrogen	0·0001647
Nitric oxide	0·000168
Oxygen	0·0001873
Carbon dioxide	0·0001431
Nitrous oxide	0·0001381
Chlorine	0·000128

Dr. Rankine gives the following values for the coefficient of viscosity of atmospheric air at various temperatures

Temperature	Coefficient of Viscosity
11·2° C	0·0001770
15·5° C	0·0001803
19·2° C	0·0001828

Dr. Rankine has also measured the coefficient of viscosity of the rare gases; these gases had been purified specially under the direction of Sir W. Ramsay.

Gas	Temperature	Coefficient of Viscosity
Helium	9·8° C.	0·0001914
Neon	10·1° C.	0·0003036
Argon	12·3° C.	0·0002168
Krypton	10·6° C.	0·0002405
Xenon	10·9° C.	0·0002180

In all cases, the coefficient of viscosity of a gas increases with the temperature. Maxwell thought that the coefficient of viscosity was proportional to the absolute temperature; but, as a matter of fact, it increases more slowly than would be the case if this law held. According to Prof. Sutherland—

$$\eta = a\frac{T^{\frac{1}{2}}}{1 + (c/T)},$$

where a and C are constants for a given gas, and T denotes the absolute temperature.

No definite laws have been deduced in connection with the viscosities of mixtures of gases. Graham found that, for a mixture of equal parts of oxygen and nitrogen, the viscosity is the arithmetical mean of the viscosities of the gases mixed. On the other hand, he found that a small amount of air added to hydrogen increased the viscosity very much, while a small amount of hydrogen added to air produced but a small effect.

Viscous drag on a body which is moving through a fluid.—When a body moves through a fluid, there is no slip between the body and the fluid immediately in contact with it; thus the fluid near to the body is in motion, while the fluid at a distance from the body is practically at rest. Hence, there is a velocity gradient in the fluid near to the body, and the motion of the body is opposed by the corresponding viscous drag called into play.

Some information as to the connection between the viscous drag and the velocity of the body can be obtained by the method of dimensions. Let the force f which opposes the motion of the body be given by the equation—

$$f = kl^x\rho^y\eta^z\mathrm{V}^n,$$

where l denotes the linear magnitude of the body (for instance, the radius, if the body has the form of a sphere), ρ denotes the density of the fluid, η denotes the coefficient of viscosity of the fluid, and V denotes the velocity of the body; k is a constant, independent of the fundamental units of length, mass, and time. Equating the dimensions of the two sides of the equation—

$$\frac{\mathrm{ML}}{\mathrm{T}^2} = \mathrm{L}^x\left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^y\left(\frac{\mathrm{M}}{\mathrm{LT}}\right)^z\left(\frac{\mathrm{L}}{\mathrm{T}}\right)^n.$$

For both sides of the equation to have equal dimensions in M—

$$1 = y + z \qquad \tag{1}$$

For both sides to have equal dimensions in L and T—

$$1 = x - 3y - z + n \qquad \tag{2}$$

$$-2 = -z - n \qquad \tag{3}$$

From (2) and (3)—

$$1 = x - 3y + n - z + n$$

$$1 = y + z - n.$$

From (1) and (3)—

$$\therefore \quad y = n - 1,$$
$$x = n,$$
$$z = -n + 2.$$

Thus—

$$f = kl^n\rho^{n-1}\eta^{2-n}\mathrm{V}^n$$

$$= k\left(\frac{\mathrm{V}l\rho}{\eta}\right)^n \cdot \frac{\eta^2}{\rho} \qquad \tag{1}$$

The method of dimensions does not suffice to determine the value of n.

Three cases now require consideration :

(1) **When the velocity of the body is small, experiment shows that the opposing force is directly proportional to the velocity.** In this case $n=1$, and—

$$f=k \,.\, \mathrm{V}l\eta.$$

Sir George Stokes proved that when a small sphere of radius r travels through a viscous fluid, the value of the opposing force f is given by the equation—

$$f=6\pi\eta r\mathrm{V}\,;$$

therefore in this case $k=6\pi$.

Let it be supposed that a very small sphere, of density σ, falls freely from rest under the action of gravity ; then it is pulled downwards by a constant force equal to $(4/3)\pi r^3(\sigma-\rho)g$, where ρ, as before, denotes the density of the surrounding fluid. The force opposing the motion of the body increases with the velocity, and so long as the downward force exceeds the opposing force, the velocity of the body increases. Thus the body will finally acquire a **terminal velocity**, such that the downward pull of gravity is just equal to the opposing force due to the viscosity of the surrounding fluid. In this case—

$$6\pi\eta r\mathrm{V}=\frac{4}{3}\pi r^3(\sigma-\rho)g\,;$$

$$\therefore\quad \mathrm{V}=\frac{2}{9}\frac{gr^2(\sigma-\rho)}{\eta}.$$

Thus, in the case of a drop of water, of 0·001 cm. radius, falling through air for which $\eta=0·00018$, the value of the terminal velocity V is given by the equation—

$$\mathrm{V}=\frac{2}{9}\cdot\frac{981\times(0·001)^2\times 1}{0·00018}=1·21 \text{ cm. per sec.}$$

For drops of other sizes, the terminal velocity is proportional to the square of the radius ; thus a drop of 0·01 cm. radius falls with a terminal velocity equal to 121 cm. per second. This result explains why the minute drops of water, which constitute a cloud, do not fall perceptibly under the action of gravity, while the much larger rain-drops fall freely.

(2) **When the velocity of the body is great, experiment shows that the opposing force is proportional to the square of the velocity.**

Thus, in the high-speed railway trials conducted at Zossen in Germany, it was found that when the speed of the train was about 100 miles per hour, the train was opposed by a force proportional to the square of the velocity.

Substituting $n=2$ in equation (1), (p. 527), it is found that—

$$f=k\left(\frac{Vl\rho}{\eta}\right)^2\frac{\eta^2}{\rho}=kV^2l^2\rho.$$

In this case, the opposing force is independent of the viscosity of the medium (p. 491), and is proportional to the density of the medium and the square of the linear magnitude of the moving body. Thus, Froude found that the frictional force acting on a ship is proportional to the square of the velocity and to the area of the wetted surface of the ship (compare p. 477).

(3) **When the velocity of the moving body is neither very great nor very small,** it has been found by Mr. Allen that the opposing force is proportional to $V^{\frac{3}{2}}$. Substituting $n=3/2$ in equation (1), (p. 527), we find that—

$$f=k\left(\frac{Vl\rho}{\eta}\right)^{\frac{3}{2}}\frac{\eta^2}{\rho}$$
$$=kV^{\frac{3}{2}}l^{\frac{3}{2}}\rho^{\frac{1}{2}}\eta^{\frac{1}{2}}$$

QUESTIONS ON CHAPTER XV

1. The plane surface of a solid body is constrained to remain parallel to, and at a distance of 1 mm. from, the surface of a plane sheet of metal of indefinite extent. The area of the plane surface of the solid body is equal to 100 sq. cm., and the space between this surface and the plane sheet of metal is filled with olive oil, of which the coefficient of viscosity is equal to 3·0 gm./(cm. sec.). Calculate the value of the force that would be needed in order to keep the solid body moving with a velocity of 10 cm./sec. in a direction parallel to the plane sheet of metal.

2. The value of the coefficient of viscosity of water at 20° C. is equal 0·010 gm./(cm. sec.); calculate the equivalent value expressed in lb./(ft. sec.).

3. What is the highest velocity at which water, at 20° C., can flow through a tube of 1 mm. diameter, without turbulence being produced.

(Coefficient of viscosity of water at 20° C. = 0·010 gm./(cm. sec.)).

4. Water at 20° C. is escaping from a cistern by way of a horizontal capillary tube 10 cm. long and 0·4 mm. in diameter, at a distance of 50 cm. below the free surface of the water in the cistern. Calculate the rate at which the water is escaping.

(Coefficient of viscosity of water at 20° C. = 0·10 gm./(cm. sec.))

5. The horizontal section of the cistern referred to in question (4) is equal to 10^4 sq. cm., and the water has been escaping for 24 hours. What was the height of the free surface of the water in the cistern, above the level of the capillary tube, at the commencement of the 24 hours?

6. The space between two co-axial cylindrical surfaces is filled with olive oil for which the coefficient of viscosity is equal to 3·0 gm./(cm. sec.). The radii of the cylinders are respectively equal to 10·0 cm. and 10·1 cm., and the axial length of either cylinder is equal to 20 cm. Calculate the value of the torque required to keep the inner cylinder rotating with an angular velocity of 2 radians per sec., the outer cylinder being fixed.

7. The perfectly plane surfaces of two bodies are constrained to remain parallel to each other. Prove that the frictional force which opposes the sliding motion of one surface over the other cannot be diminished by oiling the surfaces.

8. When a uniform spherical steel ball is set rolling on the concave surface of a lens (p. 104), the time required for the ball to come to rest is diminished by oiling the ball. Explain this.

9. In order to determine the coefficient of viscosity of air, the following modification of expt. 57 (p. 517) may be used. With the U tube in the position represented in Fig. 242, determine the time t_1 required for the descending pellet of mercury to sweep out a volume Q; then invert the U tube, so that its bent part is uppermost, and determine the time t_2 required for the descending pellet of mercury to sweep out the same volume Q as before. Prove that—

$$\frac{Q}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}\right)=\frac{(\rho-\delta)g\pi r^4}{8\eta l}.$$

The notation agrees with that used on p. 518.

10. Emery powder, comprising particles of various sizes, is stirred up in a beaker filled to a height of 10 cm. with water. Calculate the size of the largest particles that will remain in suspension after (*a*) 1 hour, and (*b*) 24 hours. (Density of emery = 4·0 gm./(cm.)^3. Coefficient of viscosity of water = 0·010 gm./(cm. sec.). The particles of emery may be assumed to be spherical.)

CHAPTER XVI

THE MOLECULAR STRUCTURE OF FLUIDS

Molecules.—Some properties of fluids can be explained, in conformity with laws derived from experimental data, without making any assumption as to the ultimate structure of the fluids. Many of these properties have been studied in the preceding chapters; but once or twice it has been found impossible to frame an explanation without assuming that fluids consist of ultimate particles or molecules (p. 351). The properties of fluids which will be studied in this chapter cannot be explained without assuming a molecular structure; the explanations deduced from this assumption are so consistent, and have led, so often, to the prediction of phenomena which have been observed subsequently, that there can be no doubt that the assumption is justified.

According to the molecular theory of matter, any homogeneous material substance consists of ultimate particles, all of which are similar; these ultimate particles are called **molecules.** In many cases a molecule may be divided into parts; but these parts possess properties which are not possessed by the molecule itself. The molecules of a substance are in rapid motion, and therefore they possess kinetic energy. When a substance is in the gaseous state, the average distance from any molecule to its nearest neighbour is great in comparison with the magnitude of the molecule itself.

It is customary to assume that a molecule is spherical in shape; this assumption is made, not because it is believed to be absolutely true, but because we have little evidence as to the real shape of molecules, and the sphere is the simplest shape which can be assumed. In some cases, where there is evidence that a molecule comprises a number of atoms, each atom is assumed to be spherical; even in this case we can obtain valuable information as to the properties of the substance by assuming that the molecules are spherical in shape.

Since molecules are very small in magnitude, it will be assumed at first that each molecule approximates to a geometrical point, although it possesses a finite mass; the extent to which the properties of material substances can be explained on this assumption will give grounds for forming some idea of the actual size of molecules; and finally the probable magnitude of a molecule will be estimated with a fair degree of exactness.

INFINITELY SMALL MOLECULES.

Pressure of a gas.—Let Fig. 245 represent a hollow cubical vessel of volume v; and let it be supposed that this vessel contains N infinitely small molecules, each of which is moving with a considerable velocity. When a molecule strikes on the walls of the vessel, it rebounds; the component velocity of the molecule, perpendicular to the wall on which it strikes, is reversed by the impact, while the component velocity parallel to the wall is left unchanged.

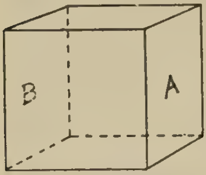

FIG. 245.—Cubical vessel containing a gas.

Since the molecules are supposed to be infinitely small, collisions between one molecule and another will be so infrequent that they may be supposed not to occur at all. The reasonableness of this assumption will be evident when it is remembered that a molecule is supposed to approximate to a geometrical point, and therefore it can pass another molecule without the occurrence of a collision, even when the shortest distance between the two is infinitely small.

Let a molecule rebound from the face A of the cubical vessel. Before the rebound, let the component velocity of the molecule perpendicular to the face A be equal to $\mathbf{V}_1$; during the rebound this velocity is reversed or changed to $(-\mathbf{V}_1)$. Let m denote the mass of the molecule; then during the rebound the component momentum of the molecule perpendicular to the face A is changed from $(+m\mathbf{V}_1)$ to $(-m\mathbf{V}_1)$, and therefore the change of momentum is equal to $2m\mathbf{V}_1$.

After the rebound, the molecule travels towards the face B with the velocity $\mathbf{V}_1$. If the molecule possesses a component velocity parallel to the face A, impacts may occur on other faces of the cube, but these impacts will not affect the component velocity $\mathbf{V}_1$ perpendicular to the faces A and B. Let the length of an edge of the cube be equal to l; then the volume v of the cube is equal to l^3. When the molecule has travelled from A to B and back to A, another impact will occur on the face A. The time required by the molecule to travel from A to B and back to A, with the velocity $\mathbf{V}_1$, is equal to $(2l/\mathbf{V}_1)$; during one second, the number of impacts of the molecule on the face A is equal to $(\mathbf{V}_1/2l)$. At each impact, the change of momentum is equal to $(2m\mathbf{V}_1)$; thus, during one second, the change of momentum of the molecule, due to its impacts with the face A, is equal to $(2m\mathbf{V}_1)\times(\mathbf{V}_1/2l)=(m\mathbf{V}_1^2/l)$.

Let the other molecules contained in the cubical vessel possess component velocities, perpendicular to A, equal to $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4$ . . . $\mathbf{V}_n$: then the total change of momentum which occurs at the face A in one second is equal to—

$$\frac{m}{l}(\mathbf{V}_1^2+\mathbf{V}_2^2+\mathbf{V}_3^2+\,.\,.\,.\,.\,.\,+\mathbf{V}_n^2)=\frac{m}{l}\Sigma\mathbf{V}^2.$$

Since change of momentum per second is equal to force (p. 19) the expression just obtained gives the force that tends to push the face A away from B. The area of the face A is equal to l^2; therefore the pressure (force per unit area) on the face A is equal to—

$$\left(\frac{m}{l}\times\frac{1}{l^2}\right)\Sigma\mathbf{V}^2=\frac{m}{v}\,.\,\Sigma\mathbf{V}^2.$$

Let the molecules possess component velocities $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$, . . . $\mathbf{U}_n$, perpendicular to the upper and lower faces of the cube; then the pressure on either of these faces is equal to—

$$\frac{m}{v}\,.\,\Sigma\mathbf{U}^2.$$

Let the molecules possess component velocities $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$

. . . $\mathbf{W}_n$, perpendicular to the front and back faces of the cube; then the pressure on either of these faces is equal to—

$$\frac{m}{v}\Sigma\mathbf{W}^2.$$

Let the velocities of the molecules be distributed in such a manner that all faces of the cube are subjected to equal pressures. Then—

$$\frac{m}{v}\,.\,\Sigma\mathbf{V}^2 = \frac{m}{v}\,.\,\Sigma\mathbf{U}^2 = \frac{m}{v}\Sigma\mathbf{W}^2 = \frac{m}{3v}\Sigma(\mathbf{U}^2+\mathbf{V}^2+\mathbf{W}^2).$$

Now $\mathbf{U}_1$, $\mathbf{V}_1$, $\mathbf{W}_1$ are the components of the velocity of a particular molecule, resolved perpendicular to the upper and lower, the right and left, and the front and back faces of the cube respectively; therefore the resultant velocity V_1 of this molecule is given by the equation—

$$V_1^2 = \mathbf{U}_1^2+\mathbf{V}_1^2+\mathbf{W}_1^2.$$

Therefore $\Sigma\,(\mathbf{U}^2+\mathbf{V}^2+\mathbf{W}^2)$ denotes the sum of the squares of the resultant velocities of the molecules, which may be written ΣV^2. The pressure p exerted on either face of the cube is given by the equation—

$$p = \frac{m}{3v}\,.\,\Sigma V^2.$$

Multiply and divide the right-hand side of this equation by N, the number of molecules contained by the vessel; then—

$$p = \frac{Nm}{3v}\,.\,\frac{\Sigma V^2}{N} = \frac{Nm}{3v}\,.\,\overline{V}^2,$$

where $\overline{V}^2$ denotes the value of $\{V_1^2+V_2^2+V_3^2+\ldots\ldots\} \div N$, or $(\Sigma V^2)/N$; that is, the average value of the square of the velocity for all the molecules, or the **mean square velocity** of the molecules. Then—

$$pv = \frac{Nm}{3}\overline{V}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It should be noticed that the result expressed by equation (1) would have been obtained by assuming that one third of the molecules travelled, with the mean square velocity $\overline{V}^2$, to and fro perpendicular to the faces A and B, while another third of the molecules travelled perpendicular to the front and back faces of the cube, and the remaining third travelled perpendicular to the upper and lower faces of the cube. This equivalent molecular distribution will be found useful in subsequent investigations.

So long as $\overline{V}^2$ remains constant, the product of the pressure p and the volume v remains constant; this is **Boyle's law**, which is complied with, to a close approximation, by gases at high temperatures.[1]

If we assume that the absolute temperature T of a gas is proportional to $\overline{V}^2$, it follows that—

$$pv = RT,$$

where R is a constant depending only on the values of N and m.

The value of the square root of the mean square velocity of a molecule can be calculated without difficulty. Let it be supposed that v is the volume of one gram of gas at 0° C., under a pressure of 76 cm. of mercury (a standard atmosphere). Then, since the density of mercury is equal to 13·6 grams per c.c., and the pull of gravity on a gram of mercury is equal to 981 dynes, it follows that—

$$p = 76 \times 13{\cdot}6 \times 981 = 1{\cdot}01 \times 10^6 \text{ dynes per sq. cm.}$$

The value of Nm is the mass of the gas, that is, 1 gram.

The volume of a gram of hydrogen, at 0° C. and atmospheric pressure, is equal to $1{\cdot}11 \times 10^4$ c.c. Thus, since—

$$pv = \frac{Nm}{3}\overline{V}^2,$$

$$(1{\cdot}01 \times 10^6) \times (1{\cdot}11 \times 10^4) = \tfrac{1}{3}\overline{V}^2;$$

$$\therefore \quad \sqrt{(\overline{V}^2)} = \sqrt{3 \times 1{\cdot}01 \times 1{\cdot}11 \times 10^{10}}$$

$$= 1{\cdot}83 \times 10^5 \text{ cm. per sec.}$$

Expressed in British units, **the root-mean-square velocity of a hydrogen molecule at 0° C. is equal to 4,100 miles per hour.**

Equipartition of energy.—Maxwell attempted to prove that, at a given temperature, the mean kinetic energy of a molecule has the same value for all substances in the gaseous state. According to this law, the mean kinetic energy of a molecule of hydrogen is equal to the mean kinetic energy of a molecule of oxygen, so long as the two gases are at equal temperatures. If

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[1] See the Author's *Heat for Advanced Students* (Macmillan), p. 203.

m_1 and m_2 denote the masses of the molecules of two substances, while $\bar{V}_1^2$ and $\bar{V}_2^2$ denote their mean square velocities at any temperature, then—

$$\tfrac{1}{2}m_1\bar{V}_1^2 = \tfrac{1}{2}m_2V_2^2 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \quad V_1/V_2 = \surd(m_2/m_1),$$

where V_1 and V_2 denote the square roots of the mean square velocities; V_1 and V_2 will be proportional, but not equal to the means of the numerical values of the velocities of the molecules.

Let two equal vessels be filled with different gases at equal pressures and temperatures; then if letters with the subscript $_1$ refer to one gas, and those with the subscript $_2$ refer to the other, we have, since the pressures are equal—

$$p = \frac{N_1 m_1}{3v}\bar{V}_1^2 = \frac{N_2 m_2}{3v}\bar{V}_2^2.$$

From (2)—

$$m_1\bar{V}_1^2 = m_2\bar{V}_2^2\,;$$

$$\therefore \quad N_1 = N_2.$$

This result is the mathematical expression of Avogadro's celebrated hypothesis, that **at equal temperatures and pressures, equal volumes of all gases comprise equal numbers of molecules.** This hypothesis forms the basis of modern chemical theory; its universal adoption is proof of its close approximation to the truth, and affords valuable support to Maxwell's law of the equipartition of molecular energy.

It follows from Avogadro's hypothesis, that **at equal temperatures and pressures, the densities of different gases are proportional to their molecular weights.** Now, if ρ_1 and ρ_2 denote the densities of two different gases under similar conditions of temperature and pressure—

$$V_1/V_2 = \surd(m_2/m_1) = \surd(\rho_2/\rho_1).$$

This equation is the mathematical expression of Graham's law, that **at equal temperatures and pressures, the velocities of effusion of different gases are inversely proportional to the square roots of the densities of the gases.**

In the majority of gases, such as hydrogen, oxygen, nitrogen, &c., each molecule comprises two atoms, so that the molecular weight is twice as great as the atomic weight.

A molecule of oxygen is 16 times as heavy as a molecule of hydrogen; thus the mass of any given number of molecules of oxygen is 16 times as great as the mass of the same number of molecules of hydrogen. Hence 32 grams of oxygen will comprise the same number of molecules, and occupy the same volume under similar conditions of temperature and pressure, as 2 grams of hydrogen. A mass of any substances, numerically equal to the molecular weight of that substance, is now generally spoken of as a **gram-molecule** of that substance.

Strictly speaking, the atomic weight of hydrogen is equal to 1·008, if the atomic weight of oxygen is equal to 16·00.

If v denotes the volume of a gram-molecule of any permanent gas at an absolute temperature T and under a pressure of p dynes per sq. cm., it follows that—

$$pv = \mathbf{R}T,$$

when **R** has a single value for all permanent gases. To calculate the value of **R**, notice that 2·016 grams of hydrogen at 0°C. and standard atmospheric pressure, occupy a volume of $2{\cdot}24 \times 10^4$ c.c. Thus—

$$(1{\cdot}01 \times 10^6) \times (2{\cdot}24 \times 10^4) = \mathbf{R} \times 273;$$

$$\therefore \quad \mathbf{R} = 8{\cdot}28 \times 10^7.$$

Thus the value of **R** is very nearly equal to twice the mechanical equivalent of heat. It should be remembered that **the gas constant R for a gram-molecule of any permanent gas is equal approximately to twice the mechanical equivalent of heat, expressed in ergs per gram calorie.**

MOLECULES OF FINITE SIZE.

Mean free path.—If the molecules of a gas are not infinitely small, collisions between them must occur inevitably. **The average distance over which a molecule can travel through a gas, without colliding with another molecule, is called the mean free path of the molecule.**

Let it be supposed that a given space contains a great number of stationary molecules distributed at random, each unit volume of the space containing N_1 molecules, all of equal size. Then any volume v, chosen from this space at random, will contain the centres of N_1v molecules; provided that the volume v is large enough to contain a considerable number of molecules. The shape of the chosen volume is immaterial.

Now let it be supposed that a molecule, of the same size as the stationary molecules, is projected at random into the space with a velocity V. The projected molecule will travel in a straight line until it collides with a stationary molecule: the distance traversed is called the **free path** of the projected molecule. When a collision occurs, let the projected molecule be removed from the space and once more projected into it; by repeating this operation a great number of times the mean value of its free paths can be determined. In the ensuing investigations, the value of the mean free path will be denoted by λ.

The value of λ can be calculated as follows. When a collision occurs, the centre of the projected molecule must be at a distance d from the centre of the molecule struck, where d denotes the diameter of a molecule. Thus, it may be supposed that the projected molecule is replaced by a circular disc of radius d, the area of the disc remaining perpendicular to the direction of projection while the centre of the disc moves along the straight line described in reality by the centre of the projected molecule. The disc will pass through the centre of a stationary molecule almost at the instant when the projected molecule would have collided with that molecule, and the free path of the disc, determined in this manner, will be approximately equal to the free path of the projected molecule (Fig. 246).

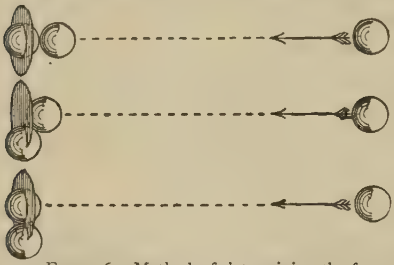

FIG. 246.—Method of determining the free path of a molecule.

Let the times which elapse during the free paths of the disc be equal to $t_1, t_2, \ldots t_n$, and let the sum of these times be equal to one second. Then the lengths of the free paths must have been equal to $Vt_1, Vt_2, \ldots Vt_n$, and the sum of these lengths must have been equal to $V \times 1$. In each free path the disc sweeps out a cylindrical volume of which the cross-sectional area is equal to πd^2, and each cylinder contains the centre of one, and only one molecule; viz., that of the molecule struck (Fig. 247). The sum of all the volumes swept out by the disc is equal to πd^2V, and this volume contains the centres of the n molecules struck. But the volume πd^2V has been chosen at random, and it therefore contains the centres of $N_1 \times \pi d^2V$ molecules. Thus $n = N_1\pi d^2V$. The sum of the lengths of the n free paths is equal to V, and therefore the mean length, λ, of the free paths is given by the equation—

$$\lambda = \frac{V}{n} = \frac{V}{N_1 \pi d^2 V} = \frac{1}{N_1 \pi d^2}.$$

Thus, the mean free path is inversely proportional to N_1, the number of molecules per unit volume; in other words, **the mean free path λ is inversely proportional to the density of the gas.** The value of λ is independent of V.

Now let it be supposed that a molecule is projected into a space, containing N_1 fixed molecules per unit volume distributed at random, and that the subsequent path of the projected molecule is observed. At every collision a change is produced in the direction of motion of the projected molecule; but if the molecules are perfectly elastic, the velocity of the projected molecule will remain constant. Thus it may be considered that at each collision the moving molecule is projected afresh, and the average distance which it traverses between two successive collisions will have the value just determined for λ.

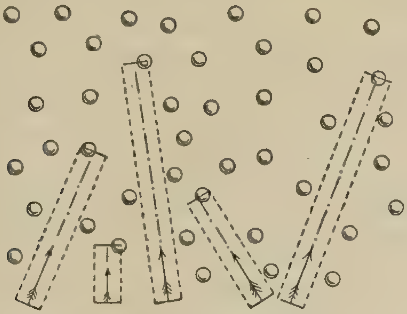

FIG. 247.—Method of determining the mean free path of a molecule.

Fig. 246 shows that the replacement of the projected molecule by a disc is legitmate when λ is large compared with *d*: when λ is of the same order of magnitude as *d*, a correction is needed which will be deduced later.

To determine accurately the mean free path of a molecule of a gas, it is necessary to suppose that a molecule is projected into a space which contains other molecules moving with various velocities in different directions. The complete investigation of this problem is far too complicated to be attempted here; but the problem becomes simpler if it be supposed that a molecule is projected, with velocity V, into a space containing N_1 molecules per unit volume, all moving in different directions with a single velocity V.

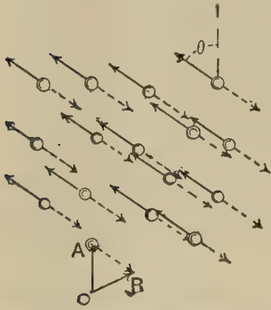

Fig. 248.—Molecule projected into a space containing molecules which are all moving in one direction with equal velocities.

As a preliminary let it be supposed that a molecule, A (Fig. 248), is projected with a velocity V, into a space containing N_1 molecules per unit volume, all moving with the velocity V, in a direction inclined at an angle θ to the direction of motion of the projected molecule. The number of collisions in a second will not be altered by impressing a velocity V, in a direction opposite to that in which the N_1 molecules are moving, on the whole system (compare p. 426). In this case the N_1 molecules will be reduced to rest, and the projected molecule will be moving with a velocity OR (Fig. 248) equal to the resultant of the impressed velocity AR and the true velocity of projection OA = V. The number of collisions of the projected molecule during a second is equal to $N_1 \pi d^2 \times OR$.

Let a given space contain $N_1/2$ molecules which are travelling from south-west to north-east with the velocity V, and $N_1/2$ molecules which are travelling from south-east to north-west with the velocity V. Let a similar molecule be projected into the space with velocity V from south to north. Then it is clear that the number of collisions in a second, between the projected molecule and the molecules which travel from south-west to north-east, is equal to the number of collisions in a second with the molecules which travel from south-east to north-west, since OR (Fig. 248) has the same numerical value in both cases. The total number of collisions per second is therefore equal to $N_1 \pi d^2 \times OR$.

Now let the molecules within a space be moving in different directions, all of which are inclined to the direction of motion of the projected molecule at an angle θ; to fix our ideas, let it be supposed that the directions of motion of the molecules in the space coincide with the generating lines of a cone, while the direction of motion of the projected molecule coincides with the axis of the cone. Then the number of collisions per second between the projected molecule and the other molecules in the space is the same as if the molecules in the space were all moving in a single direction inclined at an angle θ to the direction of motion of the projected molecule.

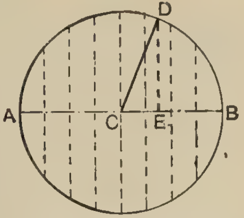

FIG. 249.—Spherical surface divided into strips having equal areas.

An important property of a spherical surface must now be deduced. **Let a diameter AB (Fig. 249) of a sphere be divided into equal parts, and through each point of division let a plane be drawn perpendicular to the divided diameter. Then these planes divide the surface of the sphere into annular strips which are equal in area.**

To prove this, draw a radius from the centre C of the sphere to D, the middle of one of the strips; and let the angle DCB $= \theta$. From D draw DE perpendicular to CB; then the average length of the strip is equal to the circumference of the circle of which DE is the radius. Thus, the average length of the strip is equal to $2\pi \times \mathrm{CD} \sin\theta$. The width of the strip is perpendicnlar to CD, and is therefore inclined to CB at an angle $\{(\pi/2) - \theta\}$. Let a be the length of each of the equal parts into which the diameter has been divided, and let w be the width of the strip; then—

$$w \cos\{(\pi/2) - \theta\} = w \sin\theta = a\,;$$
$$\therefore \quad w = a/\sin\theta.$$

The area of the strip is equal to—

$$w \times 2\pi\mathrm{CD}\sin\theta = (a/\sin\theta) \times 2\pi\mathrm{CD}\sin\theta$$
$$= 2\pi\mathrm{CD}\,.\,a\,;$$

and since this result does not contain θ, it follows that the same value would be obtained for the area of any strip.

Now let it be supposed that unit volume of a gas comprises N_1 molecules moving at random in all directions with the velocity V. To form a definite idea, let it be supposed that any point within the unit of volume is chosen, and that vectors are drawn from this point to represent, in magnitude and direction, the instantaneous velocities of all the molecules. Since the velocities of the molecules are numerically equal, it follows that the free ends of the vectors will lie on a spherical surface of radius V; and since the directions of the velocities are distributed at random, it follows that the free ends of the vectors will be distributed uniformly over the surface of the sphere.

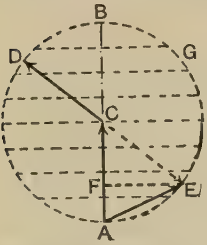

FIG. 250.—Determination of the mean free path of a molecule.

Let ADB (Fig. 250) represent a diametral section of the sphere, and let the diameter AB be the line along which a molecule is projected with the velocity V into the unit volume. Let the diameter AB be divided into a very large number n of equal parts; and let planes, perpendicular to AC, be drawn through the points of division. These planes divide the surface of the sphere into n circular strips of equal area, and on each strip terminate the free ends of N_1/n vectors. The velocities corresponding to the vectors which terminate on the strip DG are all inclined to AB at the angle $DCB = \theta$. The molecules which possess these velocities form a group, and the number of collisions between the projected molecule and this group during a second is easily determined.

From C, the centre of the sphere, draw CE equal and opposite to CD, and join AE. Then, during one second, the number of collisions between the projected molecule and the group of N_1/n molecules, is equal to—

$$\frac{N_1}{n}\pi d^2 \times AE,$$

where d is the diameter of a molecule. The truth of this result can be seen by comparing Fig. 250 and Fig. 248.

From E draw EF perpendicular to AB. Then—

$$AE^2 = AF^2 + FE^2,$$

and

$$FE^2 = AF \times FB;$$

$$\therefore \quad AE^2 = AF^2 + (AF \times FB) = AF(AF + FB)$$

$$= AF \, . \, AB.$$

Therefore the number of collisions per second between the projected molecule and the group of molecules whose velocities correspond to the vectors that terminate on the strip DE, is equal to–

$$\frac{N_1}{n}\pi d^2\sqrt{AF\ .\ AB.}$$

Let the points of division of the diameter AB be at distances x_o, x_1, x_2, x_3, . . . x_n from A; where $x_o=0$, and $x_n=AB=2V$. Let the point F lie between points of division at distances x_b and x_a from A, and let AF be denoted by ${}_bx_a$. The value of AB/n is equal to x_b-x_a; therefore the number of collisions with the group during one second is equal to—

$$\frac{N_1}{\sqrt{AB}}\cdot\pi d^2\sqrt{AF}\cdot\frac{AB}{n}=\frac{N_1}{\sqrt{AB}}\cdot\pi d^2\cdot{}_bx_a^{\frac{1}{2}}(x_b-x_a).$$

Let $x_b=\xi_b^2$, while $x_a=\xi_a^2$.

Then $$x_b-x_a=\xi_b^2-\xi_a^2=(\xi_b+\xi_a)(\xi_b-\xi_a)=2\ {}_b\xi_a(\xi_b-\xi_a),$$

where ${}_b\xi_a$ denotes the mean value of ξ_b and ξ_a. Also, ${}_bx_a={}_b\xi_a^2$ to a close approximation; therefore—

$$\begin{aligned}{}_bx_a^{\frac{1}{2}}(x_b-x_a)&={}_b\xi_a\cdot 2\ {}_b\xi_a(\xi_b-\xi_a)\\&=2\ {}_b\xi_a^2(\xi_b-\xi_a).\end{aligned}$$

In accordance with the general principle explained on p. 48, we may write—

$${}_b\xi_a^2=\frac{\xi_b^2+\xi_b\xi_a+\xi_a^2}{3};$$

thus $$2{}_b\xi_a^2(\xi_b-\xi_a)=\tfrac{2}{3}(\xi_b^2+\xi_b\xi_a+\xi_a^2)(\xi_b-\xi_a)=\tfrac{2}{3}(\xi_b^3-\xi_a^3).$$

Therefore the number of collisions per second with the group is equal to—

$$\frac{N_1}{\sqrt{AB}}\cdot\pi d^2\cdot\tfrac{2}{3}(\xi_b^3-\xi_a^3).$$

Now, let $\xi_o^2=x_o$, while $\xi_1^2=x_1$, ... $\xi_n^2=x_n$. The total number of collisions per second with the N_1 molecules comprised in unit volume is equal to the sum of the numbers of collisions with the n groups into which the molecules have been divided. Therefore the total number of collisions per second between the projected molecule and the N_1 molecules comprised in unit volume, is equal to—

$$\begin{aligned}&\frac{N_1}{\sqrt{AB}}\cdot\pi d^2\cdot\tfrac{2}{3}\left\{(\xi_1^3-\xi_o^3)+(\xi_2^3-\xi_1^3)+\ .\ .\ .\ +(\xi_n^3-\xi_{n-1}^3)\right\}\\&=\frac{N_1}{\sqrt{AB}}\pi d^2\cdot\tfrac{2}{3}(\xi_n^3-\xi_o^3),\end{aligned}$$

and $\xi_n{}^2 = x_n = AB$, so that $\xi_n{}^3 = AB^{\frac{3}{2}}$. Further, $\xi_0 = 0$. Thus, **the total number of collisions per second** is equal to—

$$N_1\pi d^2 \cdot \tfrac{2}{3}\frac{(AB)^{\frac{3}{2}}}{(AB)^{\frac{1}{2}}} = N_1 \cdot \pi d^2 \cdot \tfrac{2}{3}AB$$
$$= \tfrac{4}{3}N_1\pi d^2 \cdot V,$$

since $AB = 2V$.

The mean free path λ of the projected molecule is determined easily, for the molecule travels with a uniform velocity V, and collides $(4/3)N_1\pi d^2V$ times in a second; therefore the mean distance traversed between two consecutive collisions is equal to—

$$V \div \tfrac{4}{3}N_1\pi d^2V = \tfrac{3}{4}\frac{1}{N_1\pi d^2}.$$

Maxwell calculated the value of the mean free path when the molecules comprised in the gas are travelling in different directions with unequal velocities; the value he obtained is equal to—

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{N_1\pi d^2} = 0{\cdot}7071\frac{1}{N_1\pi d^2},$$

and this value of λ is only about 5 per cent. smaller than that obtained above.

Speed of transmission of momentum.—In obtaining the expression for the pressure of a gas in terms of the molecular bombardment of the walls of the containing vessel, it was assumed on p. 532 that the molecules are infinitely small, so that collisions between molecules do not occur. It now becomes necessary to determine the modifications introduced into the results obtained, when the molecules of a gas are supposed to be of finite magnitude. As might be supposed, the modifications depend on the collisions of the molecules with each other.

Reference to the investigations already carried out (p. 533) shows that **a molecule may be considered to be a vehicle for the transport of momentum.** The size of a molecule cannot have any effect on the momentum transported; but the size will materially affect the speed with which the momentum of the molecule is transported from one place to another. This result can be realised by referring to Fig. 251. Let the distance between the centres of the spheres A and B be denoted by λ, and at a given instant let A be projected with velocity V directly towards B. The distance over which A travels before it strikes B is equal to $(\lambda - d)$, and the time occupied in traversing this distance is equal to $(\lambda - d)/V$, where d denotes the diameter of either sphere. If the spheres possess an infinitely great elasticity, the momentum of A will be transferred to B at the instant when the collision occurs, so that the momentum of A travels through the distance λ in the time $(\lambda - d)/V$. If we suppose that a number of equal spheres lie along the straight line in which A starts to travel, and that the distance between each pair of neighbouring spheres is λ, it is clear that the momentum originally possessed by A will travel along the line of spheres at a speed $V\lambda/(\lambda - d)$. Thus the collisions of the spheres increase the speed with which momentum is transported, in the ratio $\lambda/(\lambda - d)$. Each sphere acquires the velocity of the sphere which strikes it, and travels forward until it strikes another sphere ; the time taken in transferring the momentum of a sphere through the distance λ is equal to $(\lambda - d)/V$, and therefore the speed with which the momentum is transmitted is equal to $V\lambda/(\lambda - d.)$

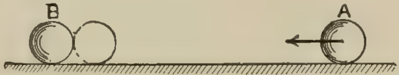

FIG. 251.—Speed at which momentum is transported by a sphere A which collides with another sphere B.

In impacts of the type just considered, the line joining the centre of the moving sphere to the centre of the sphere that is struck coincides with the direction of motion of the moving sphere at the instant of impact ; impacts of this type are said to be direct. The line joining the centres of the spheres at the instant of impact may be called the line of centres. In an impact of any kind, the component momentum of either sphere, resolved along the line of centres, is communicated to the other sphere. If the spheres are perfectly smooth, the component momentum of either sphere, perpendicular to the line of centres, is left unchanged by the impact.

Let A (Fig. 252) represent a sphere travelling along the straight line BA with the velocity V. When a collision occurs with the sphere C, the centre of A must be at a distance CA $= d$ from the centre of C, where d denotes the diameter of either sphere. Let the line of centres AC make an angle ACF $= \theta$ with the direction of motion of A before the impact; in this case, the free path of A is shortened from λ to $(\lambda - d \cos \theta)$, while in a direct impact the free path would have been shortened to $(\lambda - d)$. With C as centre and CA as radius, describe an imaginary spherical surface EFG. When A, moving in the direction BA, collides with C, the centre of A must lie for an instant in the hemisphere EFG, where EG is the trace of a plane drawn through C perpendicular to BA. To the imaginary sphere, draw a diameter FK parallel to BA, the direction of motion of A; divide FK into equal parts, and through the points of division draw planes perpendicular to FK. By this means the hemisphere EFG is divided into circular strips of equal areas, and in a great number of collisions the centre of A will be as often in one of these strips as in another.[1] From A draw AH perpendicular to CF; then in a great number of collisions the point H will lie as often in one of the equal divisions of CF as in another; therefore the average value of CH is equal to CF/2, and **the average shortening of the free path is equal to** CF/2 $= d/2$.

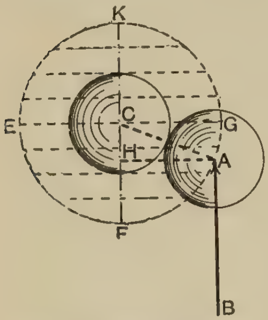

FIG. 252.—Transference of momentum during the impact of two spheres.

Let μ denote the momentum of A just before the collision occurs; then, at the instant of collision, the component momentum, resolved along the line of centres, is equal to $\mu \cos \theta$. This part of the momentum of A is transferred to the centre of B at the instant of impact; it can be resolved into a component $\mu \cos^2 \theta$, along FC, and a component $\mu \cos \theta \sin \theta$ perpendicular to FC. In determining the average momentum transmitted during a large number of impacts, it must be remembered that momentum is a vector or directed quantity; and since the component momentum perpendicular to FC is directed from right to left when the centre of the colliding sphere is to the right of FC, and from left to right when the centre of the colliding sphere is to the left of FC, it follows that, on an average, no momentum perpendicular to FC is transmitted during an impact. **The average momentum transmitted during an impact is parallel to FC, and its value is equal to $\mu \cos^2 \theta$.**

---

[1] This assumption is not strictly accurate; it is adopted here in order to simplify the calculations. From a physical point of view, the simplification can be justified, since there is small probability that the molecules are spheres, and therefore the calculation can be no more than an approximation at best.

Thus, in all impacts which are not direct, a fraction only of the momentum of the colliding molecule is transmitted to the molecule that is struck. In considering the transmission of momentum, a given quantity transmitted through one centimetre is equivalent to twice the quantity transmitted through half a centimetre. Hence, we may calculate the average distance through which the whole momentum of a molecule may be considered to be transmitted, between the instant when the molecule starts on its free path and that at which the next impact is just completed: let this distance be denoted by L.

Let an impact occur under the conditions indicated in Fig. 252; and let $\angle ACF = \theta$. Then CH $= d \cos \theta$, and the whole of the momentum μ is transmitted through the distance $(\lambda - d \cos \theta)$, while a part equal to $\mu \cos^2 \theta$ is transmitted forwards through the distance CH $= d \cos \theta$, parallel to BA. Thus, the product of the momentum transferred and the distance through which it is transferred is equal to—

$$\mu(\lambda - d \cos \theta) + \mu \cos^2 \theta \,.\, d \cos \theta.$$

Let CH $= d \cos \theta = x$; then the quantity just found is equal to—

$$\mu(\lambda - x) + \frac{\mu x^3}{d^2}.$$

Let the imaginary hemisphere EFG be divided into n strips of equal areas, and let a large number N of collisions occur; then, on an average, the centre of the colliding sphere will lie as often in one strip as in another, and the number of times it will lie in any strip will be N/n. Thus, the number of collisions that will occur with the centre of the colliding sphere in the strip that contains the point A (Fig. 252) is equal to N/n; and for these collisions, the sum of the products of the momentum transmitted and the distance through which transmission occurs is equal to—

$$\mathrm{N}\mu \cdot \frac{1}{n}\left\{\lambda - x + \frac{x^3}{d^2}\right\}.$$

Multiply and divide this quantity by d; then the result obtained is equal to—

$$\frac{\mathrm{N}\mu}{d} \cdot \frac{d}{n} \cdot \left\{\lambda - x + \frac{x^3}{d^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now let the points of division of the radius CF lie at distances x_0, x_1, x_2, x_3 . . . x_n from the centre C.

Let the point H lie between points of division at distances x_a and x_b from C. Since the elements of the radius are equal—

$$\frac{d}{n} = x_b - x_a;$$

and since x is intermediate in value between x_b and x_a, we may write (compare p. 48)—

$$x = \frac{x_b + x_a}{2},$$

$$x^3 = \frac{x_b^3 + x_b^2 x_a + x_b x_a^2 + x_a^3}{4}$$

Then $\qquad x(x_b - x_a) = \frac{1}{2}(x_b^2 - x_a^2),$

and $\qquad x^3(x_b - x_a) = \frac{1}{4}(x_b^4 - x_a^4);$

so that the quantity represented by (1) above is equal to—

$$\frac{\mathrm{N}\mu}{d} \cdot \left\{\lambda(x_b - x_a) - \frac{1}{2}(x_b^2 - x_a^2) + \frac{1}{4d^2}(x_b^4 - x_a^4)\right\} . \quad . \quad (2)$$

If we sum the quantities similar to (2), where x_1 and x_0, x_2 and x_1, . . . &c., are substituted for x_b and x_a, we obtain the sum of the products of the momentum transferred and the distance through which it is transferred, for the whole of the N collisions. Thus—

$$\mathrm{N}\mu\mathrm{L} = \frac{\mathrm{N}\mu}{d}\left\{\lambda(x_n - x_0) - \frac{1}{2}(x_n^2 - x_0^2) + \frac{1}{4d^2}(x_n^4 - x_0^4)\right\},$$

$$= \mathrm{N}\mu\{\lambda - \tfrac{1}{2}d + \tfrac{1}{4}d\} = \mathrm{N}\mu(\lambda - \tfrac{1}{4}d),$$

since $x_n = \mathrm{CF} = d$, and $x_0 = 0$;

$$\therefore \quad \mathrm{L} = \left(\lambda - \frac{d}{4}\right).$$

This result shows that, on an average, the momentum of a molecule is not transmitted through the whole distance λ, as it would be if all the impacts were direct. It may be considered that the whole momentum is transmitted through the distance $\{\lambda-(d/4)\}$ during the time between two impacts ; and as, on an average, the free path between two impacts is equal to $\{\lambda-(d/2)\}$, and the molecule traverses this distance with a velocity V, the time taken in transmitting the momentum through the distance $\{\lambda-(d/4)\}$ is equal to $\{\lambda-(d/2)\}/V$. Thus **the average speed with which momentum is transmitted forward by a molecule is equal to—**

$$V\frac{\lambda-\frac{d}{4}}{\lambda-\frac{d}{2}}=V\frac{\lambda^2-\left(\frac{d}{4}\right)^2}{\left(\lambda-\frac{d}{2}\right)\left(\lambda+\frac{d}{4}\right)}$$

$$=V\frac{\lambda}{\lambda-\frac{d}{4}}$$

when d is so small in comparison with λ, that squares and higher powers of d may be neglected.

Substituting the value of λ obtained on p. 544, we find that the speed with which momentum is transmitted forwards by a molecule

$$=V\,.\,\frac{1}{1-\frac{d}{4\lambda}}=V\frac{1}{1-\frac{d}{4}\,.\,\frac{4}{3}\pi d^2N_1}$$

$$=V\frac{1}{1-\frac{N_1\pi d^3}{3}}.$$

To interpret this result, notice that π is only about 5 per cent. greater than 3. If we take $\pi/3=1$, the result just obtained becomes equal to—

$$V\,.\,\frac{1}{1-N_1d^3},$$

and d^3 is the volume of a cube of which each side is equal to d, the diameter of a molecule. If we suppose the gas to be compressed until unit volume contains N_1 molecules piled in the manner represented in Fig. 253, it is clear that each molecule occupies a small cube of volume d^3, and N_1d^3 is the volume occupied by all of the molecules ; that is,

$N_1 d^3$ is equal to unity, and $(1 - N_1 d^3) = 0$. In this case the speed with which momentum is transmitted is infinitely great. It is easily seen that this will be the case when the molecules are in contact as represented in Fig. 253; for the molecules are supposed to possess infinite elasticity, and therefore the momentum communicated to a molecule by an impact on one side, is transmitted instantaneously to the other side of the molecule, and there transmitted to another molecule.

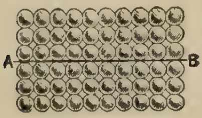

Fig. 253.—Molecules in contact (cubical piling).

Van der Waals's Equation.—Let the cube represented in Fig. 254 be supposed to contain N molecules, the volume of the cube being equal to v; then N_1, the number of molecules per unit volume, is equal to N/v. Let each molecule be of finite size, its diameter being equal to d; then if V is the velocity with which a molecule travels, it follows that the momentum of a molecule is transmitted, on an average, with the velocity—

$$V \cdot \frac{1}{1 - \frac{N\pi d^3}{3v}} = V \cdot \frac{1}{1 - \frac{b}{v}},$$

if $b = N\pi d^3/3$, which is very nearly equal to the volume that would be occupied by the molecules if they were piled in the manner represented in Fig. 253; it is easily proved that $N\pi d^3/3$ is equal to twice the sum of the actual volumes of the molecules.

The velocity of an individual molecule is changed at each collision, but the number of molecules which are travelling with a given velocity in a given direction remains constant. Thus the number of molecules which possess a component velocity $\mathbf{V}_1$ perpendicular to the face A will remain constant; let this number be denoted by n_1. The component momentum of each of these molecules, perpendicular to the face A, is equal to $m\mathbf{V}_1$; and each molecule transmits this momentum at an average speed equal to—

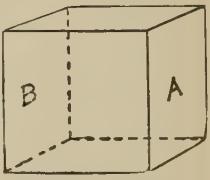

Fig. 254. — Cubical vessel containing a gas.

$$\frac{\mathbf{V}_1}{1 - (b/v)},$$

perpendicular to A.

Let an imaginary plane, parallel to A, be drawn between A and B; then at every instant $n_1/2$ molecules, each possessing the momentum $m\mathbf{V}_1$, will be travelling from right to left perpendicular to this plane, and another $n_1/2$ molecules will be travelling from left to right. Thus, momentum is streaming normally across the plane; if we describe a tube of flow of unit cross-sectional area, the momentum per unit length of this tube, due to the molecules which are travelling from right to left, is equal to $(n_1/2v)m\mathbf{V}_1$, and the velocity of transmission of momentum along the tube is equal to $\mathbf{V}_1/\{1-(b/v)\}$; therefore, (see p. 414) the momentum carried from right to left across unit area of the tube in a second is equal to—

$$\frac{n_1}{2v} \cdot m\mathbf{V}_1 \cdot \frac{\mathbf{V}_1}{1-(b/v)},$$

and an equal amount of momentum is carried from left to right. Both streams of momentum produce a pressure which tends to separate the gas on opposite sides of the plane; and the value of the pressure due to both streams is equal to—

$$\frac{m}{v} \cdot \frac{n_1\mathbf{V}_1^2}{1-(b/v)}.$$

Now let it be supposed that there are n_2 molecules possessing a component velocity $\mathbf{V}_2$ perpendicular to A, and n_3 molecules possessing a component velocity $\mathbf{V}_3$, &c.; then the resultant pressure on a plane parallel to A is equal to—

$$\frac{m}{v}\frac{1}{\{1-(b/v)\}} \cdot (n_1\mathbf{V}_1^2+n_2\mathbf{V}_2^2+ \ . \ . \ .).$$

By reasoning precisely similar to that used on p. 534, it follows that—

$$n_1\mathbf{V}_1^2+n_2\mathbf{V}_2^2+ \ . \ . \ . \ . = \frac{\mathrm{N}\overline{\mathrm{V}}^2}{3},$$

where N denotes the total number of molecules within the cube, and V^2 denotes the mean square of their velocities. Thus the pressure p of the gas is given by the equation—

$$p = \frac{\mathrm{N}m\overline{\mathrm{V}}^2}{3v\{1-(b/v)\}} = \frac{\mathrm{N}m\overline{\mathrm{V}}^2}{3(v-b)}.$$

This result may also be written in the form—

$$p = \frac{\mathrm{RT}}{v - b},$$

where RT is equal to $\mathrm{N}m\mathrm{V}^2/3$; (compare p. 535.)

With regard to the value of the constant b, the reasoning used above shows that it is equal to twice the sum of the actual volumes of the molecules. According to van der Waals, the value of b should be equal to four times the sum of the actual volumes of the molecules; according to Meyer, the value of b should be $4\sqrt{2}$ times the sum of the actual volumes of the molecules. Neither of these investigators took account of the fact that, in general, only a fraction of the total momentum of a molecule is transmitted forward at an impact; if this partial transmission were neglected in the above investigation, the result obtained would agree with that due to van der Waals.

To account for the properties of gases, it is necessary to assume, not only that the molecules are of finite size, but also that two molecules attract each other when they are separated by a distance less than a certain finite value. Let it be supposed that a molecule exerts a finite attraction on all other molecules within a certain distance c from its centre; then, if an imaginary plane be drawn in the space which contains the gas, the molecules within a distance c from the plane on one side are attracted by the molecules within a distance c from the plane on the other side. The attraction across unit area of the plane gives rise to a tensile stress which is directly proportional to the square of the density of the gas; for if the density of the gas is doubled, the number of attracted molecules on one side of the plane is doubled, and at the same time the number of attracting molecules on the other side of the plane is doubled. Hence the tensile stress may be represented by a/v^2, where a is a constant.

The tensile stress tends to prevent the expansion of the gas, while the dynamical pressure due to the motion of the molecules tends to cause the gas to expand. Hence, if p denotes the pressure exerted on the gas by the containing vessel,—

$$p = -\frac{a}{v^2} + \frac{\mathrm{RT}}{v - b}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\left(p + \frac{a}{v^2}\right)(v - b) = \mathrm{RT} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is the celebrated equation deduced by van der Waals.

Interpretation of van der Waals's equation:—Let it be supposed that a space is occupied by molecules which are very sparsely scattered. In this case the volume v of the space is very large in comparison with b, and therefore b may be neglected. Also the molecules are so far apart that the effects of their mutual attractions may be neglected, that is, a/v^2 may be neglected. In this case—

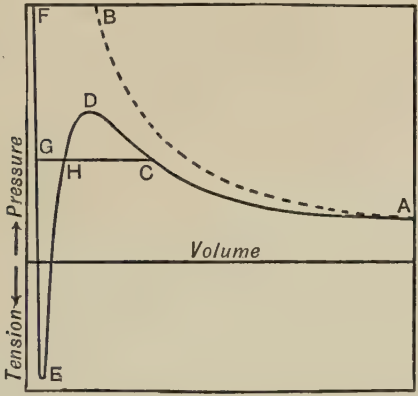

FIG. 255.—Form of the isothermals of a substance, according to van der Waals's equation.

$$p = \mathrm{RT}/v,$$

and

$$pv = \mathrm{RT},$$

so that the substance is a gas, and obeys Boyle's law.

If these conditions held for all volumes, the relation between the pressure and volume, at a constant temperature T, would be represented by the rectangular hyperbola AB (Fig. 255). But a considerable diminution in the space occupied by the substance brings the molecules nearer one to another, and the term (a/v^2) acquires a finite value which may be neglected no longer. In general, this happens before v becomes comparable with b, so that equation (2), p. 552, may be written—

$$p = -\frac{a}{v^2} + \frac{RT}{v}.$$

The curve ACD represented by this equation lies below the rectangular hyperbola AB. Between A and C, the dynamical pressure of the gas is partially neutralised by the tensile stress due to the attraction of the molecules, and the resultant pressure increases to a smaller extent than would be the case if no molecular attraction existed. As the gas is compressed still further, the tensile stress due to the molecular attractions increases still further ; at a certain point D the resultant pressure reaches a maximum value ; if a further compression occurs, the tensile stress becomes of greater importance than the dynamical pressure, and the gas contracts suddenly from D to E. This part of the curve corresponds to an unstable condition of the substance, since the tensile stress increases more and more as the molecules approach each other.

When the volume occupied by the gas has reached a value comparable with b, the resultant pressure p must be obtained from the equation—

$$p = -\frac{a}{v^2} + \frac{RT}{v-b}.$$

A small decrease in the value of v will cause a large increase in the value of the second term on the right-hand side ; in effect, the frequency with which the molecules strike on the walls of the containing vessel is increased by the proximity of the molecules, since these are of finite size ; and therefore a greater pressure is exerted. Thus, there will be some point E on the curve, which represents a minimum resultant pressure ; a further diminution of volume increases the frequency of the impacts to a greater extent than it increases the tensile stress, and thus the curve from E to F slopes upwards steeply, and a very great increase in the resultant pressure is needed to diminish the volume to a small extent. The part EF of the curve corresponds to the liquid state. We may imagine that the liquid, in the condition represented by the point F, is subjected to a diminishing pressure which ultimately becomes a tensile stress; then the point E will correspond to the state of the liquid when it can no longer sustain the tensile stress applied to it, and **the distance of the point E below the axis of volume is equal to the tensile strength of the liquid** (compare p. 282).

The curve ACDEF, as a whole, represents the isothermal relation between the pressure and volume of the substance. From A to D the substance is in the condition of a vapour. At D the vapour becomes unstable, and liquefaction occurs; the part DE of the curve cannot be realised experimentally, since it corresponds to a state in which a diminution of volume produces a diminution of pressure, and this state is essentially unstable. From E to F the substance is in the liquid state. In accordance with the assumptions made in obtaining equation (1), (p. 552), the mean square velocity of the molecules remains constant so long as the temperature is constant, and therefore **at a given temperature, the mean square velocity of the molecules of a substance in the liquid state is equal to the mean square velocity of the molecules of the same substance in the state of vapour.**

The part ACD of the curve refers to a homogeneous vapour which entirely fills the containing vessel; the part EF refers to a homogeneous liquid which entirely fills the vessel. When part of the vessel is filled with liquid, and part with vapour, the liquid and vapour being in equilibrium, the condition of the substance is represented by the horizontal straight line CG. According to Maxwell and Clausius, the line CG must be drawn so as to make the area of the loop CDHC equal to that of the loop HEGH.

At the point C, the vapour has the same pressure as the liquid at G, and the liquid and vapour may remain in the same vessel without condensation or evaporation occurring; thus the point C marks the **saturation point** of the vapour.

From C to D the vapour is supersaturated. A supersaturated vapour cannot remain in equilibrium with the liquid; but when no liquid is present, and there are no particles of dust which might serve as nuclei for condensation, a supersaturated vapour may remain in equilibrium (compare p. 363).

The higher the temperature, the less pronounced becomes the kink CDEG in the isothermal curve. At a certain temperature, called the **critical temperature**, the kink just vanishes; at higher temperatures liquefaction cannot occur, however great the pressure may be. At temperatures higher than the critical temperature, the velocity of the molecules is so great that the dynamical pressure due to their motion cannot be neutralised by the tensile stress due to their mutual attractions.

According to van der Waals, the isothermals of carbon dioxide are represented by the equation—

$$\left(p + \frac{0{\cdot}00874}{v^2}\right)\left(v - 0{\cdot}0023\right) = \frac{1{\cdot}00642\mathrm{T}}{273},$$

where the unit of volume is equal to the volume occupied by one gram of the gas at 0°C under the pressure of one atmosphere, and the

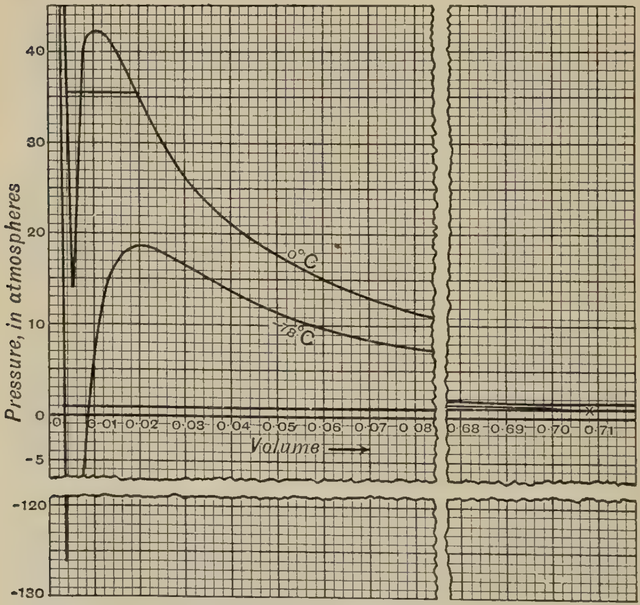

FIG. 256.—Graph of the isothermals for carbon dioxide at 0°C. and −78°C. Plotted from van der Waals's equation.

unit of pressure is one atmosphere. Fig. 256 is a graph of this equation for temperatures 0°C and −78°C. In order to avoid excessive size in the diagram, the parts corresponding to volumes between 0·08 and 0·68, and to negative pressures between −5 and −120 atmospheres, have been omitted.

At −78°C., carbon dioxide vapour commences to liquefy, under favourable conditions, when the pressure is equal to one atmosphere; the point on the curve corresponding to the commencement of liquefaction is marked by a cross, above the volume 0·7087. The curve representing supersaturation slopes upwards, and reaches a maximum height corresponding to 18·84 atmospheres, when the volume is equal to 0·0187. The curve then slopes rapidly downwards, and reaches a minimum value of −126 atmospheres at a volume 0·003807. Thus the maximum tensile strength of liquid carbon dioxide at −78°C. is equal to −126 atmospheres. The curve then becomes very nearly vertical representing the condition of the liquid. The horizontal line, drawn at a height corresponding to one atmosphere, gives the pressure of saturated carbon dioxide vapour at −78°C.

The higher curve refers to carbon dioxide at 0°C. The pressure of the saturated vapour, equal to 35·4 atmospheres, is marked by a horizontal straight line. It will be noticed that at this temperature the curve does not extend beneath the axis of volumes, and therefore the liquid has no tensile strength.

The form of these curves, which have been plotted carefully to scale, lends no support to the theory of Maxwell and Clausius, that the loops CDH and HEG (Fig. 255) are equal in area.

Determination of the constants *a* and *b*.—A method of determining the value of the constant *a*, from experimental data relating to vapour, has been explained already (p. 356). It is very difficult to obtain an accurate value of *b* from data relating to a vapour, since the value of this constant is obtained as the difference between two very large quantities, neither of which can be determined with very great accuracy. The method now to be described suffices to determine the values of both *a* and *b*, in terms of the coefficient of thermal expansion and the coefficient of compressibility of a liquid.

Let a gram of a liquid occupy a volume v_1 at an absolute temperature T_1, when the pressure is equal to p_1; and let the volume change to v_2 when the temperature falls to T_2, the pressure remaining constant. Then—

$$\left(p_1 + \frac{a}{v_1^2}\right)\left(v_1 - b\right) = \frac{T_1}{T_2}\left(p_1 + \frac{a}{v_2^2}\right)\left(v_2 - b\right) \quad . \quad (1)$$

The volume v_1 of the liquid could be changed to v_2, without changing the temperature, by increasing the pressure from p_1 to p_2; then—

$$\left(p_1 + \frac{a}{v_1^2}\right)\left(v_1 - b\right) = \left(p_2 + \frac{a}{v_2^2}\right)\left(v_2 - b\right) \quad . \quad . \quad (2)$$

Subtracting (2) from (1), we obtain the equation—

$$\left\{\left(p_2 + \frac{a}{v_2^2}\right) - \frac{T_1}{T_2}\left(p_1 + \frac{a}{v_2^2}\right)\right\}\left(v_2 - b\right) = 0.$$

Since $(v_2 - b)$ is not equal to zero, it can be divided out, when we obtain the equation—

$$\left(p_2 + \frac{a}{v_2^2}\right) - \frac{T_1}{T_2}\left(p_1 + \frac{a}{v_2^2}\right) = 0 \quad . \quad . \quad . \quad . \quad (3)$$

The quantities p_1, p_2, v_1, v_2, T_1 and T_2 are known, and therefore the value of a can be calculated. On substituting this value of a in (2), the value of b can be calculated.

Let it be required to determine the value of the constants a and b for water.

According to Chappuis[1] the density of water at 20°C. and atmospheric pressure is equal to 0·9982328. Thus for $T_1 = 273 + 20 = 293°$, and $p_1 = 1$, the value of v_1 is equal to $1/0·9982328 = 1·001770$ c.c. At 15°C. and atmospheric pressure, the density of water is equal to 0·9991285; therefore for $p_1 = 1$ and $T_2 = 273 + 15 = 288°$, the value of v_2 is equal to 1·000873 c.c. Thus, the reduction of volume due to cooling a gram of water from 20°C. to 15°C. is equal to 0·000897 c.c.

According to Pagliani and Vincentini,[2] an increase of one atmosphere of pressure on water at 20° produces a compressional strain (diminution of volume per unit volume) equal to $4·44 \times 10^{-5}$. Thus, 1·001770 c.c. of water may be compressed, at the constant temperature 20°C., so that its volume is reduced by 0·000897 c.c., if the pressure is increased by—

$$\frac{8·97 \times 10^{-4}}{1·00177 \times 4·44 \times 10^{-5}} = 20·16 \text{ atmospheres.}$$

Thus $p_2 = 1 + 20·16 = 21·16$ atmospheres.

On substituting these values in equation (3), and solving for a, it is found that $a = 1,164·1$ atmospheres. This value is of the same order of magnitude as that obtained by another method on p. 357.

On substituting the value of a and the other known quantities in equation (2), and solving for b, it is found that $b = 0·954$. This result implies that if a gram of water at 20°C. were compressed until its molecules came into contact as represented in Fig. 253, then the volume of the water would be reduced from 1·00177 c.c. to 0·954 c.c.

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[1] *Annalen der Physik*, Bd. lxiii., p. 207, 1897.

[2] *Nuovo Cimento* [3], 16, p. 27, 1884.

Hence, a molecule of water is not in contact with its neighbours except at the instant when a collision occurs ; but the distance through which a molecule moves between two successive collisions is very small—much smaller, indeed, than the diameter of a molecule.

A gram of aqueous vapour at 20°C. occupies a volume of 57,309 c.c. ; hence, if we suppose that a cube is described around each molecule, and the length of an edge of this cube is taken as unity, then the average distance from centre to centre of neighbouring molecules in the state of saturated vapour at 20°C. is equal to 38·6 units.

Since water molecules are so closely packed, it might be inferred that a molecule could not travel from place to place. It must be remembered, however, that a molecule moves with a very great velocity in the intervals between collisions; the average velocity of a molecule of water at 0°C. is equal to about 60,000 cm. per second, or 1,300 miles per hour. Further, the distribution of the molecules must alter from instant to instant, and occasionally openings, through which molecules can pass, are sure to be formed. Experiments show that molecules do travel from place to place through a liquid, but their rate of progression is exceedingly slow—much slower than the rate at which the extremity of the minute hand of a watch moves.

When a liquid is contained in a beaker, the weight of the liquid is borne by the bottom of the beaker. The molecules are not resting one on another, but are continually in motion ; when a molecule moves downwards, its speed is slightly accelerated by gravity ; when it moves upwards, its speed is diminished. Thus, the molecules comprised in any horizontal layer strike those beneath them with a slightly greater velocity than that with which they strike the molecules above, and therefore the force exerted on the molecules beneath exceeds the force exerted on the molecules above. Thus the pressure increases with the depth below the surface of the liquid, and this increase of pressure produces a small compression, which suffices to increase the number of impacts per second, so that the pressure is exerted in all directions—on the sides as well as on the bottom of the beaker.

The Molecular Structure of Gases.

Number of molecules in unit volume of a gas.—The number of molecules comprised in unit volume of a gas at a definite pressure has been determined indirectly from the results obtained in an electrical experiment due to Mr. R. A. Millikan. This method will now be described.

When air, saturated with aqueous vapour, is caused to expand suddenly, a cloud of minute drops of water is formed if dust particles, around which condensation can occur, are present (p. 364). The drops of water descend under the action of gravity, and carry the dust particles with them ; thus, if several expansions are produced one after another, all the dust particles will be removed, and a sudden expansion of moderate amount will no longer produce condensation. Mr. C. T. R. Wilson has found that if the air is now exposed to Röntgen rays, charged particles are produced which act as nuclei, around which condensation can occur ; these charged particles are called **ions.** The size of the drops of water can be determined from the rate at which they fall, under the action of gravity (p. 528) ; if V denotes the terminal velocity of a drop of radius r, while η denotes the coefficient of viscosity of the surrounding air, then—

$$V = \frac{2}{9}\frac{gr^2}{\eta},$$

since the density of the water is equal to unity, and the density of the air is so small that it may be neglected.

The mass M of a drop of water is given by the equation—

$$M = \frac{4}{3}\pi r^3 = 9\sqrt{2} \, . \, \pi\left(\frac{\eta}{g}V\right)^{\frac{3}{2}} . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Now let it be supposed that a cloud is formed in the manner just described, between two horizontal metal plates placed one above the other ; and let the plates be connected to an electrical battery, so that they are maintained at constant potentials. If a line is drawn normally from one plate to the other, the fall of potential per unit length of this line is equal to the force that would act on an electrical charge of one unit value, placed between the plates ; let the value of this force be denoted by ϕ. Let the upper plate be positive, and let it be supposed that a drop of water is formed around a negatively charged particle, the value of the charge being equal to $-q$ units. Then the electrical charge will be pulled upwards with a force equal to ϕq dynes, and pulled downwards with a force equal to Mg dynes, where M denotes the mass of the particle. If the two forces are equal, the drop will remain stationary; if the forces are unequal, the drop will move upwards or downwards according as the electrical or the gravitational force is the stronger. Thus, after a short time the space between the plates will be cleared of all drops, except those for which $\phi q = Mg$; on disconnecting the plates from the battery, the electrical force ceases to act, and the drops which have not been removed fall under the action of gravity. The terminal velocity V with which they fall can be observed by viewing them through a microscope; and when V is known, the value of the mass M of each drop becomes known from equation (1). Then—

$$q = Mg/\phi,$$

and since ϕ is known, the value of the charge q can be calculated.

Experiments in which the plates are maintained at different potentials lead to different values of q; but the values obtained are related in a very simple manner. The smallest value of q is equal to $1{\cdot}55 \times 10^{-19}$ coulomb; q may also be equal to an integral multiple of this value. This result indicates that **an electrical charge is not infinitely divisible; $1{\cdot}55 \times 10^{-19}$ coulomb is the smallest charge obtainable, or the atom of electricity.**

When a solution is electrolysed, it is found that no atom carries a charge smaller than that carried by an atom of hydrogen; hence it may be assumed that the hydrogen atom carries the minimum charge of $1{\cdot}55 \times 10^{-19}$ coulomb. There is considerable evidence in favour of this assumption, but a full discussion of the question would require more space than can be spared here.

Now, when water is electrolysed, each coulomb of electricity that passes through the water liberates $1{\cdot}045 \times 10^{-5}$ gram of hydrogen; thus, one coulomb of electricity is associated with $1{\cdot}045 \times 10^{-5}$ gram of hydrogen; therefore the minimum charge of $1{\cdot}55 \times 10^{-19}$ coulomb must be associated with $(1{\cdot}55 \times 10^{-19}) \times$

$(1·045 \times 10^{-5}) = 1·62 \times 10^{-24}$ gram of hydrogen, and this must be the **mass of an atom of hydrogen.** The molecule of hydrogen comprises two atoms, and therefore its mass is equal to $3·24 \times 10^{-24}$ gram. The masses of molecules of other substances can be calculated without difficulty; thus a molecule of water is 9 times as heavy as a molecule of hydrogen, and therefore its mass is equal to $2·92 \times 10^{-23}$ gram.

A gram of hydrogen, at 0°C. and atmospheric pressure, occupies a volume of $1·115 \times 10^4$ c.c.; therefore a cubic centimetre of hydrogen at 0°C. and atmospheric pressure weighs $8·96 \times 10^{-5}$ gram, and the **number of molecules** comprised in this mass is equal to $(8·96 \times 10^{-5}) \div (3·24 \times 10^{-24}) = 2·76 \times 10^{19}$. Since equal volumes of all permanent gases comprise equal numbers of molecules under similar conditions of temperature and pressure (p. 536), it follows that **a cubic centimetre of any permanent gas, at 0°C. and atmospheric pressure, comprises $2·76 \times 10^{19}$ molecules.**

The following comparison will help the student to form some idea of the enormous number of molecules comprised in a cubic centimetre of gas under standard conditions. The area of the surface of the earth (both dry land and sea) is equal to about $1·97 \times 10^8$ sq. miles, and about a quarter of this area, or $4·9 \times 10^7$ sq. miles, is dry land. An ordinary builder's brick is 8·75 inches long, 4 inches wide, and 2 inches thick. If $2·76 \times 10^{19}$ of these bricks were piled uniformly over the $4·9 \times 10^7$ sq. miles of the earth's dry surface, they would reach a height of 852 ft.; that is, to more than twice the height (404 ft.) of the cross of St. Paul's Cathedral.

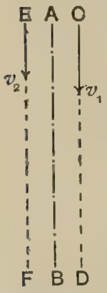

FIG. 257. Theory of the viscosity of a gas.

Viscosity of a gas.—Let AB (Fig. 257) represent the trace of a plane drawn perpendicular to the plane of the paper; and let it be supposed that the plane AB is described in the interior of a gas. If a molecule were projected from any point in AB, into the gas on either side of that plane, it would, on an average, travel a distance λ before it collided with another molecule; and since λ denotes the average distance traversed between two successive collisions (p. 538), it follows that a molecule which passes normally through the plane AB must, on an average, have travelled over a distance λ since its last collision.

We may suppose that the actual gas is replaced by three groups of molecules, one group moving with velocity V normally to AB, and the other two groups moving, in mutually perpendicular directions, parallel to the plane AB (p. 534). The molecules moving parallel to AB do not cross that plane. If N_1 denotes the number of molecules comprised in unit volume of the actual gas, each of the groups of molecules defined above will comprise $N_1/3$ molecules. Therefore, at any instant, the number of molecules per unit volume that are travelling from right to left, in a direction normal to AB, is equal to $N_1/6$; and the number of molecules per unit volume travelling from left to right is also equal to $N_1/6$.

The number of molecules that, in a second, cross unit area of AB, moving from right to left, is equal to the number of molecules moving in the direction specified, that are comprised in a rectangular prism of unit cross-sectional area and length V (p. 378); that is, to $(N_1/6)$ V. An equal number of molecules crosses unit area of AB in a second, moving from left to right.

Let two planes, represented by the traces CD and EF, be drawn parallel to, and on opposite sides of AB, the distance between either plane and AB being equal to λ, the mean free path of the molecules. Then, on an average, $(N_1/6)$ V molecules cross unit area of AB in a second, moving from one side to the other; and each of these molecules suffered its last collision in one of the planes drawn at a distance λ from AB.

Now let it be supposed that the gas in the immediate neighbourhood of the plane CD, is moving bodily, with velocity v_1, in the direction of the arrow drawn from C; and let the gas in the immediate neighbourhood of the plane EF be moving bodily, with a smaller velocity v_2, in the direction of the arrow drawn from E. Then if the bodily velocity of the gas changes continuously from point to point of a line drawn perpendicular to AB, the velocity gradient v' of the gas in the neighbourhood of the plane AB, is equal to $(v_1 - v_2)/2\lambda$. It is supposed that the bodily velocity of the gas is very small in comparison with the molecular velocity V.

Next, let it be assumed that when a molecule collides in any plane drawn parallel to AB, it acquires the velocity with which the gas in that plane is moving bodily, in addition to its molecular velocity V. Then the $(N_1/6)$ V molecules which, in a second, cross unit area of AB from right to left, all possess the velocity v_1 acquired during their collisions in the plane CD; hence the momentum, in the direction from A to B, carried across the plane AB in a second, is equal to $(N_1/6)V \times mv_1$, and this transfer of momentum is equivalent to a tangential stress equal to $(mN_1/6)Vv_1$, exerted on the gas to the left of the plane AB by the gas to the right of that plane. The $(N_1/6)V$ molecules which, in a second, cross AB from left to right, all possess the velocity v_2 acquired during their collisions in the plane EF; and the momentum, directed from A to B, which is transferred by these molecules across unit area of AB in a second, produces a stress equal to $(mN_1/6)Vv_2$, exerted by the gas on the left of AB on the gas to the right of that plane; an equal but opposite stress acts on the gas to the left of AB, since action and reaction are equal and opposite. Thus, the gas to the left of AB is acted upon by the tangential stress $(mN_1/6)Vv_1$, directed from A to B, together with the tangential reaction stress $(mN_1/6)Vv_2$, acting from B to A. Hence, the resultant tangential stress exerted on unit area of AB is equal to—

$$\frac{mN_1V}{6}(v_1 - v_2).$$

The coefficient of viscosity η of the gas is equal to the ratio of the tangential stress to the velocity gradient (p. 488); and the velocity gradient in the neighbourhood of the plane AB is equal to $(v_1 - v_2)/2\lambda$. Thus—

$$\eta = \frac{mN_1V}{6} \cdot (v_1 - v_2) \div \frac{v_1 - v_2}{2\lambda}$$

$$= \frac{mN_1V\lambda}{3}.$$

In this equation, N_1 denotes the number of molecules per unit volume, and m denotes the mass of each molecule. Therefore N_1m is equal to the density ρ of the gas, and—

$$\eta = \frac{\rho V\lambda}{3}.$$

It was proved on p. 539 that the mean free path λ is inversely proportional to the density of the gas, and therefore $\rho\lambda$ is independent of the density. Thus **the value of η, for a given gas, is**

independent of the pressure to which the gas is subjected. This result, which has been alluded to previously (p. 514), was predicted by Maxwell from the results of an investigation similar to that just carried out; its subsequent experimental verification has always been regarded as affording strong evidence for the truth of the kinetic theory of gases.

The law that the viscosity of a gas is independent of the density, holds only within certain limits, which may be inferred from the method used in deducing the law. Thus, **if a gas is being forced through a capillary tube, the pressure gradient necessary to drive a given volume through in a second is independent of the density of the gas, so long as the mean free path of the molecules is small in comparison with the radius of the tube.**

The **value of** λ can now be calculated. At 0°C., the value of η for hydrogen is equal to $8{\cdot}64 \times 10^{-5}$ gm./cm. sec. The value of V, the root-mean-square velocity of hydrogen at 0°C., is equal to $1{\cdot}84 \times 10^{5}$ cm. per sec.; and the density of hydrogen, at 0°C. and atmospheric pressure, is equal to $8{\cdot}96 \times 10^{-5}$ gm./(cm.)^3. Thus—

$$\lambda = \frac{3\eta}{\rho V} = \frac{3 \times 8{\cdot}64 \times 10^{-5}}{8{\cdot}96 \times 10^{-5} \times 1{\cdot}84 \times 10^{5}}$$

$$= 1{\cdot}57 \times 10^{-5} \text{ cm.}$$

The **number of collisions** suffered by a hydrogen molecule in a second, when the temperature of the hydrogen is 0°C. and its pressure is one atmosphere, is equal to—

$$(1{\cdot}84 \times 10^{5}) \div (1{\cdot}57 \times 10^{-5}) = 1{\cdot}17 \times 10^{10}.$$

The **diameter** d **of a hydrogen molecule** can be determined from the equation for the mean free path, obtained on p. 544.

$$\lambda = \frac{3}{4} \cdot \frac{1}{N_1 \pi d^2};$$

therefore

$$d = \frac{1}{2}\sqrt{\frac{3}{N_1 \pi \lambda}},$$

where N_1 has the value determined on p. 562. Thus—

$$d = \frac{1}{2}\sqrt{\frac{3}{2{\cdot}76 \times 10^{19} \times 3{\cdot}14 \times 1{\cdot}57 \times 10^{-5}}}$$

$$= 2{\cdot}3 \times 10^{-8} \text{ cm.}$$

Some idea of the size of a molecule of hydrogen can be formed by the following method of comparison. The mean diameter of the earth is equal to $1{\cdot}27 \times 10^9$ cm. or, in round numbers, 10^9 cm. Thus, if hydrogen, contained in a spherical vessel one centimetre in diameter, were magnified so that the vessel appeared to be as large as the earth, each molecule of hydrogen would appear to be 23 cm. in diameter, that is, a little larger than a man's head.

Thermal conductivity of a gas.—Let the temperature of a gas fall along a given direction; and let the fall of temperature per unit length be called the temperature gradient. Then, if a plane be drawn perpendicular to the direction along which the temperature gradient is measured, the quantity of heat transmitted per second across unit area of the plane is proportional to the temperature gradient. **The ratio of the quantity of heat transmitted per second across unit area, to the temperature gradient, is called the coefficient of thermal conductivity of the gas.** This constant will be denoted by κ. Let s_v denote the specific heat of a gas at constant volume; then s_v is the thermal equivalent of the increase in the total kinetic energy of one gram of the gas, due to a rise of temperature of 1°C. If unit mass of the gas comprises N_1 molecules, each of mass m, it follows that the thermal equivalent of the total kinetic energy of a molecule at the absolute temperature T is equal to ms_vT. Now let AB (Fig. 258) represent the plane across which heat is being transmitted, and let the temperatures of the gas at the planes CD and EF be denoted by T_1 and T_2; then, if each of these planes is at a distance λ from AB, it follows that the temperature gradient in the neighbourhood of AB is equal to $(T_1 - T_2)/2\lambda$.

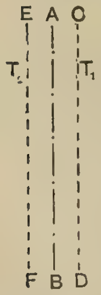

Fig. 258. Theory of the thermal conductivity of a gas.

Let the molecules be divided into groups as before, and let it be assumed that each of the $(N_1/6)V$ molecules which, in a second, cross unit area of the plane AB from right to left, possesses the kinetic energy corresponding to the temperature T_1; while each of the equal number of molecules which cross from left to right possesses the kinetic energy corresponding to the temperature T_2. The temperatures T_1 and T_2 are nearly equal, and therefore the rate at which the molecules cross the plane is scarcely affected by the slightly different values of the translational velocity corresponding to T_1 and T_2. The thermal equivalent of the energy carried, in a second, across unit area of AB from right to left is equal to—

$$\frac{N_1}{6} V \, . \, ms_v T_1,$$

and the thermal equivalent of the energy carried from left to right is equal to—

$$\frac{N_1}{6} V \, . \, ms_v T_2,$$

so that the net amount of heat carried, in a second, from right to left across unit area of AB, is equal to—

$$\frac{N_1}{6} V ms_v \, . \, (T_1 - T_2).$$

When this quantity is divided by the temperature gradient $(T_1 - T_2)/2\lambda$, the value of κ, the coefficient of thermal conductivity is obtained. Thus—

$$\kappa = \frac{N_1}{6} \, . \, V ms_v (T_1 - T_2) \div \frac{T_1 - T_2}{2\lambda}$$

$$\therefore \quad \kappa = \frac{N_1 m}{3} V s_v \lambda.$$

Further, $N_1 m = \rho$, the density of the gas; for N_1 is the number of molecules comprised in unit volume, and m is the mass of each molecule. Thus—

$$\kappa = \frac{\rho V s_v \lambda}{3} = s_v \eta,$$

where η is the coefficient of viscosity of the gas. It is very difficult to determine the value of κ directly, but the above investigation shows that its value can be calculated in terms of the coefficient of viscosity and the specific heat at constant volume. Also, it follows that **the coefficient of thermal conductivity is independent of the density of the gas.**

Diffusion of gases.—Let the lower half of a vessel be filled with a heavy gas, such as oxygen, and the upper half with a light gas, such as hydrogen; then, if the vessel is closed and left undisturbed, it will be found that ultimately the gases become mixed uniformly throughout the vessel. This result cannot be due to gravity, for the heavier gas has risen and the lighter gas has descended; it must be ascribed to the motion of the gaseous molecules.

In the following investigation, the effect of gravity will be ignored. It will be assumed that the molecules of both gases move with velocities equal to V, and that their mean free paths are both equal to λ. The results obtained will be approximately true when the molecular weights of the two gases are nearly equal.

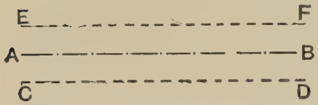

FIG. 259.—Theory of the diffusion of a gas.

Let AB (Fig. 259) be the trace of a plane perpendicular to the plane of the paper; and let CD and EF be the traces of parallel planes described on opposite sides of AB at distances equal to λ from that plane, where λ is the mean free path common to the two gases. In the neighbourhood of the plane CD, let there be N_1 molecules of one gas X per unit volume, and n_1 molecules of another gas Y per unit volume, while in the neighbourhood of EF, let there be N_2 molecules of X per unit volume, and n_2 molecules of Y per unit volume.

By reasoning similar to that used in connection with the determination of the viscosity of a gas, the total number of molecules of X that pass upwards through unit area of AB in a second is equal to $VN_1/6$, and the total number that pass downwards is equal to $VN_2/6$. The net number of molecules of X that pass upwards through unit area of AB in a second is thus equal to $V(N_1 - N_2)/6$.

The mass of the gas X contained in unit volume is called the **concentration** of that gas; and the change of concentration per unit distance is called the **concentration gradient** of the gas. The concentration gradient, c_1, of X is thus equal to $m_1(N_1 - N_2)/2\lambda$, where m_1 is the mass of a molecule of the gas X. The mass of the gas X that passes upwards through unit area of AB in a second is equal to—

$$\frac{m_1(N_1 - N_2)V}{6} = \frac{c_1 V}{6} \cdot 2\lambda = \frac{c_1 V\lambda}{3};$$

thus, **the rate of diffusion is proportional to the concentration gradient**

c_1. The ratio of the mass of gas that diffuses per second through unit area, to the concentration gradient, is called the **coefficient of diffusion**; let this be denoted by Δ. Then—

$$\Delta = \frac{V\lambda}{3}.$$

The mean free path varies inversely as the density (p. 539), and at a given pressure the density varies inversely as the absolute temperature T. Thus, at a given pressure, λ varies directly as T. The root-mean-square velocity V varies as $\sqrt{T}$, and therefore at a given pressure the coefficient of diffusion Δ varies as $T^{\frac{3}{2}}$. Loschmidt and von Obermayer have found that Δ varies as a power of T somewhat greater than (3/2).

The radiometer.—The radiometer invented by Sir William Crookes is represented in Fig. 260. Four aluminium vanes are mounted on a light framework which is pivoted centrally on a needle point. One surface of each vane is blackened, while the other surface is left bright. The vanes are arranged so that if the blackened surfaces are seen on the right-hand side of the instrument, the bright surfaces are seen on the left-hand side (Fig. 260). The vanes, mounted in the manner described, are enclosed in a glass vessel which is highly exhausted. When a hot body is placed near to the instrument, the vanes revolve; their direction of motion indicates that the blackened surfaces are repelled from the hot body. Although a high degree of exhaustion is needed before the vanes will revolve, a perfect vacuum is not required. Indeed, in a perfect vacuum the true radiometer action ceases; this circumstance indicates that the motion of the vanes is due to some action exercised by the highly attenuated gas in the enclosing vessel.

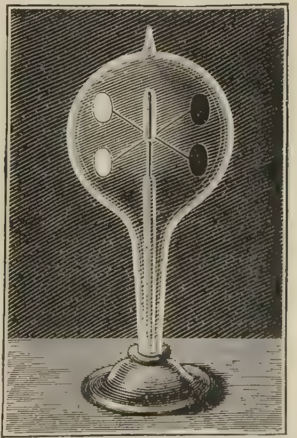

FIG. 260.—Sir William Crookes's radiometer.

The elementary theory of the radiometer is as follows. The radiation from the hot body is absorbed by the black surfaces, while most of it is reflected from the bright surfaces. Thus the black surfaces become hot, and the molecules rebound from them with augmented velocities. The effect is the same as if equal numbers of molecules were projected from both surfaces of a vane, those projected from the black surface having the greater velocities. The recoil due to the projection of a molecule from the black surface of a vane, is thus greater than that due to the projection of a molecule from the bright surface ; hence, the black surfaces are subjected to a pressure which is greater than that exerted on the bright surfaces.

This theory is incomplete, since it gives no explanation of the fact that the vanes will not revolve unless the gas surrounding them is highly attenuated. The complete theory of the radiometer, due to Prof. Osborne Reynolds, must now be discussed briefly.

Let AB (Fig. 261) represent the section of a vane of which the width is great in comparison with the mean free path of the surrounding gas ; and let the blackened surface of the vane be uppermost. Let CD represent an imaginary plane described parallel to the blackened surface of AB, and at a distance λ from it ; where λ denotes the mean free path of the surrounding gas. Molecules will be projected in various directions from any part of the surface AB, and will collide, in the plane CD, with molecules of the surrounding gas. Since the width of the surface AB is great in comparison with λ, the force exerted on an element of area of AB, due to the projection of molecules from that element, will be exactly equal to the force exerted on an equal element of the plane CD, due to the molecules that collide in that element. The gas, on the side of the plane CD remote from AB, will be

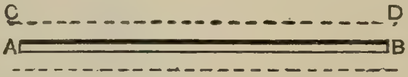

FIG. 261.—Vane of which the width is great in comparison with the mean free path of the surrounding gas.

driven away from AB with a pressure exactly equal to that exerted on AB; consequently the gas between AB and CD must be more rarefied than that on the bright side of the vane, and the number of impacts on AB must be less than that on the bright surface. Thus, a small number of molecules are projected with high velocity from the black face of the vane, while a larger number of molecules are projected with smaller velocity from the bright face. It is evident that the pressures exerted on both faces of the vane must be equal; for if the pressure on the black face were greater than that on the bright one, AB (Fig. 261) would move downwards, and the gas above CD would have to follow, in order to prevent the formation of a vacuum between AB and CD; and in this case the vane would be set in motion by the action of forces which react on the gas above the plane CD, and the action and reaction would both produce motions in the same direction. This, of course, is impossible; for the effect of the reaction must always be opposite to that of the action (p. 24).

Let it now be supposed that the gas surrounding the vane is so attenuated that the mean free path of the molecules extends right up to the walls of the enclosing vessel; in this case the reaction is exerted on the fixed walls, and therefore the action can set the vanes in motion; it is clear that the gas on the black side of a vane cannot be rarefied by the impact of its molecules on the walls of the containing vessel.

Prof. Schuster suspended a radiometer by a fibre which acted as a torsional control, and found that when the vanes revolved in one direction under the action of the radiation received from a neighbouring hot body, the enclosing vessel suffered an angular deflection in an opposite direction about its suspension.

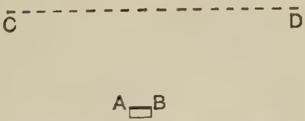

FIG. 262.—Vane of which the width is small in comparison with the mean free path of the surrounding gas.

Now let it be supposed that the width AB (Fig. 262) of a vane is very small in comparison with the mean free path of the surrounding gas. Molecules will be projected from the black surface *in various directions*, and the reaction, due to their collisions in the plane CD, will be distributed over an area far greater than that of AB. Hence, the pressure (force per unit area) exerted on the gas above the plane CD will be far less than that exerted on AB, and AB will tend to move away from CD. Thus, Prof. Osborne Reynolds found that a very fine spider thread, suspended in a vessel containing air at a pressure of about a third of an atmosphere, was repelled from a neighbouring hot body. Sir Oliver Lodge drew attention to an instance of a similar repulsion, which produces effects that most people must have observed. When a hot-water pipe is fixed at some distance from a wall, a black deposit is always formed on the part of the wall which faces the pipe. This is due to the radiometric repulsion of minute dust particles suspended in the air; these particles are driven away from the hot pipe to the cold wall, just as, in a partial vacuum, the vanes of a radiometer are driven away from a neighbouring hot body.

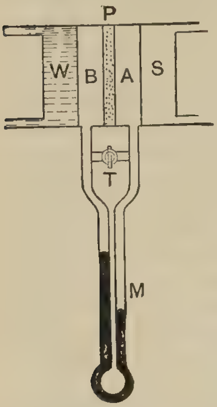

FIG. 263.—Apparatus for the observation of thermal transpiration.

Thermal transpiration.—Let it be supposed that the small vane AB (Fig. 262) is fixed. When its blackened surface is heated, the projected molecules that collide in the imaginary plane CD produce a pressure that drives the gas away from AB. Thus the gas will stream past AB, from the cold to the hot side.

Next, let it be supposed that a great number of minute vanes are arranged in a plane, with their blackened surfaces all turned in one direction: then, when the blackened surfaces are heated, the gas will stream through the plane, from the cold to the hot side. But the numerous minute vanes arranged in a plane constitute a perforated diaphragm: hence, we conclude that **if a porous diaphragm is surrounded by gas, and one side of the diaphragm is heated, the gas will stream through the diaphragm from the cold to the hot side.**

Prof. Osborne Reynolds observed this effect, by means of the experimental arrangement represented diagrammatically in Fig. 263. Two vessels, A and B, are separated by a porous diaphragm P of meerschaum. The two vessels are connected to the ends of the mercury manometer M. The side of the vessel A remote from P is heated by steam led through the chamber S; the side of the vessel B remote from P is cooled by tap water led through the chamber W. Thus the side of the diaphragm which faces the steam chamber S is heated by radiation, and the other side of the diaphragm remains cold. Gas streams through the diaphragm from the cold vessel B to the hot vessel A, and the manometer indicates that the pressure of the gas in A becomes much greater than that of the gas in B. The pressures in the two vessels can be equalised by opening the stop-cock T; but directly this stop-cock is closed, the pressure in A commences to rise above that in B, and finally a definite difference of pressure is established.

The final difference of pressure between the gas in the vessels A and B can be calculated as follows. Let there be N_1 molecules per unit volume in the hot vessel A, and let the root-mean-square velocity of these molecules be denoted by V_1; and let N_2 and V_2 represent the number of molecules per unit volume, and the root-mean-square of their velocities, with respect to the cold vessel B. Then by reasoning similar to that used on p. 563, the number of molecules that stream through unit area of the porous diaphragm in a second, from the hot to the cold side, is equal to $(N_1/6)\,V_1$; and the number that stream from the cold to the hot side is equal to $(N_2/6)\,V_2$. So long as the number of molecules that stream in one direction through unit area of the diaphragm in a second, is different from the number that stream in the opposite direction, the difference of pressure between the vessels A and B will increase; the difference of pressure will become constant when—

$$\frac{N_1}{6}V_1 = \frac{N_2}{6}V_2,$$

that is, when $N_1V_1 = N_2V_2$.

Now let p_1 denote the final steady pressure of the gas in A, while p_2 denotes the pressure of the gas in B. Then (p. 534)—

$$p_1 = \frac{mN_1}{3}V_1^2,$$

$$p_2 = \frac{mN_2}{3}V_2^2;$$

$$\therefore \frac{p_1 - p_2}{p_2} = \frac{(N_1V_1)V_1 - (N_2V_2)V_2}{(N_2V_2)V_2}$$

$$= \frac{V_1 - V_2}{V_2},$$

since $N_1V_1 = N_2V_2$. Also, if T_1 and T_2 denote the absolute temperature of the air in the vessels A and B respectively—

$$\frac{V_1^2}{V_2^2} = \frac{T_1}{T_2};$$

$$\therefore \quad \frac{p_1 - p_2}{p_2} = \frac{\sqrt{T_1} - \sqrt{T_2}}{\sqrt{T_2}}.$$

This law was verified by Prof. Osborne Reynolds.

OSMOTIC PRESSURE.

Semi-permeable membranes.—If a pig's bladder, filled with alcohol, be tightly closed and then immersed in water, the bladder commences to swell and may ultimately burst. On the other hand, if a pig's bladder, filled with water, be immersed in alcohol, the bladder shrinks. These phenomena are due to the circumstance that water can pass through the substance of the bladder, but alcohol cannot. A membrane which will allow some substances, but not others, to pass through it, is called a **a semi-permeable membrane.** The selective transmission of a liquid through a semi-permeable membrane is called **osmosis.**

If some grocer's currants, in their shrunken and dried-up condition, are placed in water, they soon commence to swell, and finally become globular. In this case the organic substances, contained within the currants, cannot pass outwards into the water, but the water can pass into the interior of the currants. This is a familar instance of osmosis

Osmotic pressure.—The laws of osmosis were first systematically studied by Pfeffer. This investigator filled a porous pot (similar to that used in many electric cells) with cupric sulphate solution, and immersed the cell in a dilute solution of potassium ferro-cyanide. A membrane of cupric ferro-cyanide was formed in the perforations of the cell wall, where the solutions came into contact; and this membrane was found to be permeable to water, but impermeable to many substances, such as sugar, which can be dissolved in water. Accordingly, the cell was washed and filled with an aqueous solution of sugar, and its mouth was closed with a well-fitting stopper pierced to receive a long vertical tube (Fig. 264). The cell was then immersed in pure water, and it was found that the solution rose in the vertical tube. From this it may be concluded that water passed through the semi-permeable membrane into the cell. When the liquid attained a certain height in the vertical tube, no more water entered the cell; the hydrostatic pressure due to the head of liquid prevented the further inflow of water. The internal pressure which will just prevent the inflow of water is called the **osmotic pressure** of the solution contained in the cell. This pressure may be very great in the case of concentrated solutions, and therefore it is best to measure it by means of a mercury manometer; in this case the solution becomes less diluted, since only a small volume of water enters the cell before the maximum internal pressure is attained.

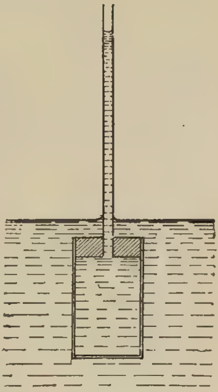

FIG. 264.—Apparatus for the observation of osmotic pressure.

Pfeffer found that, for dilute sugar solutions at a constant temperature, **the osmotic pressure is proportional to the concentration of the solution**; in other words, the osmotic pressure is proportional to the mass of sugar dissolved in each unit of volume of the solution. Thus, if the internal volume of the cell is equal to v, and m grams of sugar are dissolved in the contained water, the osmotic pressure p is given by the equation—

$$p = k\frac{m}{v},$$

where k is a constant depending only on the temperature of the solution. Thus—

$$pv = km,$$

a relation similar to that which obtains between the pressure and volume of a given mass of a perfect gas at a constant temperature.

Van't Hoff analysed Pfeffer's experimental results, and found that **for a given concentration of the solution, the osmotic pressure**

is proportional to the absolute temperature. Thus, the relation between the osmotic pressure p, the volume v in which a given mass of the sugar is dissolved, and the absolute temperature T, is given by the equation—

$$pv = \mathbf{R}T.$$

Further, it was found that **if a gram-molecule of sugar is dissolved in the volume v, the constant R has the same value as the corresponding constant for a gram-molecule of a perfect gas** (p. 537). This relation holds so long as the solution is dilute; that is, so long as the volume v in which the gram-molecule of sugar is dissolved, is very large.

Thus it may be stated that **the osmotic pressure of a sugar solution is equal to the pressure that would be exerted if the sugar were distributed, in the form of a gas, in an otherwise empty space having the same volume as the solution.**

With regard to solutions of substances other than sugar, the same law holds if a suitable semi-permeable membrane can be obtained. The cupric ferro-cyanide membrane, which is impermeable to sugar, is partially permeable to potassium nitrate, and therefore cannot be employed in connection with solutions of that salt. Further, many substances when dissolved in water, form electrically conducting solutions; in such cases the substances are dissociated more or less completely, and the osmotic pressure is equal to the pressure that would be exerted by the substance, dissociated to the same extent, in the state of a gas.

Explanation of the action of a semi-permeable membrane.—Before any attempt can be made to frame a dynamical theory of osmotic pressure, some definite idea must be formed as to the way in which a semi-permeable membrane acts.

It is known that, at high temperatures, hydrogen can be absorbed by palladium, while other gases, such as nitrogen, cannot be absorbed. Acting on this knowledge, Sir William Ramsay filled a heated palladium vessel with nitrogen, and immersed the vessel in an atmosphere of hydrogen. He found that hydrogen passed into the vessel, until the internal pressure rose to $(p_1 + p_2)$, where p_1 denotes the initial pressure of the nitrogen, and p_2 denotes the pressure of the surrounding hydrogen.

A clear idea of the way in which this result is brought about, may be formed as follows. Let it be supposed that a certain fraction, k, of the hydrogen molecules which strike on the palladium vessel, pass into the palladium ; and that the rate at which hydrogen molecules escape from either surface of the vessel is equal to the average rate at which they are absorbed by both surfaces. Now, let there he N_1 molecules of hydrogen per unit volume outside the vessel, and N_2 molecules of hydrogen per unit volume inside the vessel; the temperature inside the vessel is equal to that outside, and therefore the velocity V_1 of the hydrogen molecules is the same inside and outside. From reasoning similar to that employed on p. 563, the number of molecules of hydrogen that strike on unit area of the external surface of the vessel in a second is equal to $(N_1/6)V_1$, and the number absorbed is equal to $k(N_1/6)V_1$. Similarly, the number of molecules of hydrogen absorbed by unit area of the internal surface of the vessel is equal to $k(N_2/6)V_1$. Therefore the total number of molecules of hydrogen absorbed during a second, by one sq. cm. of the walls of the vessel, is equal to $(kV_1/6)(N_1+N_2)$, and the number of molecules that escape from a sq. cm. of the internal surface is equal to half this value, viz., $(kV_1/12)(N_1+N_2)$, while an equal number escapes from a sq. cm. of the external surface. Thus, the interior of the vessel is gaining hydrogen molecules at the rate of $(kV_1/12)(N_1+N_2)$ per sec. per sq. cm. of the internal surface of the vessel, and losing hydrogen molecules at the rate of $(kV_1/6)\ N_2$ per sec. per sq. cm. of the internal surface. When a steady state is reached, the number of molecules gained must be equal to the number lost ; therefore—

$$\frac{kV_1}{12}(N_1+N_2) = \frac{kV_1}{6} . N_2 ;$$

$$\therefore\ N_1+N_2=2N_2, \text{ and } N_1=N_2.$$

Thus, in the steady state, the concentration of the hydrogen inside the vessel is equal to that outside. Let the number of nitrogen molecules per c.c inside the vessel be equal to n, while the mass of each of these molecules is equal to m, and its mean-square velocity to V ; then (p. 534) the impacts of the nitrogen molecules on the internal surface of the vessel produce a pressure p_1 equal to $(nm/3)V^2$; while the impacts of the hydrogen molecules produce a pressure p_2 equal to $(N_1m_1/3)V_1^2$, where m_1 denotes the mass of a hydrogen molecule. The total internal pressure is equal to—

$$p_1+p_2=\frac{nm}{3}V^2+\frac{N_1m_1}{3}V_1^2.$$

Thus, the heated palladium plays the part of a membrane, permeable to hydrogen but not to nitrogen. If a soap-bubble, blown with air, is immersed in ether vapour, the bubble swells owing to the ether which dissolves in the walls of the bubble, and escapes into the interior of the bubble; the soap bubble forms a membrane, permeable to ether vapour but not to air.

In order to explain the action of the cupric ferro-cyanide membrane used by Pfeffer, it is only necessary to assume that water molecules can pass into this membrane, while sugar molecules cannot. Owing to the close packing of the water molecules, the number of impacts per second on unit area of the membrane will be enormously great; let it be supposed that a certain fraction k of the water molecules which strike unit area of either surface of the membrane in a second are absorbed, while water molecules escape from both surfaces as quickly as they are absorbed into the membrane. Let it be supposed that n_1 water molecules strike unit area of the external surface per second, while $(n_1 - n_2)$ strike unit area of the internal surface; where n_2 denotes the number of sugar molecules that strike unit area of the internal surface per second. This amounts to the assumption that each sugar molecule displaces a water molecule, the total number of molecules of both kinds remaining unchanged. Then the number of water molecules absorbed by a sq. cm. of the membrane in a second is equal to $\{kn_1 + k(n_1 - n_2)\}$, and half this number escapes from unit area of either surface in a second. Thus the sugar solution is losing water molecules at the rate of $k(n_1 - n_2)$ per second per sq. cm. of the internal surface of the membrane, and gaining water molecules at the rate of $\{kn_1 + k(n_1 - n_2)\}/2 = k\{n_1 - (n_2/2)\}$. The net rate at which water molecules are gained by the sugar solution is equal to $kn_2/2$; in other words, **the rate at which water molecules pass through the membrane into the sugar solution, is equal to** $kn_2/2$.

Dynamical theory of osmotic pressure.—So long as the number of water molecules that strike unit area of the internal surface of the membrane in a second is less than the number that strike unit area of the external surface in the same time, so long will water flow into the solution. In the arrangement represented in Fig. 264, the flow of water into the sugar solution must be accompanied by a rise of the solution in the vertical tube, and this rise must produce an increased internal pressure. But an increase in the internal pressure compresses the solution: the compression is very small, far too small to be observed directly; but the molecules are so close together that a very slight compression appreciably increases the number of impacts on the internal surface of the membrane (p. 554), and the increase in the number of impacts causes the inflow of water to diminish. **Water will cease to flow in when equal numbers of water molecules strike on the internal and external surfaces of the membrane in a second.** In this case, the component pressure on the internal surface of the membrane, due alone to the impacts of the water molecules, is exactly equal to the total pressure on the external surface of the membrane; but the impacts of the sugar molecules produce an additional pressure on the internal surface, and this additional pressure is the osmotic pressure of the solution.

So far, the production of osmotic pressure has been accounted for by a train of reasoning which presents no special difficulties. It yet remains to be proved that the pressure due to the impacts of the sugar molecules is identical with that which would be produced if the water molecules were absent, and the sugar molecules were distributed in the state of a gas.

In the first place, it may be assumed that at a given temperature the velocity of motion of a sugar molecule is equal to that of a gas molecule of the same mass (p. 535). If this be admitted, it follows that the pressure, due to the impacts of the sugar molecules, will be identical with that due to the same molecules in the gaseous state, if the number of impacts per second is the same in both cases. In other words, the observed phenomena of osmotic pressure will be accounted for, if it can be shown that the sugar molecules will strike the internal surface of the membrane as frequently as if the water molecules were removed.

There can be no doubt that a sugar molecule moves from place to place through the solution, in spite of its frequent collisions with the water molecules. If we suppose that its path consists of a series of zig-zags (that is, that the sugar molecule is not generally constrained to travel to and fro along the same free path) it follows at once that **the number of collisions in a second, between one sugar molecule and the other sugar molecules in the solution, is the same as if the water molecules were absent.** For, let a disc of radius d, equal to the diameter of a sugar molecule, be carried along the zig-zag path actually described by the molecule, the area of the disc being always perpendicular to the direction in which it is moving ; then, the disc will sweep out a volume equal to $\pi d^2 V$ in a second, where V denotes tbe velocity of the molecule. If there are n_2 sugar molecules distributed at random in each unit of volume of the solution, it follows that the centres of $n_2 \pi d^2 V$ sugar molecules will lie in the volume swept out by the disc ; and therefore the number of collisions of a sugar molecule with other sugar molecules is the same as if the water molecules were absent, and the sugar were distributed in the state of a gas (p. 538). This being so, there appears to be no reason to doubt that the sugar molecules strike the diaphragm as frequently as if the water molecules were absent. The progress of a sugar molecule, *in any definite direction*, is impeded by frequent collisions with water molecules ; but these collisions merely jostle the sugar molecule about at random, and cause a collision with another sugar molecule or with the diaphragm, as often as they prevent one. Thus it appears that the laws of osmotic pressure can be explained in a satisfactory manner.

It may be noticed that the free surface of an aqueous solution of a non-volatile substance is equivalent to a semi-permeable membrane ; for water molecules can pass through the surface from the solution to the space above it, or in the reverse direction; but the non-volatile substance must remain in the solution. The sugar molecules exert an outwardly directed pressure on the free surface, and therefore tend to make the solution expand. The osmotic flow through a semi-permeable membrane is sometimes explained by this expansive property of the solution ; according to this view, water is sucked into the solution through the membrane, just as if the walls of the vessel tended to expand and so exerted a tensile stress on the contained liquid.

The "Brownian motion."—In the year 1827, the botanist Brown observed a phenomenon which affords direct experimental evidence as to the molecular structure of liquids. An aqueous solution of mercuric sulphide or gum mastic appears to be a perfectly clear liquid, which may be filtered and subjected to other similar modes of treatment without showing any signs that would distinguish it from an ordinary solution. If, however, a drop of the liquid be placed on a cover glass and examined under a microscope, numberless small particles in continual motion are observed. A solution of this character is called a **colloidal solution**, in order to distinguish it from ordinary solutions, in which no such particles are visible. Light is scattered from the particles, and if they are small enough, each is represented in the field of the microscope by a small luminous disc surrounded by diffraction rings; in this case the particles are not seen, in the strict sense of the word, any more than a distant lamp is seen when it presents the appearance of a star. When the particles have diameters lying between 0·001 mm. and 0·0001 mm., they can be seen and their sizes can be estimated with some degree of accuracy. Perrin has studied the motion of these particles. He has found that the mean velocity with which a particle moves is such that the average kinetic energy due to its linear motion is equal to the average kinetic energy due to the linear motion of a hydrogen molecule at the temperature of the experiment. This result confirms Maxwell's hypothesis as to the equipartition of energy (p. 535) as applied to liquids. If we suppose that a particle is an aggregate of molecules each of the size of a molecule of hydrogen (p. 562), it follows that a spherical particle 0·0001 mm. in diameter will comprise about 6×10^7 molecules. Hence, each particle is virtually a very large molecular complex; and since the hypothesis of equipartition of energy applies to these molecular complexes, there can be little doubt that it applies to the simple molecules of a substance dissolved in a liquid (p. 579).

The particles collide one with another, and the frequency with which a particle collides with other particles can be found by the method explained on pp. 537–544; thus we reach the conclusion stated on p. 580. Near the bottom of the solution, Perrin found the average distance between two neighbouring particles to be smaller than the corresponding distance near to the top of the solution; further, he found that the number of particles per unit volume in the neighbourhood of any point in the liquid varies with the depth of the point below the free surface, in exactly the same way as the density of a gas varies with the height of the superincumbent gas, as in the case of the atmosphere. In fine, then, **the distribution of the particles, their velocities, and the frequency with which they collide one with another, are practically the same**

as if they were in the gaseous state and the surrounding water were removed. This result affords the strongest possible support to the conclusion reached on p. 580, in connection with the dynamical explanation of osmotic pressure.

Osmotic pressure, and diffusion in liquids.—When a substance is dissolved in a pure liquid, the substance is called the **solute,** the pure liquid is called the **solvent,** and the mixture obtained by dissolving the solute in the solvent is called the **solution.** The number of gram-molecules of the solute dissolved in a c.c. of the solution is called the **concentration of the solution.** Sometimes the concentration is measured in grams per c.c.

Let a solution be contained in the compartment A, which forms part of the containing vessel represented in Fig. 265; and let this compartment be separated from the remainder of the vessel, B, by a fixed semi-permeable diaphragm which allows the solvent and not the solute to pass through it. Let the space B be filled with the pure solvent; then the solvent will pass through the diaphragm, from B into A, until the pressure in A is greater than that in B by an amount equal to the pressure that would be exerted by the solute if it were distributed in the state of a gas in the space A. Thus, if the concentration of the solution in A is equal to c_1 gram-molecules per c.c., the osmotic pressure p_1 in the compartment A is equal to $c_1\mathbf{R}T$, where **R** is the gas constant of a gram-molecule of gas (p. 537), and T is the absolute temperature. Owing to the small compressibility of liquids (p. 281), the amount of the solvent that must pass through the diaphragm, in order to establish this excess of pressure, is very small; and therefore the full osmotic pressure p_1 will be established almost instantaneously.

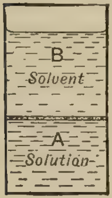

FIG. 265.—Osmotic pressure of a solution.

If the semi-permeable diaphragm is not fixed, but can slide like a piston inside the containing vessel, it will move upwards under the action of the excess of pressure p_1 exerted on it from below. Thus, **the solution tends to expand just as a gas would if it filled the space actually occupied by the solution.**

Now let it be supposed that the closed vessel represented in Fig. 266 is divided into three compartments by means of two semi-permeable diaphragms, the compartment A being filled with a solution of concentration c_1, and the compartment C with a similar solution of concentration c_2; the intervening compartment B being filled with the pure solvent. Then the osmotic pressure p_1 in the compartment A is equal to $c_1\mathbf{R}$T as before, and the osmotic pressure p_2 in C is equal to $c_2\mathbf{RT}$. If the diaphragms take the form of pistons which can move individually, they will tend to approach each other; but if the pistons are rigidly connected with each other, so that the space between them must remain constant, both pistons will move upwards if $c_1 > c_2$; the resultant force which produces the motion being equal to $a(c_1 - c_2)\mathbf{R}$T, where a is the area of either piston. It should be noticed that this result is independent of the volume of the compartment B; if the volume of B is infinitely small, the two pistons virtually coalesce into a single piston which divides the cylinder into two compartments, and the forces acting on opposite sides of this piston are respectively equal to $ac_1\mathbf{R}$T and $ac_2\mathbf{R}$T. Thus, **the osmotic pressure exerted on one side of the piston is independent of the concentration of the solution on the other side of the piston.**

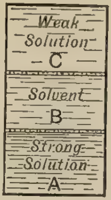

FIG. 266.—Osmotic pressure of a solution.

Next, let it be supposed, as before, that the compartments A and C of the vessel represented in Fig. 266 are filled with similar solutions of concentrations c_1 and c_2; but that the middle compartment B is filled with a solution of which the concentration varies from c_1 near the lower piston, to c_2 near the upper piston. Equal and oppositely directed forces will now act on the opposite faces of each piston, but the solute in the compartment B will be acted upon by a resultant force $a(c_1 - c_2)\mathbf{R}$T, owing to the excess of the upwardly directed force exerted on the lower piston, as compared with the downwardly directed force exerted on the upper piston. In this case, both pistons will tend to move upwards, carrying with them the solute which lies between them.

Attention must now be directed to the diffusion of a solute through a solution. To fix our ideas, let it be supposed that some crystals of sugar are placed at the bottom of a cylindrical vessel which contains water (Fig. 267); the crystals dissolve in the water immediately in contact with them, and the dissolved sugar gradually diffuses upwards. The concentration of the solution varies with the height in the containing vessel, as can be observed without difficulty, since the refractive index of the solution varies with its concentration. If an imaginary transverse plane AB be described across the vessel, molecules of the solute will cross this plane in both directions, but in a given time the number that pass upwards through it will exceed the number that pass downwards. We may suppose, however, that the imaginary plane AB moves upwards with a velocity V, such that equal numbers of molecules of the solute cross the plane in opposite directions during a second. In this case, for every molecule of the solute that passes upwards through the moving plane, another molecule of the solute must simultaneously pass downwards. If the imaginary plane were replaced by a semi-permeable diaphragm moving with the velocity V, the conditions would not be altered materially; for the motion of a molecule of the solute up to the moving diaphragm, and its immediately subsequent rebound, would correspond to the simultaneous passage of two molecules through the imaginary plane, in opposite directions. If the concentration just beneath the moving diaphragm AB is equal to c_1, the pressure on the lower surface of this diaphragm will be equal to $c_1\mathbf{R}T$; and if another semi-permeable diaphragm, also moving upwards with the velocity V, were substituted for a neighbouring imaginary plane CD where the concentration is equal to c_2, the pressure on the upper surface of this diaphragm would be equal to $c_2\mathbf{R}T$. Thus, the solute comprised between the two moving diaphragms would be urged upwards by a resultant force equal to $a(c_1 - c_2)\mathbf{R}T$, where a is the area of either diaphragm; and the solute comprised between the two imaginary planes AB and CD must be urged upwards by an equal force.

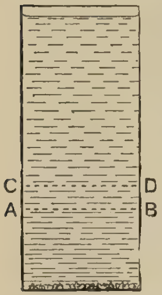

FIG. 267.—Diffusion of a solute through a solution.

Let it be assumed that, by the application of a constant force, a uniform motion in the direction of the force can be superimposed on the erratic zig-zag motion of a molecule; also, let it be assumed that the uniform superimposed velocity is proportional to the applied force. Let f_1 denote the sum of the forces that would have to be applied to the molecules comprised in a gram-molecule of the solute, in order to keep them all moving with unit superimposed velocity; then the sum of the forces required to keep the same molecules moving with a velocity V would be equal to f_1V. Between the imaginary planes AB and CD, there must be cad molecules of the solute, if c denotes the average concentration in the space between AB and CD, and d denotes the linear distance between these planes. In order that these molecules may be constrained to move upwards with the velocity V, the sum of the forces exerted on them must be equal to f_1cadV. Equating this force to the resultant force due to the osmotic pressure below AB and that above CD, we have—

$$f_1cadV = a(c_1 - c_2)\mathbf{R}T,$$

$$\therefore \quad cV = \frac{\mathbf{R}T}{f_1} \cdot \frac{c_1 - c_2}{d}.$$

Now $(c_1 - c_2)/d$ is the value of the concentration gradient between the planes AB and CD; $\mathbf{R}$ is an absolute constant, and $\mathbf{R}T/f_1$ will be constant for a given solute at a given temperature. Let $\mathbf{R}T/f_1$ be denoted by K. The quantity cV is equal to the number of gram-molecules of the solute that would diffuse upwards during a second through unit area of an imaginary transverse plane which remains in a fixed position between AB and CD; for c is the average number of gram-molecules of the solute per unit volume between AB and CD, and V is the average velocity superimposed on the molecules of the solute which occupy that space. Thus, the equation

$$cV = K\frac{c_1 - c_2}{d}$$

is equivalent to **Fick's law of diffusion**, which states that **the mass of solute which diffuses through unit area in a second is proportional to the concentration gradient.** The constant K is called the **coefficient of diffusion. The coefficient of diffusion is defined as the ratio of the number of gram-molecules of the solute that diffuse through a sq. cm. in a second, to the concentration gradient measured in gram-molecules per c.c. per cm.** If both sides of the above equation be multiplied by the molecular weight of the solute, the left-hand side becomes equal to the number of grams of the solute that diffuse through a sq. cm. in a second, and the concentration-gradient on the right-hand side becomes expressed in grams per c.c. per cm. Thus the value of the coefficient of diffusion is the same, whether the unit of mass is the gram-molecule or the gram.

Fick's law has been verified for a great number of dilute solutions. In electrically conducting solutions the solute is more or less dissociated into electrically charged ions, and the rate of diffusion cannot be determined theoretically without taking into account the forces called into play by the electric charges on the ions. A theory taking account of these forces has been developed by Nernst.[1]

When the coefficient of diffusion of a substance in a given liquid is known, we can calculate the value of f_1, the force needed to propel the molecules comprised in a gram-molecule of the solute, at a speed of 1 cm. per sec., through the solvent. Thus, at 18°C., 0·42 gram-molecule of cane sugar diffuses through a sq. cm. in a day, under a concentration gradient of one gram-molecule (342 gm.) per c.c. per cm. Thus, $K = 0{\cdot}42/86400$, using c.g.s. units; and the value of **R** is given on p. 537. Therefore

$$f_1 = \frac{\mathbf{R}T}{K} = \frac{2 \times 4{\cdot}2 \times 10^7 \times 273 \times 86{,}400}{0{\cdot}42}$$

$$= 4{\cdot}7 \times 10^{15} \text{ dynes.}$$

This force is equal, roughly, to the pull exerted by gravity on 4·7 million tons of matter.

Work done by osmotic pressure.—Let Fig. 268 represent a cylinder provided with a frictionless piston, the lower end of the cylinder being closed by a semi-permeable membrane. Let the cylinder be filled with a solution, and immersed externally in the pure solvent; then the osmotic pressure of the solution will force the piston outwards if the external force urging the piston inwards is slightly less than the internal force due to the osmotic pressure. As the piston moves outwards against the external force, work is done; the value of the work is the same as if the solute were in the state of a gas and the solvent were removed.

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[1] See *Theoretical Chemistry*, by W. Nernst, p. 322 (Macmillan); or *A Treatise on the Theory of Solutions*, by W. C. Dampier Whetham, p. 376 (Cambridge University Press).

If the solution is maintained at a constant temperature, the external work is performed at the expense of heat which enters the solution; similarly, the external work done by a gas which expands isothermally is derived from the heat which enters the gas.[1]

When a solute diffuses through a solvent, the conditions are somewhat different. In this case there is no external work done, and the conditions are essentially similar to those of a gas which is forced through a porous plug.[2] In both cases the molecules are urged forwards, the kinetic energy due to their forward motion being derived from heat which disappears. As soon as they gain any appreciable kinetic energy due to their forward motion, this energy is dissipated in the form of heat; the heat thus generated is then used up again in urging the molecules forwards, and so on.

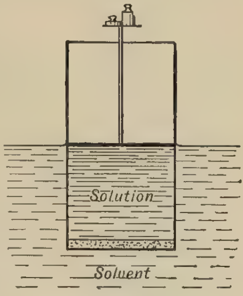

FIG. 268.—Work done by osmotic pressure.

The process of solution.—The molecules of a solid such as sugar exert a great attraction one on another, and thus a molecule near to the surface is unable to escape into empty space (compare p. 358). To account for the escape of molecules when water is in contact with the surface of the solid, it is only necessary to assume that the molecules of the water exert a sufficiently great attraction on the molecules of the solid, and there is independent evidence of such an attraction (p. 282).

Osmotic pressure and vapour pressure of a solution.—Let a solution, contained in the tall cylindrical vessel A (Fig. 269) be separated by a semi-permeable diaphragm from the pure solvent contained in the vessel B. For equilibrium to exist, the free surface of the solution must stand at a certain height h above the free surface of the pure solvent. Let both vessels be contained in an isothermal enclosure, which is empty but for the presence of the vapour of the solvent. Then, if p denotes the vapour pressure immediately above the free surface of the solvent in B, it follows that the pressure at a height h above that surface is equal to $(p - g\sigma h)$, where σ denotes the density of the vapour. The vapour in contact with the surface of the solution must be in equilibrium with the solution; for if the pressure were less than the vapour pressure of the solution, the solvent would evaporate from the solution and condense on the surface of the pure solvent, and more of the solvent would have to pass into the vessel A through the semi-permeable membrane; in other words, a type of perpetual motion would be realised. Thus, the vapour pressure p' of the solution must be equal to $(p-g\sigma h)$. The semi-permeable membrane represented in Fig. 269 is at a distance h below the surface of the solution, and immediately beneath the free surface of the solvent; therefore the osmotic pressure P of the solution must be equal to $\{(p'+g\rho h)-p\}$, where ρ denotes the density of the solution; that is, $P=g(\rho-\sigma)h$, and $h=P/g(\rho-\sigma)$. Thus, the vapour pressure p' of the solution is less than that of the pure solvent by $g\sigma h=P\sigma/(\rho-\sigma)$. Since P is directly proportional to the concentration of the solution, and is constant for solutions of equal molecular concentrations, we obtain Raoult's law that **the depression of the vapour pressure is directly proportional to the number of molecules of the solute dissolved in unit volume of the solution, and is independent of the chemical constitution of the molecules.** Thus, a solution containing a gram-molecule of sugar dissolved in a given volume of water will have the same vapour pressure as a solution containing a gram-molecule of lactic acid dissolved in the same volume of water. When the solution is a good electrical conductor, allowance must be made for the fact that the molecules of the solute are dissociated into ions, and each of these ions plays the same rôle in connection with osmotic pressure as an undissociated molecule.

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[1] See the Author's *Heat for Advanced Students*, p. 298 (Macmillan).
[2] See the Author's *Heat for Advanced Students*, p. 380 (Macmillan).

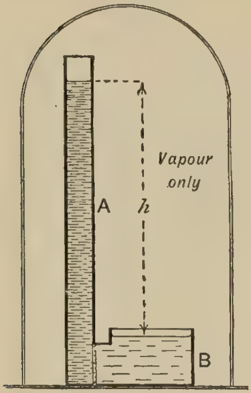

FIG. 269.—Osmotic pressure and vapour pressure of a solution.

Influence of external pressure on the vapour pressure of a liquid.—Let Fig. 270 represent two vessels, containing respectively a solution and the pure solvent, separated by a diaphragm M permeable to the solvent and not to the solute. The space above the solution is occupied solely by the vapour of the solvent, while the space above the solution is occupied by vapour of the solvent and an inert gas such as nitrogen. A diaphragm M′ permits the vapour of the solvent to pass freely from one vessel into the other, but is impermeable to the inert gas.

Let it be supposed that the pressure of the inert gas is adjusted so that the solution on one side of the diaphragm M is in equilibrium with the solvent on the other side of that diaphragm. Then the resultant pressure of the vapour and the inert gas above the surface of the solution, must exceed the pressure of the vapour above the solvent, by P, the osmotic pressure of the solution.

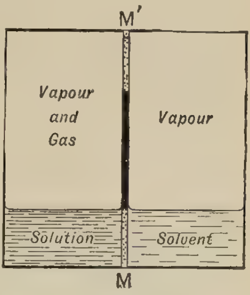

FIG. 270.—Influence of external pressure on the vapour pressure of a liquid.

When these conditions are realised, the vapour on one side of the diaphragm M′ must be in equilibrium with the vapour on the other side of that diaphragm. If this were not the case, vapour would flow through the diaphragm M′, to be condensed on the liquid in the vessel entered; thus, the equilibrium of the liquids would be destroyed, so that a flow of liquid would occur through the diaphragm M into the vessel which the vapour leaves. Hence, we should have a continual flow of vapour from one vessel to the other, and a flow of liquid in the opposite direction; and this continual flow would occur in a system at a uniform temperature, which is impossible.

For the vapour in one vessel to be in equilibrium with that in the other, the density of the vapour must have an identical value in both vessels (p. 577), and therefore the vapour pressure must have equal values in the two vessels. Thus, the pressure (say **p**) of the inert gas must have increased the vapour pressure of the solution so far as to make it equal to the vapour pressure of the pure solvent. In other words, the pressure **p** has increased the vapour pressure of the solution from $\{p - \mathrm{P}\sigma/(\rho - \sigma\}$ to p. Now the pressure **p** of the inert gas must be equal to the osmotic pressure P of the solution, since the vapour has equal pressures in the two vessels; thus the pressure **p** of the inert gas increases the vapour pressure of the solution by

$$\frac{\mathbf{p}\sigma}{(\rho - \sigma)}.$$

This increase in the vapour pressure, due to subjecting a liquid to pressure, has been noticed already (p 360) and a formula equivalent to that just obtained has been deduced. For a physical explanation of this process, see p. 360.

Osmotic pressure, and boiling point of a solution.—A liquid can boil when its vapour pressure is equal to the pressure to which the liquid is subjected. The effect of dissolving a non-volatile substance in a solvent is to diminish the vapour pressure, and therefore it may be inferred that the boiling point of a solution of a non-volatile substance must be higher than that of the pure solvent. The relation between the elevation of the boiling point and the osmotic pressure of a solution must now be investigated.

Fig. 271 represents two vessels, A and B, of which A is immersed in a bath maintained at a constant temperature T°, while B is immersed in a bath at a constant temperature (T + *t*)°, both temperatures being measured in absolute units. The vessel A is closed, and contains only a pure solvent and its vapour; if *p* denotes the pressure of the vapour in A, then, by definition, T denotes the boiling point of the solvent under a pressure *p*. The vessel B contains a solution of a non-volatile substance in the solvent; let (T + *t*)° be the boiling point of this solution under the pressure *p*. Two tubes extend from the vessel B to the vessel A. The lower tube communicates with the two vessels beneath the surfaces of the contained liquids, and is provided with a stop-cock C in which the free-way is replaced by a semi-permeable membrane. Thus, when C is opened, the pure solvent and the solution are separated from each other only by the semi-permeable membrane; when C is closed, the liquids are isolated from each other. The upper tube opens into B at a point just above the surface of the solution, and opens into A at a point above the surface of the solvent; this tube is provided with an ordinary stop-cock C′. The vessel B is provided with a piston D

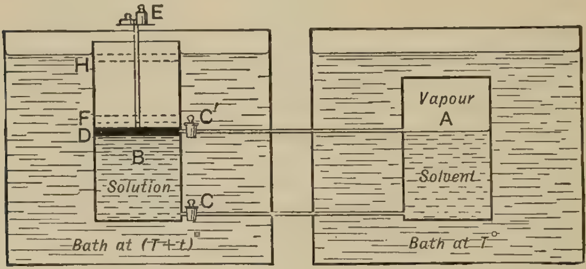

FIG. 271.—Osmotic pressure and boiling point of a solution.

which can move freely up or down; by placing weights on a scale-pan E attached to this piston, the contents of the vessel B can be subjected to any required pressure.

In the first place, let both of the stop-cocks C and C′ be closed, and let the piston D rest on the surface of the solution. Let the scale-pan E be loaded until the pressure exerted on the solution by the piston D is equal to $(P + p)$, where P denotes the osmotic pressure of the solution. Thus, the pressure to which the solution is subjected, exceeds the pressure to which the pure solvent in A is subjected, by P. If the stop-cock C be opened, the solution and the solvent will be in equilibrium. By making an infinitesimal reduction in the load of the scale-pan E, the piston D can be caused to ascend, some of the pure solvent passing from the vessel A into the vessel B, through the semi-permeable membrane in the stop-cock C. Let ρ be the density of the solvent at the temperature T; and let a small volume v of the solvent enter B, while the piston rises from D to F. The work done in raising the piston against the force of gravity[1] is equal to $(P + p)v$. Of this work, the part Pv is performed at the expense of heat which enters the vessel B from the surrounding bath, while the solution virtually expands like a gas (p. 586); the remainder pv is performed by the solvent which enters B under a pressure p.

In the second place, let the stop-cock C be closed, so that the solution is isolated from the pure solvent; and let the load on the scale-pan E be adjusted so that the pressure exerted on the solution is equal to p, the vapour pressure of the pure solvent at T° or of the solution at $(T + t)°$. Allow a mass of the solvent, equal to the mass ρv that previously entered the vessel B, to evaporate from the solution. During this process, the piston ascends to H and sweeps out a volume (say V) equal to the space between F and H. The work done in raising the piston against the force of gravity is equal to pV: an amount of heat, equal to the amount that becomes latent during the evaporation of the solvent from the solution, passes into B from the surrounding bath. Let Q_1 denote the total quantity of heat that has entered B during the osmotic expansion of the solution and the subsequent evaporation of the solvent.

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[1] For simplicity, it is assumed tacitly that there is a vacuum above the piston, so that the pressure exerted on the solution is due to the pull of gravity on the piston and the loaded scale-pan attached to it. Also, the rise of temperature of the solvent which enters B produces an expansion which is neglected.

Next, let the stop-cock C′ be opened, and let all the vapour previously evaporated from the solution pass into the vessel A, there to be condensed on the pure solvent. During this process the piston descends to its original position D, sweeping out the volume $(V + v)$; an amount of work equal to $p(V+v)$ is done by gravity. The net work, done against gravity, now has the value

$$\{(P+p)v+pV-p(V+v)\} = Pv.$$

While the vapour is condensing on the surface of the pure solvent in A, a quantity Q_2 of heat is given out and passes into the bath at T°; the value[1] of Q_2 is equal to $L\rho v$, where L denotes the latent heat of vaporisation of the solvent at T°.

It is unnecessary to take account of the quantity of heat absorbed by the solvent while its temperature was being raised from T° to $(T + t)°$, previous to entering B; or of the heat given out by the vapour in cooling from $(T + t)°$ to T°, previous to entering A. Both these quantities will be negligibly small, since the specific heat of the liquid solvent, and that of its vapour, are both small in comparison with the latent heat of vaporisation L.

The contents of both A and B are now precisely in the same condition as at first; therefore a complete cycle has been traversed, and obviously this cycle is reversible.[2] During the cycle, a net amount of external work equal to Pv has been performed; an unknown quantity Q_1 of heat has been absorbed at $(T + t)°$, and a quantity $Q_2 = \rho\, v\, L$ of heat has been rejected at T°. Thus—

$$\frac{Q_1 - Q_2}{Q_1} = \frac{(T+t) - T}{T+t},$$

$$\therefore \quad 1 - \frac{Q_2}{Q_1} = 1 - \frac{T}{T+t},$$

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[1] While the solvent was passing from A into B through the semi-permeable membrane, some of the solvent was evaporated in order to fill the space left vacant in A, and heat was absorbed from the bath which surrounds A. During the final condensation of the vapour which is returned from B to A, an additional quantity of vapour, equal to that previously evaporated, is condensed on the solvent, and an amount of heat, equal to that previously absorbed from the bath which surrounds A, is returned to that bath. These two heat exchanges are equal and are both effected at the temperature T, and therefore one cancels the other.

[2] For the thermo-dynamics of reversible cycles, see the Author's *Heat for Advanced Students*, pp. 335–347 (Macmillan).

so that

$$\frac{Q_1}{Q_2} = \frac{T+t}{T},$$

$$\therefore \quad \frac{Q_1 - Q_2}{Q_2} = \frac{(T+t)-T}{T} = \frac{t}{T}.$$

Now $(Q_1 - Q_2)$ is the amount of heat that has disappeared during the completion of the cycle, and this heat must be equivalent thermally to the net external work performed during the cycle. Thus, $(Q_1 - Q_2) = Pv/J$, where J denotes the mechanical equivalent of unit quantity of heat. Then, since $Q_2 = \rho v L$,—

$$\frac{Pv}{J\rho v L} = \frac{P}{J\rho L} = \frac{t}{T},$$

and

$$t = \frac{PT}{J\rho L}.$$

This equation suffices to determine the value of t, the elevation of the boiling point produced by dissolving a non-volatile substance in the solvent, so as to produce a dilute solution with an osmotic pressure equal to P.

Problem. *A gram of cane sugar (molecular weight* = 342) *is dissolved in* 100 *c.c. of water. Determine the boiling point of the solution under a pressure of a standard atmosphere.*

The solution has a concentration, equivalent to one gram-molecule (342 gm.) dissolved in $342 \times 100 = 3{\cdot}42 \times 10^4$ c.c. Thus, at the absolute temperature $T = (273 + 100) = 373°$, the osmotic pressure P of the solution is given (p. 576) by the equation—

$$P \times 3{\cdot}42 \times 10^4 = 8{\cdot}4 \times 10^7 \times 373,$$

$$\therefore \quad P = \frac{8{\cdot}4 \times 10^7 \times 373}{3{\cdot}42 \times 10^4} = 9{\cdot}15 \times 10^5 \text{ dyne/(cm.)}^2.$$

The boiling point T of the pure solvent (water) is 373° absolute; at this temperature its density ρ is equal to 0·96 gm./(cm.)^3. The value of L is equal (roughly) to 537 gram-calories per gram. The elevation t of the boiling point due to the sugar dissolved in the water is given by the equation—

$$t = \frac{PT}{J\rho L} = \frac{9{\cdot}15 \times 10^5 \times 373}{4{\cdot}2 \times 10^7 \times 0{\cdot}96 \times 537}$$

$$= 0{\cdot}0158°\text{C}.$$

As a result of experiments, t is found to be 0·0158°C., which is in excellent agreement with the value just obtained.

The elevation of the boiling point for an aqueous solution, containing one gram-molecule of sugar per 100 c.c. of water, is equal to $342 \times 0{\cdot}0158 = 5{\cdot}40^\circ$C. In accordance with theory, it has been found that the elevation of the boiling point depends only on the number of molecules of the solute per unit volume of the solution, and is independent of the chemical nature of the molecules.

Osmotic pressure, and freezing point of a solution.—The relation between the freezing point and the osmotic pressure of a solution can be determined from the study of an ideal thermodynamic cycle somewhat similar to that which has just been used. The ideal apparatus required is similar to that represented in Fig. 271, with the following modifications. The vessel A which contains the solvent is open to the atmosphere, and the upper tube extending from the vessel A to the vessel B is not required. Further, the bath surrounding the vessel A is maintained at a constant temperature T° equal to the freezing point of the pure solvent under atmospheric pressure; the vessel B, which contains the solution, is surrounded by a bath which is maintained at a temperature $(T-t)^\circ$, equal to the freezing point of the solution under atmospheric pressure.

In the first place, the stop-cock C, which contains the semi-permeable membrane, is closed. The piston D is allowed to rest on the solution, and the scale pan E is loaded until the pressure exerted on the solution is equal to $(P+p)$, where P denotes the osmotic pressure of the solution and p denotes the atmospheric pressure. On opening the stop-cock C, the solution and the pure solvent are in equilibrium, since the pressure of the solution in B exceeds that of the solvent on the opposite side of the semi-permeable membrane, by P, the osmotic pressure of the solution.

Let a volume v of the solvent pass through the semi-permeable membrane into the vessel B. The piston D rises, and an amount of work equal to $(P+p)v$ is done against gravity: simultaneously the surface of the solvent in A sinks, and the work done on the system by the atmospheric pressure is equal to pv. Thus the net amount of external work done is equal to $\{(P+p)v-pv\} = Pv$. Heat is absorbed by the solution from the bath at the temperature $(T-t)^\circ$ which surrounds B.

Now let the stop-cock C be closed, and let the piston D be removed so that the solution is at atmospheric pressure. If the solution is allowed to begin to freeze, only the solvent will separate out: let a mass of the solvent equal to ρv be frozen out of the solution, where ρ denotes the density of the liquid solvent. During this process, heat is given out by the solution, and absorbed by the bath at the temperature $(T-t)^\circ$. Let Q_2 denote the net amount of heat transferred from the solution to the surrounding bath, during the freezing of the mass ρv of the solvent and the osmotic expansion of the solution.

Let the solid formed in B be removed, placed in the pure solvent contained in A, and there allowed to melt. Meanwhile, an amount of heat Q_1, equal to $\rho v L$, is absorbed from the bath at T°.

The whole system is now in precisely the same condition as at first, and a reversible cycle has been traversed. In completing this cycle, a net amount of external work equal to Pv has been performed; the changes of volume which occurred during the freezing and melting of the mass ρv of the solvent were numerically equal but opposite in sign, and both were effected at atmospheric pressure, so that one cancels the other. An amount of heat $Q_1=\rho v L$ has been absorbed at the temperature T°, and an unknown quantity Q_2 of heat has been rejected at $(T-t)^\circ$. Therefore—

$$\frac{Q_1-Q_2}{Q_1}=\frac{T-(T-t)}{T}=\frac{t}{T}.$$

Also, (Q_1-Q_2), the amount of heat which has disappeared, must be thermally equivalent to the net external work Pv which has been performed. Thus $(Q_1-Q_2)=Pv/J$, and—

$$\frac{Pv}{J\rho v L}=\frac{P}{J\rho L}=\frac{t}{T},$$

$$\therefore\quad t=\frac{PT}{J\rho L}.$$

This equation suffices to determine t, the depression of the freezing point produced by dissolving a non-volatile substance in the solvent so as to produce a solution with an osmotic pressure P.

Problem. *A gram of cane sugar (molecular weight* = 342) *is dissolved in* 100 *c.c. of water. Determine the freezing point of the solution under a pressure of a standard atmosphere.*

The osmotic pressure P of the solution at 0°C. (T = 273°) is given by the equation—

$$P \times 3{\cdot}42 \times 10^4 = 8{\cdot}4 \times 10^7 \times 273,$$

$$\therefore \quad P = \frac{8{\cdot}4 \times 10^7 \times 273}{3{\cdot}42 \times 10^4} = 6{\cdot}7 \times 10^5 \text{ dyne/(cm.)}^2.$$

The density of water at O°C. is equal to 0·99987, or, roughly, 1 gm./(cm.^{3}). The latent heat L of water = 80 calories per gm. Thus

$$t = \frac{6{\cdot}7 \times 10^5 \times 273}{4{\cdot}2 \times 10^7 \times 80} = 0{\cdot}0543^\circ\text{C}.$$

As a result of experiments, it is found that the depression of the freezing point of a solution of the above composition and strength is equal 0·0543°C. The depression of the freezing point for a solution comprising 1 gram-molecule per 100 c.c. of water is equal to 18·58°C.

Questions on Chapter XVI

1. A mass of oxygen equal to 0·2 gm. is contained in a vessel of 3 litres capacity. Calculate the value of the pressure of the oxygen at 27°C.

2. A certain gas has a density of 0·001 gm. per c.c. under a pressure of 50 cm. of mercury. Calculate the root-mean-square velocity of the molecules of the gas.

(Density of mercury = 13·6 gm. per c.c.)

3. Assuming the law of equipartition of energy, calculate the root-mean-square velocity of a molecule of oxygen at 0°, if the root-mean-square velocity of a molecule of hydrogen is equal to $1{\cdot}83 \times 10^5$ cm./sec. at 0°C.

4. Calculate the mean free path in helium at 0°C. and standard atmospheric pressure, being given that the number of molecules per c.c. is equal to $2{\cdot}76 \times 10^{19}$, and the diameter of a helium molecule is equal to $3{\cdot}36 \times 10^{-8}$ cm.

5. The coefficient of viscosity of argon at 12·3°C. is equal to $2{\cdot}168 \times 10^{-4}$ gm./cm. sec. Calculate the value of the diameter of a molecule of argon, being given that the molecular weight of argon is equal to 39·9, that of hydrogen being 2·016; while the number of

molecules in a c.c. of hydrogen at 0°C. and standard atmospheric pressure is equal to $2{\cdot}76 \times 10^{19}$.

(Density of hydrogen at 0°C. and standard atmospheric pressure = $8{\cdot}96 \times 10^{-5}$ gm./(cm.^{3}).)

6. A cloud of water particles is observed through a microscope, and it is found that the particles descend with a velocity equal to 0·12 cm. per sec. Calculate the radius of the drops from the formula (p. 528).

$$V = \frac{2}{9}\frac{gr^2}{\eta}.$$

(At the temperature of the experiment, value of η for air = $1{\cdot}8 \times 10^{-4}$ gm./cm. sec.)

7. Assuming that each of the molecules in liquid water has the constitution denoted by H_2O, calculate the number of molecules in a spherical raindrop 2 mm. in diameter, being given that the mass of an atom of hydrogen is equal to $1{\cdot}62 \times 10^{-24}$ gm.

8. From the data given in question 7, calculate the number of molecules in a gram of water.

9. If the value of b in van der Waals' equation is equal to 0·954 (see p. 559), calculate the diameter d of a molecule of water from the formula (p. 550)—

$$b = N\pi d^3/3,$$

where N denotes the number of molecules in a gram of water (see question 8).

10. A gram of sugar is dissolved in a litre of water. Calculate the value of the osmotic pressure of the solution (*a*) at 0°C., and (*b*) at 100°C.

11. The boiling point of ethyl ether is 34·6°C. Determine the elevation of the boiling point due to dissolving one gram-molecule of carbon hexachloride (molecular weight = 237) in 1000 grams of ether.

Latent heat of vaporisation of ethyl ether = 81·49 calories per gm.

Density of ether at 34·6°C. = 0·6944 gm./(cm.)^3.

12. It has been found that 1·065 gm. of iodine dissolved in 30·14 gm. of ethyl ether, raises the boiling point by 0·296°C. Prove that each molecule of the dissolved iodine must consist of two atoms.

Atomic weight of iodine = 127
Boiling point of ethyl ether = 34·6°C.
Latent heat of vaporisation of ethyl ether = 81·49.
Density of ethyl ether at 34·6°C. = 0·6944 gm./(cm.)^3.

13. The vapour pressure of water at 40°C., when in contact only with its own vapour, is equal to 5·55 cm. of mercury. Calculate the

vapour pressure of water at 40°C., when it is enclosed in a vessel which also contains nitrogen at 100 atmospheres pressure.

(Vol. of one gram of saturated aqueous vapour at 40° = 19,442 c.c.
Vol. of one gram of water at 40° = 1·0077 c.c.)

14. The freezing point of benzene is equal to 5·58°C. Determine the depression of the freezing point, due to dissolving one gram-molecule of methyl iodide in 1,000 gm. of benzene.

Latent heat of fusion of benzene = 29·089 calories per gm.
Density of liquid benzene at 5·58° = 0·900 gm./(cm.)^3.

15. One gram-molecule of acetic acid ($C_2H_4O_2$) is dissolved in 1,000 grams of benzene, and it is found that the freezing point of the solution is 2·53°C. lower than that of pure benzene. Prove that the constitution of the molecules dissolved in the benzene is $2(C_2H_4O_2)$.

ANSWERS TO QUESTIONS

CHAPTER I

(1) $3{\cdot}94 \times 10^{4}$ poundals, $1{\cdot}22 \times 10^{3}$ pounds, $5{\cdot}45 \times 10^{8}$ dynes.
(2) $3{\cdot}84 \times 10^{4}$ poundals, $1{\cdot}19 \times 10^{3}$ pounds, $5{\cdot}32 \times 10^{8}$ dynes.
(3) $3{\cdot}63 \times 10^{5}$ poundals, $1{\cdot}13 \times 10^{4}$ pounds, $5{\cdot}01 \times 10^{9}$ dynes.
(4) $3{\cdot}60 \times 10^{3}$ poundals, 112 pounds, $4{\cdot}88 \times 10^{7}$ dynes.
(5) Body would move upwards relatively to the lift chamber.

(Suggestion : If the elastic filament were cut at the instant when the lift chamber is set free, the body and the lift chamber would fall with equal accelerations.)

(6) Let m = mass of body. When filament is nearly vertical, its tension is equal to mg dynes (nearly). The horizontal component of this tension is equal to $mg\theta$, when θ is small. Force needed to produce acceleration of body = ma.

$$\therefore\ mg\theta = ma, \text{ and } \theta = a/g.$$

(7) $3{\cdot}2$ ft./(sec.)^2.

(8) Depth of well, calculated on supposition that time taken for sound to travel from bottom to top is negligible, = 4,410 cm. Sound takes $0{\cdot}13$ sec. to travel over this distance ; thus, to a closer degree of approximation, depth of well = $(\frac{1}{2}) \times 981 \times (2{\cdot}87)^{2}$ = 4,030 cm.

(9) See p. 456.

(10) Centripetal force per unit mass of the earth = $1{\cdot}945 \times 10^{-2}$ poundal per lb., or $0{\cdot}59$ dyne per gram.

(11) Velocity of cord = $\sqrt{2gh}$, where h = height of table.

(12) $0{\cdot}273$ ft.-lb., or $3{\cdot}70 \times 10^{6}$ ergs.

(13) Rod is in stable equilibrium when it hangs with its length in a vertical straight line passing through point of support.

CHAPTER II

(1) $\dfrac{7m}{5}r^2$.

(2) Let m be the mass of a circular ring cut from either cone, at a perpendicular distance x from the vertex. Then, moment of inertia of this ring about axis of symmetry of cone = mr^2, where r = radius of

ring. Moment of inertia of same ring, about a line through vertex and perpendicular to axis of symmetry of cone $= m\left(\frac{r^2}{2} + x^2\right)$, and this is equal to mr^2, if $(r/x)^2 = 2$.

(3) $\frac{5ml^2}{18}$.

(4) Angular velocity, $\omega = \sqrt{(3g/l)}$.

(5) Force pressing body against plane $= mg\cos\theta$. When body is on point of sliding, tangential force at point of contact $= \mu mg\cos\theta$, and this force exerts torque $\mu mgr\cos\theta$ about centre of body. If $\alpha =$ linear acceleration of body, angular acceleration of body $= \alpha/r$, and rate of increase of moment of momentum $= mk^2\alpha/r$. Then

$$\frac{mk^2\alpha}{r} = \mu mgr\cos\theta.$$

Linear acceleration $\alpha = g\sin\theta - \mu g\cos\theta$,

$$\therefore\quad k^2(g\sin\theta - \mu g\cos\theta) = \mu gr^2\cos\theta.$$

(6) Kinetic energy due to linear velocity = 105 ft.-poundals. Kinetic energy due to angular velocity = 105 ft.-poundals. Total kinetic energy = 210 ft.-poundals, or 6·52 ft.-lbs.

CHAPTER III

(3) $T = 2\pi\sqrt{\frac{2r}{g}}$.

(4) $T = 2\pi\sqrt{\frac{2l}{3g}}$.

(5) $T = 2\pi\sqrt{\left(\frac{2\sqrt{l^2+b^2}}{3g}\right)}$

($l =$ length, and $b =$ breadth of rectangular sheet.)

(6) Number of oscillations lost $= 0{\cdot}00006n$, where $n =$ number of oscillations completed in 24 hours at lower temperature.

(7) $T = 2\pi\sqrt{\left(\frac{\frac{2\cdot 6}{15}r}{g}\right)}$.

(8) When bar tilts through small angle θ, its point of contact with cylinder moves through distance $r\theta$; line joining original point of contact to centre of gravity of bar, turns through angle θ, and the centre of gravity moves through distance $d\theta/2$, in direction of displacement of point of contact. Hence, in tilted position of bar, centre of gravity is at distance $\{r\theta - (d\theta/2)\}$ from new point of contact, and restoring torque $= mg\{r - (d/2)\}\theta$.

$$T = 2\pi\sqrt{\left(\frac{l^2 + \frac{5}{4}d^2}{12g(r - d/2)}\right)}.$$

(9) Restoring torque per unit twist = $mg \cdot cd/4l$.

$$T = 2\pi \cdot \frac{2k}{\sqrt{(cd)}} \sqrt{\frac{l}{g}}.$$

l = length of each filament, k = radius of gyration of suspended body, c = diameter of circle on which upper points of attachment of filaments lie, and d = diameter of circle on which filaments are attached to body.

(15) Period T of oscillation of body is given by equation—

$$gl\left(\frac{T}{2\pi}\right)^2 = k^2 + l^2.$$

After loading with mass m, at distance L_1 from axis of support, period T_1 is given by equation—

$$\left(\frac{T_1}{2\pi}\right)^2 = \frac{M(k^2+l^2)+mL_1^2}{g(Ml+mL_1)} = \frac{k^2+l^2+\frac{m}{M}L_1^2}{g\left(l+\frac{m}{M}L_1\right)},$$

where M = the unknown mass of the body when unloaded.

$$\therefore \quad g\left(l+\frac{m}{M}L_1\right)\left(\frac{T_1}{2\pi}\right)^2 = k^2+l^2+\frac{m}{M}L_1^2$$

$$= gl\left(\frac{T}{2\pi}\right)^2 + \frac{m}{M}L_1^2 \quad . \quad . \quad (1)$$

On loading with same mass m at distance L_2 from axis of support, period T_2 is given by equation—

$$g\left(l+\frac{m}{M}L_2\right)\left(\frac{T_2}{2\pi}\right)^2 = gl\left(\frac{T}{2\pi}\right)^2 + \frac{m}{M}L_2^2 \quad . \quad . \quad . \quad (2)$$

The only unknown quantities contained in equations (1) and (2) are l and $\frac{m}{M}$, and therefore the value of l can be calculated from these equations.

CHAPTER IV

(1) In equation on p. 133, the various symbols have the following values :—

$I = 5 \times 10^4$ gm. (cm.)^2.
$I' = 6{\cdot}69 \times 10^6$ gm. (cm.)^2.
$M = 10^3$ gm.
$L = 75$ cm.
$\omega = 62{\cdot}8$ radians per sec.

$$\text{Therefore } \frac{2\pi}{T} = \pm 0{\cdot}23 + 3{\cdot}32.$$

$$= 3{\cdot}55 \text{ and } 3{\cdot}09.$$

$\therefore$ periods of oscillation = 1·77 sec. and 2·03 sec.

(2) In formula at bottom of p. 135, the various symbols have the following values :—

$$k^2 = 50.$$
$$K^2 = 25.$$
$$L = 10.$$

Smallest angular velocity for steady spin = 43·1 radians per sec. Value of ω, corresponding to 10 rotations per sec. = 62·8 radians per sec.

The smaller value of p is given by the equation—

$$p = 12{\cdot}56 - \sqrt{(158 - 78{\cdot}5)} = 3{\cdot}59 \text{ radians per sec.}$$

This corresponds to one conical rotation in 1·75 sec.

(4) Minimum speed = 13·2 ft./sec., or 9 miles per hour.

CHAPTER V

(2) Let original amplitude $= a_0$, and let amplitudes at ends of successive intervals of time, each equal to 5 min., be denoted by

$$a_1, a_2, a_3, \ldots a_n.$$

Then $$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots \frac{a_{n-1}}{a_n} = k \text{ (say)},$$

$$\therefore \quad \frac{a_0}{a_1} \times \frac{a_1}{a_2} \times \frac{a_2}{a_3} \times \ldots \times \frac{a_{n-1}}{a_n} = \frac{a_0}{a_n} = k^n.$$

In problem under consideration, $a_0 = 5$ cm., and $a_1 = 3$ cm.

$$\therefore \quad k = 5/3 = 1{\cdot}67.$$

Also, $a_n = 0{\cdot}1$ cm.

$$\therefore \quad \frac{5}{0{\cdot}1} = 50 = (1{\cdot}67)^n.$$

$$n \log 1{\cdot}67 = \log 50.$$

$$n = \frac{\log 50}{\log 1{\cdot}57} = \frac{1{\cdot}699}{0{\cdot}1959} = 8{\cdot}67.$$

Therefore, required time $= 8{\cdot}67 \times 5 = 43{\cdot}35$ min.

(3) $4{\cdot}66 \times 10^6$ cm./sec.

CHAPTER VI

(1) $\frac{mG\sqrt{3}}{l^2}$.

(2) $T = 2\pi\sqrt{\frac{3}{4\pi\rho G}}$.

(3) $\frac{2}{3}mG\pi\rho r^2$ ergs, if m, G, ρ, and r are measured in c.g.s. units.

(4) $1{\cdot}98 \times 10^{33}$ gm., or 2×10^{27} tons (nearly).

(5) $5{\cdot}97 \times 10^{27}$ gm., or 6×10^{21} tons (nearly).

(6) $\dfrac{\text{Mass of sun}}{\text{Mass of earth}} = 3{\cdot}51 \times 10^5.$

(7) In the absence of the sphere of platinum, the surface of the water will form part of a sphere with its centre at the centre of the earth. Now let it be supposed that a spherical mass of water, of radius r, becomes solid; no change will be produced in the shape of the free surface of the water. Next, let it be supposed that the solid sphere increases in density from 1 to σ, when σ denotes the density of platinum. Let the sphere drawn with C as centre (Fig. 272) represent the solid sphere, and let AK be a straight line drawn through the centres of the sphere and the earth. At the point B, at a small horizontal distance x from A, the gravitational attraction due to the earth is equal to $G\frac{4}{3}\pi R^3\rho/R^2 = G\frac{4}{3}\pi\rho R$, where ρ is the mean density, and R is the radius of the earth; this attraction acts along the straight line drawn from B to the centre of the earth. The attraction due to the increased density of the solidified sphere is equal to $G\frac{4}{3}\pi r^3(\sigma - 1)/r^2 = G\frac{4}{3}\pi(\sigma - 1)r$; this attraction acts along the line BC. Let BR denote the resultant attraction at B; the surface at B must be perpendicular to BR, as otherwise there would be a tangential force acting along the surface of the water, and the surface would not be in equilibrium. Produce BR to cut AK in K; then K is the centre of curvature, and BK is the radius of curvature, of the water just above the sphere. Let BK = R′, and let the resultant force BR be denoted by F. Then, resolving F and its components perpendicular to AK, we have—

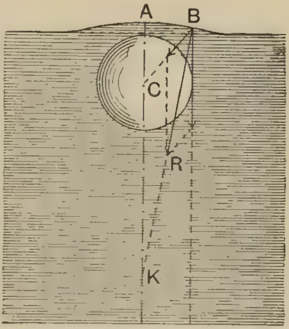

FIG. 272.—Curvature of the surface of a liquid, due to gravitational attraction of a massive submerged sphere.

$$F\frac{x}{R'} = \frac{4}{3}G\pi\rho R\frac{x}{R} + \frac{4}{3}G\pi(\sigma - 1)r\frac{x}{r},$$

$$\therefore \quad \frac{F}{R'} = \frac{4}{3}G\pi(\rho + \sigma - 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

F and its components are all very nearly parallel to AK, since x = AB is supposed to be small in comparison with the radius of the solid sphere. Thus—

$$F = \frac{4}{3}G\pi\{\rho R + (\sigma - 1)r\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Dividing (1) by (2), and neglecting the small quantity $(\sigma - 1)r$ in comparison with the much larger quantity ρR, we have—

$$\frac{1}{R'} = \frac{1}{R}\left(1 + \frac{\sigma - 1}{\rho}\right).$$

This equation determines the value of R′, the radius of curvature of the water surface just above the sphere. When the sphere has a density $\sigma = 21{\cdot}5$ gm./(cm.)^3, we have—

$$\frac{1}{R'} = \frac{1}{R}\left(1 + \frac{20{\cdot}5}{5{\cdot}53}\right) = \frac{1}{R} \cdot \frac{26{\cdot}03}{5{\cdot}53},$$

$$\therefore \quad R' = \frac{R}{4{\cdot}71}.$$

(8) See p. 210.

(9) The following definitions must be remembered. The **angle of a cone** is the angle that a generating line (p. 138) makes with the axis. Let a sphere be described with the vertex of the cone as centre ; the area cut off from this sphere by the cone, divided by the square of the radius of the sphere, is a constant called the **solid angle** of the cone.

Call the chosen point P, and draw a perpendicular PO from P to the plane. About PO as axis, describe two cones with slightly different angles θ and $(\theta + \phi)$; these cones cut off an annular strip of area a from the plane lamina. The gravitational attraction exerted at P by the annular strip will be perpendicular to the plane lamina ; for equal elements at opposite ends of a diameter of the annulus exert forces whose components parallel to the lamina are equal and opposite. Let r denote distance from strip to P ; then resultant attraction at P due to strip $= (Gma/r^2)\cos\theta$. Now $a\cos\theta/r^2 =$ the difference between the solid angles of the two cones. Let the solid angles of the cones be equal to ω_a and ω_b ; then resultant attraction at P due to annular strip $= Gm(\omega_b - \omega_a)$.

Describe cones with solid angles $\omega_0, \omega_1, \omega_2, \ldots \omega_n$, where ω_0 approximates to zero, and $\omega_n = 2\pi$, the solid angle subtended by a hemisphere at its centre. Then, summing the attractions exerted at P by the annular strips into which the plane lamina is divided, the resultant attraction is found to be equal to $2\pi Gm$.

CHAPTER VII

(1) $k = 3{\cdot}5 \times 10^{11}$ c.g.s. units, $\sigma = 0{\cdot}167$.

(2) $n = 5{\cdot}37 \times 10^{10}$ poundals per sq. ft., or $1{\cdot}16 \times 10^7$ pounds per sq. in.

$k = 1{\cdot}00 \times 10^{11}$ poundals per sq. ft., or $2{\cdot}16 \times 10^7$ pounds per sq. in.

(3) A uniform dilatational strain A is equivalent to an elongation A/3

parallel to each of the axes. A shear θ_1 in the plane x, y, is equivalent to an elongation $\theta_1/2$ parallel to x, and a linear compression $\theta_1/2$ parallel to y. A shear θ_2 in the plane x, z, is equivalent to an elongation $\theta_2/2$ parallel to z and a linear compression $\theta_2/2$ parallel to x. Then—

$$a = \frac{A}{3} + \frac{\theta_1}{2} - \frac{\theta_2}{2}.$$

$$b = \frac{A}{3} - \frac{\theta_1}{2}.$$

$$c = \frac{A}{3} + \frac{\theta_2}{2}.$$

$$\therefore \quad a + b + c = A.$$

$$\frac{\theta_1}{2} = \frac{A}{3} - b; \quad \therefore \ \theta_1 = \left\{(2/3)(a+b+c) - 2b\right\}.$$

$$\frac{\theta_2}{2} = -\frac{A}{3} + c \quad \therefore \ \theta_2 = -\left\{(2/3)(a+b+c) - 2c\right\}.$$

(4) Dilatational strain = 1×10^3.

This strain could be produced by tensile stresses, each equal to $15 \times 10^{11} \times 1 \times 10^{-3} = 1{\cdot}5 \times 10^9$ dyne/(cm.)^2, parallel respectively to the axes of x, y, and z.

Shear θ_1 in plane x, y, $= 2{\cdot}67 \times 10^4$. This shear could be produced by a tensile stress parallel to x, equal to $8 \times 10^{11} \times 2{\cdot}67 \times 10^{-4} = 2{\cdot}14 \times 10^8$ dyne/(cm.)^2, and an equal compressive stress parallel to y.

Shear θ_2 in plane x, z, $= -3{\cdot}33 \times 10^{-4}$. This shear could be produced by a tensile stress parallel to x, equal to $8 \times 10^{11} \times 3{\cdot}33 \times 10^4 = 2{\cdot}66 \times 10^8$ dyne/(cm.)^2, and an equal compressive stress parallel to z.

Resultant tensile stress parallel to $x = (15 + 2{\cdot}14 + 2{\cdot}66) \times 10^8$
$= 1{\cdot}98 \times 10^9$ dyne/(cm.)^2.

Resultant tensile stress parallel to $y = (15 - 2{\cdot}14) \times 10^8$
$= 1{\cdot}29 \times 10^9$ dyne/(cm.)^2.

Resultant tensile stress parallel to $z = (15 - 2{\cdot}66) \times 10^8$
$= 1{\cdot}23 \times 10^9$ dyne/(cm.)^2.

(5) In the answer to question (3), let $b = 0$, and $c = 0$. Then—

$$A = a.$$

$$\theta_1 = (2/3)a.$$

$$\theta_2 = -(2/3)a.$$

To produce dilatational strain A, a tensile stress parallel to x and equal to $kA = ka$ is needed. To produce each of the shears θ_1 and θ_2, a tensile stress parallel to x equal to $(2/3)na$ is needed. Thus—

$$f_1 = ka + \tfrac{4}{3}na,$$

and

$$\frac{f_1}{a} = k + \tfrac{4}{3}n.$$

The value of f_1/a is the modulus of longitudinal elasticity when the conditions are such that no lateral contraction or expansion can occur.

(6) $4{\cdot}8 \times 10^8$ dynes per sq. cm., or half a ton per sq. cm. (nearly).

(7) Required period $= 2^4 \times \mathrm{T}$.

(8) $(1 - 0{\cdot}0078)$ N.

(9) $$\frac{\text{Sag of tube}}{\text{Sag of rod}} = \frac{\mathrm{R}^4}{\mathrm{R}^4 - r^4} = \frac{1}{1 - \left(\dfrac{r}{\mathrm{R}}\right)^4}.$$

(10) Spring must be lengthened to $(721/720)^2$ of its original length.

CHAPTER VIII

(1) Treating the scale as a cantilever, force per unit displacement of free end $= \dfrac{3\mathrm{EK}}{l^3}$ (p. 258), and $\mathrm{K} = bd^3/12$, where b is the breadth, and d the thickness of the scale. Therefore (p. 94), if mass of heavy body $= \mathrm{M}$,—

$$\mathrm{T} = 2\pi\sqrt{\frac{4\mathrm{M}l^3}{\mathrm{E}bd^3}}.$$

(2) Treating the scale as a twisted strip, torque per unit twist $= nbd^3/3l$ (p. 276). Thus (p. 108) if I denotes the moment of inertia of the heavy body—

$$\mathrm{T} = 2\pi\sqrt{\frac{3\mathrm{I}l}{nbd^3}}.$$

(4) 104 dyne-centimetres.

(5) Let the rod be divided in imagination into very short elements of length. When the stress f_1 is applied to one end of the rod, the effect is to compress the first element of the rod until its strain is equal to f_1/E, where E is the elastic modulus (Young's modulus, if the rod can expand laterally; $(k + \frac{4}{3}n)$ if the rod is prevented from expanding laterally, p. 605). The first element then transmits the stress f_1 to the next element, which becomes compressed in its turn; and so the strain travels along the rod. After an element has been compressed, it moves forward with uniform velocity, since its front and rear surfaces are acted on by equal and opposite forces. In t sec. let the strain reach the point B (Fig. 273), where AB = Vt. The longitudinal compression

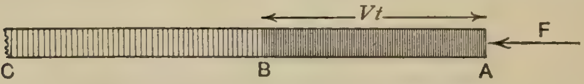

FIG. 273.—Compressional strain transmitted along rod.

of the portion AB of the rod is equal to $Vt \times f_1/E$; thus the end A has moved towards C through the distance $Vt \times f_1/E$, and therefore each element between A and B is moving forward with the uniform velocity Vf_1/E.

Let a = cross-sectional area of rod, and let $f_1 a = F$. During time t, work done by applied force F is equal to $F \times VtF/aE$. If the portion AB of the rod were brought to rest, and then allowed to expand to its original length, the energy given up during the expansion would be equal to $\frac{1}{2}\, VtF^2/aE$, for meanwhile the stress would fall off uniformly from f_1 to zero. Thus, **the potential energy stored in the compressed portion AB of the rod is equal to half the work done by the applied force** F. Therefore, the remaining half of the work done by the applied force has been converted into kinetic energy, and **the total energy of the compressed portion is equally divided between the potential and kinetic types.**

Mass of portion AB = $Vta\rho$, where ρ = density of rod. Kinetic energy of portion AB = $(1/2)\, Vta\rho \times (VF/aE)^2$.

$$\therefore \quad (1/2)Vta\rho \times (VF/aE)^2 = (1/2)VtF^2/aE,$$

$$\therefore \quad V^2\rho/E = 1,$$

$$\text{and } V = \sqrt{(E/\rho)}.$$

(6) Let rod be divided in imagination into elements, as described in answer to previous question. Let AC (Fig. 273) represent the rod, and suppose that a torque τ is applied to end A. The first element is strained until its restoring torque is equal to τ, and then this element rotates with uniform angular velocity while the other elements are strained in turn. When the torsional strain has travelled to B, where $AB = Vt$, the end A has rotated through an angle $(\tau \times 2Vt/n\pi r^4)$ (p. 237), and work done by torque = $\tau^2 \times 2Vt/2\pi r^4$ (p. 41). Half of this work has been converted into the potential energy of the strained portion AB, (compare answer to question 5), and therefore the kinetic energy of the portion AB is equal to $\tau^2 Vt/2\pi r^4$. The portion AB is rotating with angular velocity $2\tau V/n\pi r^4$, and the moment of inertia of AB = $\rho Vt\pi r^2 \,.\, r^2/2$. Thus, kinetic energy of AB = $1/2\, (\rho Vt\pi r^4/2)(2\tau V/n\pi r^4)^2$.

$$\therefore \quad (1/2)(\rho Vt\pi r^4/2)(2\tau V/n\pi r^4)^2 = \tau^2 Vt/n\pi r^4,$$

$$\therefore \quad V^2 \frac{\rho}{n} = 1,$$

$$\text{and } V = \sqrt{(n/\rho)}.$$

(7) Let ABC (Fig. 274) represent the rod, bent so that its neutral axis assumes the form of a sinusoidal curve, by opposite collinear forces, each equal to F, applied to its ends. The restoring torque called into play at the section P is equal (p. 247) to $EK/R = EK \times (2\pi/\lambda)^2 PQ$.

The force F at C exerts a torque equal to F × PQ at P. Therefore, for equilibrium—

$$F \times PQ = EK(2\pi/\lambda)^2 PQ,$$
$$\therefore \quad F = EK(2\pi/\lambda)^2 = EK(\pi/l)^2.$$

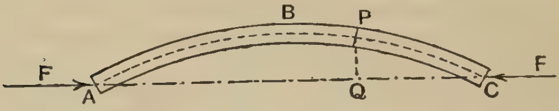

FIG 274.—Rod bent by collinear forces applied to its ends.

(8) Let Fig. 275 represent a flexural wave travelling with velocity V from left to right along the rod. The neutral axis of the rod is bent into a number of loops of a sine curve. Impress a velocity equal to V on the whole of the rod, in the direction from right to left; the waves now become stationary in space, and the rod moves past them. The portion of the rod which at any instant forms a loop of the curve, such as ABC, needs equal and opposite forces, parallel to the axis XX′, applied to its ends; each of these forces is numerically equal to EK $(2\pi/\lambda)^2$ (see answer to question 7).

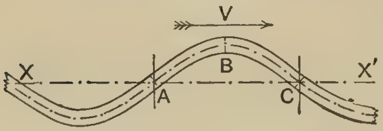

FIG. 275.—Flexural wave travelling along a rod.

Through A and C draw imaginary planes perpendicular to XX′; then, if m denotes the mass per unit length of the rod, the mass that crosses the plane through C in a second, moving from right to left with the velocity V, is equal to mV; and the space to the left of the plane through C is gaining momentum at the rate of mV^2 units per second. This is equivalent to a force equal to mV^2 acting from right to left on the part of the rod to the left of C; and the reaction on the part of the rod to the right of C is equivalent to a force mV^2 acting from left to right. Similarly, a force equal to mV^2 acts on the part of the rod to the right of A. Then, for the equilibrium of the part ABC—

$$mV^2 = EK\left(\frac{2\pi}{\lambda}\right)^2,$$
$$\therefore \quad V = \frac{2\pi}{\lambda}\sqrt{\frac{EK}{m}}.$$

Let the cross-sectional area of the rod $= a$; then $m = a\rho$, where ρ is the density of the rod. Also, let $K = ak^2$, where k denotes the radius of gyration (p. 58) of the cross-section of the rod about its line of intersection with the neutral surface. Then—

$$V = \frac{2\pi k}{\lambda}\sqrt{\frac{E}{\rho}}.$$

(9) Pressure = $a\rho x$, where ρ = density of rod.

(10) Let ABC (Fig. 276) represent a transverse disturbance travelling with the velocity V from left to right along the cord. Impress on the whole of the cord a velocity V from right to left; the disturbance now remains stationary in space, and the cord travels past it. Let DBE be a short element of the cord, of length l, passing the crest of the disturbance. Complete the circle of which DBE forms a part; then the centre K of this circle is the centre of curvature of the curve at B. Let BK = R. If m denotes the mass per unit length of the cord, the element DBE must be urged towards K with a force equal to mlV^2/R (p. 26).

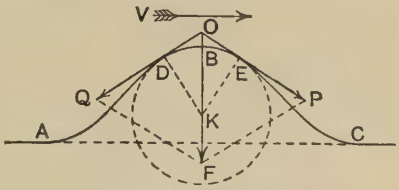

Fig. 276.—Transverse disturbance travelling along a stretched cord.

Join KE and KD, and through E and D draw lines perpendicular to KE and KD, and let these lines meet at O. Along OE produced, mark off a length OP equal to the tension f of the cord; and mark off an equal length OQ from OD produced. The effective force acting on the element DBE is the resultant of the two forces OP and OQ acting normally across the ends of the element; the resultant OF is the diagonal of the parallelogram of which OP and OQ form conterminous sides. If the element DBE is very short, the length of a straight line drawn from D to E will be equal to l; the triangles DEK and OFP will be similar, so that—

$$OF/OP = DE/EK.$$

$$\therefore\quad OF = fl/R.$$

Thus—

$$mlV^2/R = fl/R$$

and

$$V = \sqrt{\frac{f}{m}}$$

The same result may be obtained more quickly by applying the principle explained in the answer to question 8. The transfer of momentum past the points E and D produces opposite compressive forces, each equal to mV^2, acting on the ends of the element DBE. These compressive forces must be equal to the tensions acting across the ends of the elements, and therefore $mV^2 = f$.

CHAPTER IX

(1) 2·64 cm.

(2) 0·0195° C.

(3) 544 ergs per sq. cm.

(5) $1{\cdot}14 \times 10^6$ dyne/(cm)^2.

(6) 0·066 cm.

(7) $1{\cdot}88 \times 10^5$ ergs.

(8) Let Q = total charge of bubble. Then outward pull normal to each sq. cm. of the surface $= 2\pi(Q/4\pi r^2)^2 = (1/8\pi)V^2(1/r)^2$.

$$\therefore \quad \frac{1}{8\pi r^2}V^2 = \frac{4S}{r}.$$

(9) 100·16° C. (nearly).

CHAPTER X

(1) 180 dyne/(cm.)^2.

(2) Radius of cylindrical bubble must be half that of spherical bubble.

(3) Thickness d of disc of mercury $= 1/(13{\cdot}6\pi \times 25) = 9{\cdot}4 \times 10^{-4}$ cm. Radius of curvature R of a vertical section, through middle of disc of mercury, is given by equation—

$$2R \sin(140^\circ - 90^\circ) = 9{\cdot}4 \times 10^{-4}.$$

$$\therefore \quad R = 6{\cdot}15 \times 10^{-4} \text{ cm.}$$

Excess of pressure inside mercury $= 7{\cdot}3 \times 10^5$ dyne/(cm.)^2.

Force exerted on upper glass plate $= 5{\cdot}75 \times 10^7$ dynes, or $5{\cdot}85 \times 10^4$ gm. (130 lbs. nearly).

(5) $4{\cdot}12 \times 10^4$ dynes.

(6) Draw a horizontal line to represent the level of the plane part of the surface of the drop. Calculate the thickness H of the drop (p. 309), and draw another line to represent the horizontal surface of the solid on which the drop rests (Fig. 277). Let the surface of the water leave the solid at A. The radius of curvature R of the surface of the liquid where it leaves the solid is given by the equation—

$$S/R = g\rho H.$$

From A measure a distance R vertically upwards, and with the point so chosen describe a small circular arc of radius R, to represent the lowest arc of the profile. The remaining arcs of the profile are drawn in the manner indicated in Fig. 277, the radius of curvature of each arc being calculated from the mean depth of the arc beneath the horizontal free surface of the liquid. Fig. 277 is drawn to scale for water, density = 1 gm./(cm.)^3, and S = 70 dyne/cm.

(7) The section of the surface is drawn in the manner indicated in the answer to question 6. When the water wets the plate, the section of

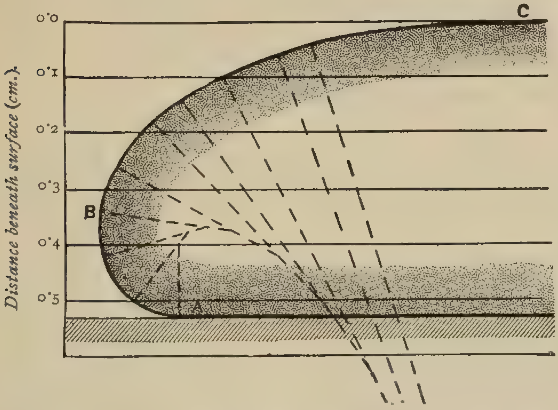

FIG. 277.—Profile of a drop of water resting on a plane surface which it does not wet.

the surface is found by inverting the part BC of Fig. 277, the liquid being on the convex side of the curve. When the liquid does not wet the plate, the section of the surface is similar to the part BC of Fig. 277.

(9) Let Fig. 278 represent a transverse section of a needle, of radius r, floating on water. Let the surface of the water leave the needle tangentially at B, where the radius CB of the needle makes an angle θ with the vertical. Let the point B be at a distance h below the flat surface of the water. The upwardly directed force acting on unit length of needle due to the surface tension of the water, is equal to 2S sin θ. If the portion of the needle below the horizontal plane AB were removed, the hydrostatic pressure of the water would exert an upward force equal

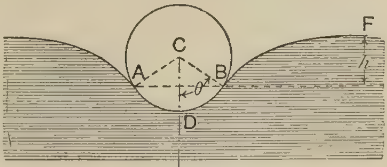

FIG. 278.—Needle floating on water.

to $g\rho h \times 2r \sin\theta$ per unit length of the needle; ρ denotes the density of the water. The additional upward force due to the portion of the needle below AB is equal to the weight of the water displaced by this portion of the needle; that is, to $\{r^2\theta - (r^2 \sin 2\theta)/2\}g\rho$. The downward pull of gravity on unit length of the needle is equal to $g\pi r^2\delta$, where δ denotes the density of the needle. Thus—

$$2S\sin\theta + 2g\rho h r\sin\theta + \left\{r^2\theta - \frac{r^2\sin 2\theta}{2}\right\}g\rho = g\pi r^2\delta.$$

For the liquid FB to be in equilibrium—

$$S(1-\cos\theta) = \tfrac{1}{2}g\rho h^2 \text{ (see p. 307)},$$

$$\therefore \quad h = 2\sin\frac{\theta}{2}\sqrt{\frac{S}{g\rho}}\,.$$

Substitute this value of h in the first equation; on giving their numerical values to the various constants, and simplifying, we obtain the equation—

$$r^2(24{\cdot}2 + 0{\cdot}5\sin 2\theta - \theta) - 2r \times 0{\cdot}534\sin\theta\sin\frac{\theta}{2} = 0{\cdot}1426\sin\theta.$$

This is a quadratic in r; solving it, and effecting a slight simplification, we obtain the value of r in the form—

$$r = \frac{\sin\theta\sin(\theta/2)}{45{\cdot}2 + 0{\cdot}932\sin 2\theta - 1{\cdot}86\theta}$$
$$+\sqrt{\frac{0{\cdot}267\sin\theta}{45{\cdot}2+0{\cdot}932\sin 2\theta - 1{\cdot}86\theta} + \left(\frac{\sin\theta\sin(\theta/2)}{45{\cdot}2+0{\cdot}932\sin 2\theta - 1{\cdot}86\theta}\right)^2}\,.$$

Fig. 279 shows the values of r for values of θ lying between 0° and 120°. It will be noticed that the maximum value of r corresponds to $\theta = 100°$, and is equal to 0·0995 cm., or roughly, 1 mm. Thus the diameter of the largest needle that can float on water is equal to 2 mm. very nearly. The position of the needle, relatively to the surface, is shown in Fig. 280, which is drawn to scale. The section of the water surface is merely a part of the profile represented in Fig 277.

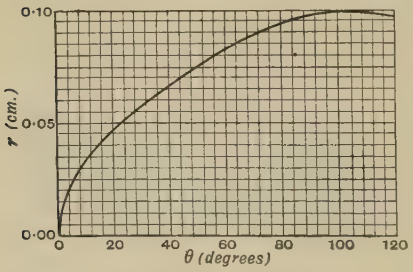

FIG. 279.—Graph exhibiting relation between r and θ.

If a steel needle 1·7 mm. in diameter be heated and coated thinly with paraffin wax, and dropped into some lycopodium powder before the wax has solidified, this needle can be floated on water by placing it on a piece of blotting paper and then floating this on water (see p. 300).

(10) In this case, a glance at Fig. 279 shows that θ must be small. Substituting θ for sin θ, etc., in the original equation, and discarding terms involving θ^2, we find that—

$$(0{\cdot}02)^2(24{\cdot}2+\theta-\theta)=0{\cdot}1426\,\theta,$$

$$\therefore\quad \theta=0{\cdot}068 \text{ radian}=3{\cdot}8^\circ$$

$$h=\theta\sqrt{\frac{S}{g\rho}}=0{\cdot}068\times0{\cdot}267=0{\cdot}0183 \text{ cm.}$$

(11) Mass of drop $=k\dfrac{SR}{g}$,

where S = surface tension of water, and R = radius of sphere.

(12) Decrease in surface energy $=8\pi r E_1\delta$, where E_1 denotes the energy per unit area of the surface (p. 295). The thermal energy needed to evaporate a layer of thickness δ is equal to $4\pi r^2\rho L\delta$, where L denotes the latent heat of vaporisation. For the layer to evaporate without the supply of heat from external sources—

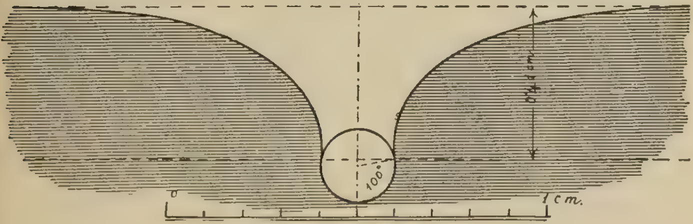

FIG. 280.—Needle floating on the surface of water.

$$8\pi r E_1\delta>4\pi r^2\rho L\delta J,$$

$$\therefore\quad r<\frac{2E_1}{\rho LJ}.$$

(13) Radius r of largest drop of water that could evaporate at 0° C. without heat being supplied to it, is given by the equation

$$r=\frac{2\times117}{1\times606\times4{\cdot}2\times10^7}=9{\cdot}2\times10^{-9}\text{ cm.}$$

CHAPTER XI

(1) No. The attraction of gravity acts on each particle of matter directly; the forces which produce the acceleration dealt with on p. 369, act on the walls of the tube, and are transmitted by these to the enclosed liquid.

(5) Let B = difference of velocity-potential between any two neighbouring equipotential surfaces. Let d = distance between two neighbouring equipotential surfaces, and V = velocity of liquid between these surfaces. Then $Vd = B$. Also, if a = sectional area of tube of flow, $Va = A$, where A is a constant. Therefore $V^2ad = AB$. The kinetic energy of the liquid enclosed by the tube of flow and the two equipotential surfaces, is equal to $(1/2)\rho ad . V^2 = (1/2)\rho AB$, which is constant throughout the liquid, since ρ is constant.

(6) See p. 425.

(7) Let A and B (Fig. 281) represent the cross-sections of the vortex filaments. Let C be a point at the distance d from A; and with C as centre, let a circle of radius R be described. **In order that this circle may be a line of flow, there must be no component velocity perpendicular to any element of the circle.** Thus, at the point D the component velocity parallel to the radius DC must be equal to zero.

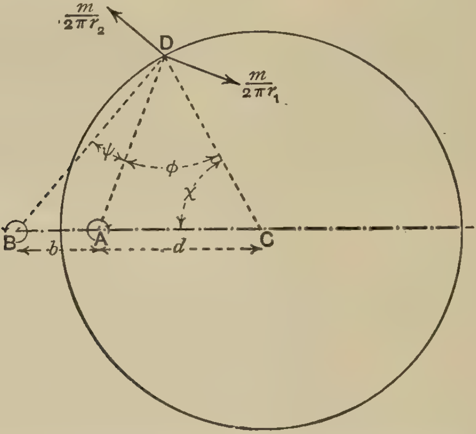

FIG. 281.—Line of flow due to two equal but dissimilar vortex filaments.

Let $AD = r_1$, $BD = r_2$. Let $\angle ADB = \psi$, $\angle ADC = \phi$, and $\angle DCA = \chi$. Then, the velocity at D due to the vortex filament at A is equal to $m/2\pi r_1$ (p. 395), and its direction is perpendicular to AD; the component of this velocity, parallel to DC, is equal to $(m/2\pi r_1)\sin\phi$. The component velocity parallel to DC, due to the other filament, is equal to $(m/2\pi r_2)\sin(\phi+\psi)$. Thus—

$$\frac{m}{r_1}\cdot\sin\phi = \frac{m}{r_2}\sin(\phi+\psi).$$

Now

$$d/r_1 = \sin\phi/\sin\chi$$

$$(d+b)/r_2 = \sin(\phi+\psi)/\sin\chi,$$

$$\therefore\quad \left(\frac{md}{r_1^2} - \frac{m(d+b)}{r_2^2}\right)\sin\chi = 0.$$

Since χ may have any value, it follows that—

$$\frac{d}{r_1^2} = \frac{(d+b)}{r_2^2}.$$

Also,

$$r_1^2 = R^2 + d^2 - 2Rd\cos\chi,$$

and

$$r_2^2 = R^2 + (d+b)^2 - 2R(d+b)\cos\chi,$$

$$\therefore\quad d(R^2 + (d+b)^2 - 2R(d+b)\cos\chi) = (d+b)(R^2 + d^2 - 2Rd\cos\chi),$$

$$\therefore\quad bR^2 - bd(d+b) = 0,$$

and

$$R^2 = d(d+b).$$

(9) Let a source of strength $(+q)$ be situated at A (Fig. 282), while a sink of strength $(-q)$ is situated at B. Let it be supposed that A and B are in reality at an infinitesimal distance apart. From a point P in the surrounding fluid, draw PC to the point midway between A and B, and let $PC = r$, while $\angle\ PCA = \theta$. With P as centre, describe the circular arc AD; this will approximate to a straight line perpendicular to PB, when the distance AB is infinitesimally small.

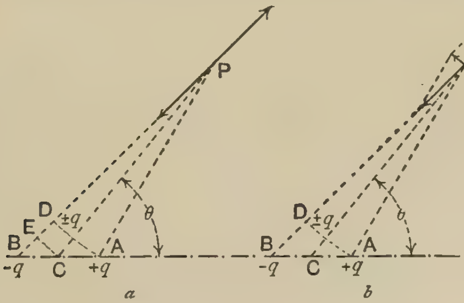

FIG. 282.—Velocity of flow due to a doublet.

To determine the component velocity at P, **in a direction parallel to** PC, let a source of strength $(+q)$ be superimposed on a sink of strength $(-q)$ at the point D. Then the source $(+q)$ at A, together with the sink $(-q)$ at D, produce no appreciable velocity at P, parallel to PC. The component velocity at P, parallel to PC, is due to the source $(+q)$ at D, combined with the sink $(-q)$ at B. With P as centre, and radius $PC = r$, describe the circular arc CE. Then, if $AB = l$, it follows that $BE = ED = (l\cos\theta)/2$. Thus (Fig. 282, a), the velocity at P due to $(+q)$ at D and $(-q)$ at B is equal to—

$$\frac{q}{4\pi\rho\left(r-\dfrac{l\cos\theta}{2}\right)^2}-\frac{q}{4\pi\rho\left(r+\dfrac{l\cos\theta}{2}\right)^2}=\frac{2qlr\cos\theta}{4\pi\rho\left\{r^2-\left(\dfrac{l\cos\theta}{2}\right)^2\right\}^2}$$

$$\doteq\frac{2s\cos\theta}{4\pi\rho r^3},$$

when s is written for ql, and l is so small that $(l\cos\theta/2)^2$ can be neglected in comparison with r^2.

To determine the component velocity at P, **in a direction perpendicular to** PC, notice that the source $(+q)$ at D and the sink $(-q)$ at B produce no appreciable velocity perpendicular to PC; therefore we must determine the resultant velocity V at P due to $(+q)$ at A and $(-q)$ at D. For this purpose we may take $QA = QD = r$ (Fig. 282, b); and the required velocity, V, is the resultant of the velocity $(q/4\pi\rho r^2)$ along AP produced, and the equal velocity along PD. Then—

$$V\bigg/\frac{q}{4\pi\rho r^2}=\frac{AD}{r},$$

and

$$V=\frac{qAD}{4\pi\rho r^3}=\frac{s\sin\theta}{4\pi\rho r^3}$$

(10) The flow is obviously perpendicular to the linear source; if we describe a cylinder of radius r coaxial with the source, the mass of fluid q_1 generated by unit length of the source will pass normally through unit length of the cylindrical surface.

Thus— $$q_1 = 2\pi r\rho V.$$

(11) The answer to this question is obtained by a procedure essentially similar to that employed in answer to question 9, taking account of the result obtained in answer to question 10.

CHAPTER XII

(1) Let θ = inclination of tube to horizontal. Then—

$$\tan\theta=\frac{e}{\rho g}.$$

(2) During a second, a mass $\rho a V$ of liquid loses its velocity normal to the surface. Therefore, rate of change of momentum $= \rho a V^2$. This result gives the force exerted on that part of the plane, between

the point cut by the axis of the jet, and a concentric circle drawn through the points at which the normal velocity of the jet has just disappeared; **it does not give the pressure** (force per unit area) **on the plane.**

(3) Force f exerted across orifice of nozzle $= \rho a V^2$. Also, $V^2 = 2gh$, where h is the height above the orifice of the free surface of the liquid contained in the tank. Therefore, $f = a \times 2g\rho h$.

(4) Velocity of escaping kerosene $= 2{\cdot}41 \times 10^6$ cm./sec.

(5) Impress a velocity $(-V)$ on the locomotive and the water in the reservoir; then the water in the reservoir flows with velocity V up to the orifice of the horizontal portion of the pipe. Describe a tube of flow extending up to the orifice of the pipe, and along the pipe to the interior of the tank where the water is flowing with a velocity v. Let A be the sectional area of the tube of flow in the reservoir where the velocity is equal to V, and let a be the cross-sectional area of the pipe where the water enters the tank with velocity v; then, if $P =$ the atmospheric pressure, both above the reservoir and in the tank—

$$va(P + \tfrac{1}{2}\rho v^2) - VA(P + \tfrac{1}{2}\rho V^2) = -\rho va \,.\, gh,$$

$$\therefore \quad v^2 = V^2 - 2gh.$$

(6) In this case $V = 58{\cdot}7$ ft./sec. The virtual height h of the delivery pipe $= (4/3) \times 8 = 10{\cdot}7$ ft. Then—

$$v = \sqrt{\{(58{\cdot}7)^2 - 2 \times 32{\cdot}2 \times 10{\cdot}7\}} = 52{\cdot}5 \text{ ft./sec.}$$

Required time = 43·7 sec.

(7) In question 5, let h be such a height that $v = 0$. Then—

$$h = \frac{V^2}{2g}.$$

(8) Assume constricted pipe to be horizontal. Then, pressure in constriction = 8·8 cm. of mercury. Water would boil under this pressure at a temperature of 49·1° C. This temperature should be substituted for 21·5° in question. Water at 21·5° C. would boil in the constriction if it entered the pipe under a head of 11 ft.

(10) Water enters and leaves the pipe at atmospheric pressure, and loses gravitational energy in flowing through the pipe. Hence, the velocity of the water must increase as it traverses the pipe, and this would be impossible if the water filled the pipe at all levels.

CHAPTER XIII

(1) Force per unit area of the spherical boundary of the source $= P + q^2/32\rho\pi^2 r^4$, where $P =$ statical pressure at a great distance from the source.

(2) Dynamical pressure on boundary of a spherical source of radius

$r = q^2/32\rho\pi^2r^4$. The force per unit area on the surface of a sphere charged with s electrostatic units of electricity per unit area $= 2\pi s^2 = q^2/8\pi r^4$, if the total charge on the sphere is equal to q. Therefore the force acting across the transverse section of a tube of flow $= 1/4\pi\rho \times$ the force acting across the transverse section of a similar electric tube of force. Let the number of tubes of force leaving a charge q be equal to the number of tubes of flow leaving a source of strength q. Then, from the similarity between tubes of force and tubes of flow (p. 423), it follows that the attraction of two sources of strength q_1 and q_2, when placed at a distance d apart, is equal to—

$$\frac{q_1q_2}{4\pi\rho d^2},$$

since the repulsion of two electric charges, equal respectively to q_1 and q_2 electrostatic units, placed at a distance d apart, is equal to q_1q_2/d^2.

(3) Maximum outward velocity of a point on the surface of either body $= (2\pi a/\mathrm{T})$ (p. 86). When either spherical body is expanding with maximum velocity, the flow of fluid away from it is equal to that due to a point source of strength $q = 4\pi\rho r^2(2\pi a/\mathrm{T})$. Therefore, maximum attraction of the pulsating bodies

$$= q^2/4\pi\rho d^2 = 16\pi^3\rho r^4a^2/\mathrm{T}^2d^2.$$

(4) Let any system of tubes of flow, which gives the correct flow normal to the boundary, be denoted by A. If there be another system of tubes of flow due to the same sources and sinks, which will also produce the correct flow normal to the boundary, let this be denoted by B. Reverse the flow B and superimpose it on A. There is now no flow normal to the boundaries, and the sources and sinks are all annihilated. Thus, if there are any resultant tubes of flow in the fluid, they must be closed ; for they cannot cross the external boundaries of the fluid, and they cannot start or end within the fluid. But closed tubes of flow would possess a definite curl, and this would imply that vortex filaments had been created by superimposing a flow due entirely to sources and sinks on another similar flow, and this is impossible.

(5) The boundary conditions are, that there can be no flow normal to the plane. In Fig. 182 (p. 385), describe a plane, perpendicular to the straight line joining the two sources and to the plane of the paper, through the point midway between the sources. There will be no flow normal to this plane, and therefore we may suppose the plane to become a rigid sheet, without altering the flow on either side of it. Subsequently, the fluid may be removed from one side of the sheet without altering the flow on the other side. We now have obtained a system of tubes of flow due to a source placed at a definite distance

from a rigid plane; and since this system satisfies the boundary conditions, it is the only possible system of tubes of flow (see answer to question 4). Thus, the force f exerted on a source q placed at a distance d from the plane boundary of the fluid, has the same value as the force exerted by a source q on another equal source placed at a distance $2d$; that is—

$$f = \frac{q^2}{16\pi\rho d^2} \text{ (see answer to question 2).}$$

(6) Let APB (Fig. 283) represent a diametral section of the sphere, moving from left to right in the direction of the diameter AB with the velocity V. Draw a radius PC making an angle θ with AB. Then the small element of area immediately surrounding P is moving with the velocity V parallel to AB; and the component velocity, normal to the area of the element, is equal to V cos θ. Hence, the fluid near to P, and just outside the sphere, must possess a component velocity V cos θ perpendicular to the element of area immediately surrounding P. Since the spherical surface is the only boundary of the fluid, it follows that we know the component velocity of the fluid normal to the boundary.

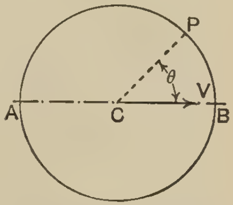

FIG. 283.—Sphere moving through an incompressible perfect fluid.

Let the sphere be removed, and let a doublet of strength s (p. 398) be placed at the position previously occupied by the centre C of the sphere. Let the axis of the doublet lie along AB, with the source on the right-hand side; then, at P the component velocity parallel to CP is equal to $2s \cos\theta/4\pi\rho R^3$, if $CP = R$ (p. 616). The flow due to this doublet will satisfy the boundary conditions due to the motion of the sphere with velocity V, if—

$$\frac{2s}{4\pi\rho R^3} = V.$$

Hence, $s = 2\pi\rho R^3 V$, and the instantaneous flow in the space outside the moving sphere is identical (p. 618) with that which would be produced if the sphere were removed, and a doublet of strength s were placed at the position previously occupied by the centre of the sphere. The lines of flow are represented in Fig. 284.

(7) In order to solve this problem, we must determine the total kinetic energy of the fluid set in motion by the sphere. Let an imaginary sphere MNO (Fig. 285) be described concentrically with the moving sphere APB. Then, if the radius, NC $= r$, makes an angle θ with the direction of motion CB of the sphere APB, it follows that the

velocity v of the fluid at N has the component $2s\cos\theta/4\pi\rho r^3$ parallel to CN, and the component $s\sin\theta/4\pi\rho r^3$ perpendicular to CN (p. 616). Thus—

$$v^2 = \{(2s\cos\theta)^2 + (s\sin\theta)^2\}/(4\pi\rho)^2 r^6$$
$$= \frac{s^2}{(4\pi\rho)^2 r^6}(1+3\cos^2\theta).$$

The particles of fluid, at a common distance r from C, together possess an amount of kinetic energy equal to the product of half their combined mass and the average value of v^2. The only variable part of v^2 is $\cos^2\theta$; thus, we must determine the average value of $\cos^2\theta$ for equidistant points distributed over the surface of the sphere MNO. This average value is *not* equal to the average value of $\cos^2\theta$ for equidistant points distributed around a circle (p. 53).

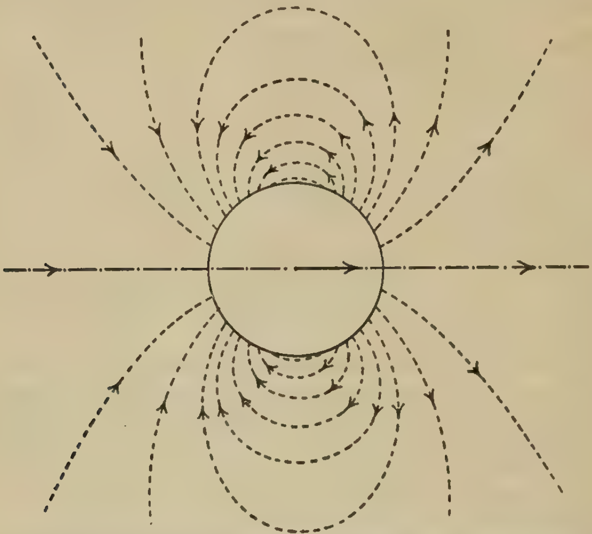

FIG. 284.—Lines of flow in an incompressible perfect fluid, due to the motion of a sphere. (To obtain the tubes of flow, rotate the diagram about the straight line extending from left to right.)

Describe a sphere concentric with MNO and of slightly greater radius. This sphere, together with MNO, cut off a spherical shell of the fluid. Let MCO be the diameter of the inner sphere, coinciding with the direction of motion of APB. Divide the straight line MO into a large number of equal parts, and through the points of division draw planes perpendicular to MO. These planes divide the spherical surface MNO into annular strips with equal areas (p. 541), and they divide the spherical shell of fluid into rings with equal masses. Let N be a point at the middle of one of the annular strips into which MNO has been divided, and from N drop the perpendicular NK on to MO. Let CK = x. If ∠ NCO = θ, $\cos^2\theta = (r^2\cos^2\theta/r^2) = x^2/r^2$; and since the various rings of fluid are all equal in mass, the average value of x^2/r^2 for the equal elements of the radius CO will give the average value of $\cos^2\theta$ for all the particles of fluid comprised in the spherical shell of fluid. Now, the average value of x^2/r^2 is equal to $\frac{1}{n}\sum\frac{x^2}{r^2}$, where n denotes the number of elements in the radius CO, and x is the distance from C to the centre of an element. Multiply and divide this sum by CO = r. Then the required average value $= \frac{r}{n}\sum\frac{x^2}{r^3}$, and r/n is equal to the length of any one of the n equal elements of the radius CO. Let the points of division of CO be at distances $x_0, x_1, x_2, \ldots x_n$ from C. Then for the element lying between the points x_0 and x_1, x^2 (which is intermediate in value between x_1^2 and x_0^2) can be written in the form $(x_0^2+x_0x_1+x_1^2)/3$ (p 48), and the values of x^2 for the other elements can be written in a similar form.

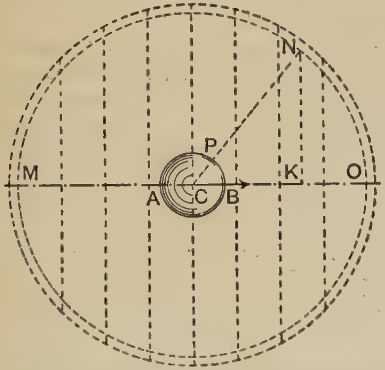

FIG. 285.—Kinetic energy of an incompressible perfect fluid through which a sphere is moving.

Also, $r/n = x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1}$. Thus—

$$\frac{r}{n}\sum\frac{x^2}{r^3} = \frac{1}{r^3}\left\{(x_1-x_0)\frac{(x_1^2+x_1x_0+x_0^2)}{3}+\ldots\right\}$$

$$= \frac{1}{3r^3}\left\{(x_1^3-x_0^3)+(x_2^3-x_1^3)+\ldots(x_n^3-x_{n-1}^3)\right\}$$

$$= \frac{1}{3r^3}(x_n^3-x_0^3) = \tfrac{1}{3},\ \text{since } x_0 = 0 \text{ and } x_n = r.$$

Thus, the average value of $\cos^2\theta$ for all particles of fluid comprised in the spherical shell of fluid, is equal to 1/3, and the average value of v^2 for these particles is equal to—

$$\frac{s^2}{(4\pi\rho)^2r^6}\left\{1+\left(3\times\tfrac{1}{3}\right)\right\}=\frac{2s^2}{(4\pi\rho)^2r^6}.$$

Now let the fluid surrounding APB be divided into spherical shells by means of concentric spheres of radii r_0, r_1, r_2, . . . r_n, where r_0 is equal to the radius R of the moving sphere, and r_n is infinitely great. The mass m_1 of the spherical shell of fluid bounded by the spheres of radii r_1 and r_0 is equal to $(4/3)\pi\rho(r_1^3-r_0^3)$. The kinetic energy of the shell of fluid is equal to—

$$\tfrac{1}{2}m_1\frac{2s^2}{(4\pi\rho)^2r^6},$$

and we may write $r^6=r_1^3r_0^3$ (compare p. 193). Thus, the kinetic energy of the shell of fluid is equal to—

$$\tfrac{1}{2}\cdot\frac{4\pi\rho}{3}\left(r_1^3-r_0^3\right)\times\cdot\frac{2s^2}{(4\pi\rho)^2r_1^3r_0^3}=\frac{s^2}{3(4\pi\rho)}\left(\frac{1}{r_0^3}-\frac{1}{r_1^3}\right).$$

Writing down the values of the kinetic energy for the other shells, and summing the results, we find that the total kinetic energy of the fluid surrounding the moving sphere APB is equal to—

$$\frac{s^2}{3(4\pi\rho)}\left\{\left(\frac{1}{r_0^3}-\frac{1}{r_1^3}\right)+\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)+\ldots\left(\frac{1}{r_{n-1}^3}-\frac{1}{r_n^3}\right)\right\}$$
$$=\frac{s^2}{3(4\pi\rho)}\,\frac{1}{R^3}.$$

From p. 619,

$$s=2\pi\rho R^3V.$$

Thus, the total kinetic energy of the fluid is equal to

$$\frac{(2\pi\rho R^3V)^2}{3(4\pi\rho)R^3}=\tfrac{1}{4}\left(\tfrac{4}{3}\pi\rho R^3\right)V^2,$$

and this is equal to one half of the kinetic energy of the mass $(4/3)\pi\rho R^3$ of fluid that would fill the space occupied by the sphere, if this mass were moving with a uniform velocity V. Thus, the effect of the surrounding fluid is to increase the inertia of the moving sphere by an amount equal to one half of the mass of fluid that is displaced by the sphere (see p. 20).

(8) This problem is solved by a method essentially similar to that employed in solving question 7. The following are the results of the various stages of the investigation. The instantaneous velocity of the fluid is the same as if the moving rod were replaced by a linear doublet (p. 398) of strength $s_1=2\pi\rho R^2V$, where R is the radius of the rod. The mutually perpendicular components of the velocity, at a distance r from the axis of the rod, are equal to $s_1\cos\theta/2\pi\rho r^2$ and $s_1\sin\theta/2\pi\rho r^2$

(p. 398), and the resultant value of the square of the velocity is equal to $s_1^2/(2\pi\rho r^2)^2$. The kinetic energy of the fluid bounded by two planes perpendicular to the rod and at unit distance apart, is equal to—

$$\tfrac{1}{2}\,\pi\rho(r_1^2 - r_o^2)\frac{s_1^2}{(2\pi\rho)^2 r_1^2 r_o^2} + \dots$$

$$= \tfrac{1}{4}\cdot\frac{s_1^2}{2\pi\rho}\left\{\left(\frac{1}{r_o^2} - \frac{1}{r_1^2}\right) + \left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) + \dots \left(\frac{1}{r_{n-1}^2} - \frac{1}{r_n^2}\right)\right\}$$

$$= \tfrac{1}{4}\,\frac{s_1^2}{2\pi\rho}\,\frac{1}{R^2}$$

$$= \tfrac{1}{2}\,.\,\pi\rho R^2 V^2,$$

which is equal to the kinetic energy of the mass of fluid displaced by unit length of the rod, if it were moving with the velocity V.

(9) The cross-section of the internal surface of the cylindrical vessel must be a line of flow. If the vessel were removed, and an equal vortex filament of opposite sign were placed in the straight line passing through the centre of the cylinder and the filament, and at a distance b from the latter, we should obtain a similar line of flow if $R^2 = d(d + b)$ (p. 398), and the lines of flow within the cylindrical space would be identical in both cases. Thus, the instantaneous velocity of the vortex filament is the same as if it were placed at a distance $b = (R^2 - d^2)/d$ from an equal filament of opposite sign, in an infinite ocean of fluid. For the rest, see p. 430.

(10) If a plane be drawn perpendicular to the paper through the straight line extending from the top to the bottom of Fig. 205 (p. 430), the plane may be supposed to be converted into a rigid sheet without altering the flow on either side of it. The motion of the filament on one side of the sheet will be the same as before the sheet was introduced. (Compare p. 618).

(11) This is a problem on the escape of a compressed gas from a small aperture in its containing vessel. If the channel through which the gas escapes is short, the effects of viscosity may be neglected, and the results of the investigation on pp. 437–447 may be used. The pressure at the orifice is equal to one-half of the pressure within the containing vessel, to a sufficient degree of approximation. The velocity V of escape, given by equation (2) (p. 440), is equal to—

$$\sqrt{\frac{2\gamma}{\gamma - 1}\,\frac{p_1}{\rho_1}\left\{1 - (0{\cdot}5)^{\frac{\gamma-1}{\gamma}}\right\}}$$

Also, $\sqrt{\dfrac{\gamma p_1}{\rho_1}}$ is equal to the velocity of sound in the interior of the containing vessel; at the average room temperature, this is equal to 340 metres per sec. Substituting $\gamma = 1{\cdot}41$, the velocity of escape at the orifice is found to be equal to 0·95 times the velocity of sound under the conditions pertaining to the interior of the vessel. The density of the escaping gas is equal to—

$$\rho_1 \times (0{\cdot}5)^{\frac{1}{\gamma}} = 0{\cdot}61\rho_1.$$

If a denotes the area of the aperture through which the gas escapes, the mass of gas that escapes per second is equal to

$$a \times 0{\cdot}61\rho_1 \times 0{\cdot}95 \times \textit{velocity of sound.}$$

Thus, the mass that escapes in a given time is proportional to the density of the gas within the vessel, and therefore to the pressure of the gas within the vessel.

Let M = mass of air contained by air tube at atmospheric pressure.

Then, PM = mass of air contained by air tube at P atmospheres pressure.

Let pressure in air tube when gas commences to escape = P_0 atmospheres.

Let m = mass of gas expelled per unit time when the internal pressure = P_0 atmospheres.

The mass of gas expelled per unit time when the internal pressure is P atmospheres $= \dfrac{Pm}{P_0}$.

Owing to the expulsion of the air, the internal pressure falls ; the ratio of the pressures, at the end and at the beginning of unit time, is equal to the ratio of the masses of gas contained at these times ; hence—

$$\left.\begin{array}{l}\text{Pressure at unit time after the}\\ \quad\text{pressure was equal to P}\end{array}\right\} = \frac{PM - \dfrac{P}{P_0}m}{PM}P$$

$$= \left(1 - \frac{m}{P_0M}\right)P.$$

Thus, to determine the pressure at the end of unit time, we must multiply the pressure at the beginning of that time by $\left(1 - \dfrac{m}{P_0 m}\right)$.

Pressure at commencement of experiment = P_0.

Pressure after lapse of first unit of time $= \left(1 - \dfrac{m}{P_0M}\right)P_0$.

,, ,, ,, second ,, ,, $= \left(1 - \dfrac{m}{P_0M}\right)^2 P_0$.

. . . . = ...

,, ,, ,, nth ,, ,, $= \left(1 - \dfrac{m}{P_0M}\right)^n P_0$.

The air in the bubble is practically at atmospheric pressure, and the volume of the bubble is equal to 0·0082 c. in., or, in round numbers,

8×10^{-3} c. in. Thus, if we take half an hour as the unit of time, the volume of air at atmospheric pressure that escapes in unit time = 2·4 c. in. Thus, $m/\mathrm{M} = 2{\cdot}4/80 = 0{\cdot}03$. At the end of 48 hours (96 units of time), the air escapes at half its original rate, and therefore the pressure = $\mathrm{P}_0/2$. Thus—

$$\mathrm{P}_0\left(1 - \frac{0{\cdot}03}{\mathrm{P}_0}\right)^{96} = \frac{\mathrm{P}_0}{2},$$

$$1 - \frac{0{\cdot}03}{\mathrm{P}_0} = (0{\cdot}5)^{\frac{1}{96}} = 0{\cdot}9927.$$

$$\therefore \quad \mathrm{P}_0 = \frac{0{\cdot}03}{1 - 0{\cdot}9927} = 4{\cdot}1 \text{ atmospheres.}$$

(12) Let ρ = density of air at atmospheric pressure; then, density of air contained by tyre at commencement of experiment = $4{\cdot}1\rho$, and density of escaping gas = $0{\cdot}61 \times 4{\cdot}1\rho = 2{\cdot}5\rho$. Velocity of gas at orifice = $0{\cdot}95 \times 34{,}000$ cm./sec. = 12,700 in./sec. The gas escapes at the rate of $(8 \times 10^{-3})/6 = 1{\cdot}3 \times 10^{-3}$ c. in. per sec. Thus, if a = area of orifice, measured in sq. in.—

$$\rho \times 1{\cdot}3 \times 10^{-3} = a \times 4{\cdot}1\rho \times 12{,}700,$$

$$\therefore \quad a = 2{\cdot}5 \times 10^{-8} \text{ sq. in.}$$

CHAPTER XIV

(4) 0·67 cm.

(5) 12·7 ft./sec., or 8·65 miles per hour.

(6) Let β = height of ridge on bed of canal, and let α = height of wave on surface over ridge, measured from undisturbed level of surface. Vertical accelerations may be neglected (p. 482) and velocity is uniform across any transverse section. Thus, velocity over ridge on bed = $\mathrm{VH}/(\mathrm{H}+\alpha-\beta)$.

Consider tube of flow just above bed of canal. For this tube of flow we have—

$$g\rho(\mathrm{H}+\alpha-\beta) + \tfrac{1}{2}\rho\mathrm{V}^2\left(\frac{\mathrm{H}}{\mathrm{H}+\alpha-\beta}\right) - (g\rho\mathrm{H} + \tfrac{1}{2}\rho\mathrm{V}^2) = -g\rho\beta,$$

$$\therefore \quad \mathrm{V}^2\left\{\frac{1}{\left(1+\dfrac{\alpha-\beta}{\mathrm{H}}\right)^2} - 1\right\} = -2g\alpha,$$

$$\therefore \quad \mathrm{V}^2 \times 2\frac{\alpha-\beta}{\mathrm{H}} = 2g\alpha,$$

$$\text{and} \quad \alpha = \frac{\mathrm{V}^2\beta}{\mathrm{V}^2 - g\mathrm{H}}.$$

Thus, α is positive or negative, according as $(V^2 - gH)$ is positive or negative. If $V^2 = gH$, α would be infinitely great; but this result cannot be trusted, since it has been tacitly assumed in the course of the investigation that α is small.

CHAPTER XV

(1) 3×10^4 dyne.
(2) $6{\cdot}72 \times 10^{-4}$ lb./ft. sec.
(3) 200 cm./sec.
(4) $3{\cdot}08 \times 10^{-2}$ c.c. per sec.

(5) Let H = initial height of free surface in cistern above level of tube. From answer to question 4, it follows that the water that escapes during an hour has a volume of $H \times 2{\cdot}2$ c.c., and during this hour the level of the surface in the cistern is reduced by $H \times 2{\cdot}2 \times 10^{-4}$ cm. Therefore, at end of hour, height of surface above tube $= H(1 - 2{\cdot}2 \times 10^{-4})$. After 24 hours the height of the surface will be equal to $H(1 - 2{\cdot}2 \times 10^{-4})^{24}$ (see p. 624).

$$\therefore \quad H(1 - 2{\cdot}2 \times 10^{-4})^{24} = 50,$$
$$\text{and } H = 50{\cdot}24 \text{ cm.}$$

(6) $7{\cdot}15 \times 10^6$ dyne-cm.
(10) (*a.*) $r = 1{\cdot}87 \times 10^{-4}$ cm. (*b.*) $r = 3{\cdot}82 \times 10^{-5}$ cm.

CHAPTER XVI

(1) $5{\cdot}2 \times 10^4$ dyne/(cm.)^2.
(2) $\overline{V} = 4{\cdot}46 \times 10^4$ cm./sec.
(3) $\overline{V} = 4{\cdot}57$ cm./sec.
(4) $7{\cdot}7 \times 10^{-5}$ cm.
(5) $\lambda = 8{\cdot}65 \times 10^{-6}$ cm., $d = 3{\cdot}16 \times 10^{-8}$ cm.
(6) $r = 3{\cdot}14 \times 10^{-4}$ cm.
(7) $1{\cdot}44 \times 10^{20}$.
(8) $3{\cdot}42 \times 10^{22}$.
(9) $d = 2.99 \times 10^{-8}$ cm.
(10) At 0° C., $P = 6{\cdot}7 \times 10^6$ dyne/(cm.)^2.
At 100° C., $P = 9{\cdot}15 \times 10^6$ dyne/(cm.)^2.

(11) Elevation of boiling point = 2·32° C. This value is in close agreement with experimental results.

(12) On the assumption that each dissolved molecule consists of one atom (that is, that the molecular weight of the dissolved iodine = 127),

the calculated value of the elevation of the boiling point = 0·64° C. This is roughly twice as great as the observed value. Notice that the solution is fairly concentrated, and therefore the calculated value of the osmotic pressure (which is obtained on the assumption that the solution is dilute) will not be quite accurate.

(13) The vapour pressure will be increased by 0·42 cm. of mercury.

(14) Depression of freezing point = 5·35° C.

APPENDIX I

THE GYROSTATIC COMPASS

THE earth is rotating about its polar axis, which possesses an approximately constant direction in space ; a plane including this axis and any given point on the earth is called the **meridian plane** at that point, and the line of intersection of this plane with the surface of the earth is called a **meridian of longitude.** In specifying a direction at any point on the surface of the earth, it is necessary to state the angle which the direction

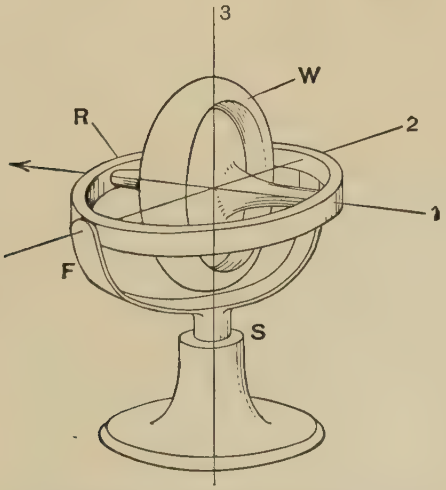

FIG. 286.—Gyrostat with three degrees of rotational freedom.

makes with the meridian of longitude passing through that point. The magnetic compass does not directly indicate the true meridian of longitude, since a suspended magnet comes to equilibrium with its axis on the **magnetic meridian,** which, in general, is inclined to the true meridian of longitude; thus troublesome corrections are needed in order to determine a direction correctly by the aid of a magnetic compass. Further, in large steel vessels, and particularly in submarine vessels, the magnetic compass is rendered inaccurate by the magnetism of the steel in its neighbourhood. Hence it is highly desirable to have an instrument which indicates the true meridian of longitude at any place, and this end is attained by the gyrostatic compass.

Consider a fly-wheel W (Fig. 286) rotating about the axis 1. Let the axle of the wheel be pivoted in the ring R, which in its turn is pivoted in the fork F so that it can rotate freely about the axis 2; and let the fork F be pivoted on the stand S so that it can rotate freely about the axis 3. Further, let it be supposed that the wheel, the ring, and the fork are balanced perfectly so that gravity exerts no torque on the wheel. Then if the wheel is set spinning its axle will maintain a constant direction in space. Such a rotating wheel might be used to measure directions, but several difficulties would be encountered. The direction of the axle of the wheel relative to some fixed direction in space would have to be determined, and as the meridian of longitude moves with the earth, the axle would not make a constant angle with the meridian plane; further, if the rotation of the wheel were accidentally allowed to cease, the axle might acquire a new constant direction when the wheel was set in rotation once more. It is therefore desirable to modify the arrangement represented diagrammatically in Fig. 286 so that the axle of the fly-wheel tends to set itself in the meridian plane, instead of pointing in a constant direction in space. This end is attained by the **Anschütz Gyrostatic Compass.**

Let it be supposed that a small weight is fixed to the ring R so that gravitation tends to keep the ring horizontal, *i. e.* with its plane perpendicular to that radius of the earth which passes through the common centre of the ring and fly-wheel. For simplicity let it be supposed that the fly-wheel is set rotating at the equator, and that the axis of rotation (p. 61) initially points from East to West. Then, looking along the initial direction of the axis (from East to West, A, Fig. 287) the fly-wheel would be seen rotating in a clockwise direction. As the fly-wheel is carried toward the East owing to the rotation of the earth, the direction of the axle would remain constant were it not for the weight attached to the ring; that is, after a short time the fly-wheel would be in the position B (Fig. 287), the plane of the ring being no longer perpendicular to the radius of the earth. The weight fixed to the ring tends to set the ring perpendicular to the radius of the earth, and thus generates a torque acting in an anti-clockwise direction about a perpendicular to the plane of the paper through B; and Lanchester's rule (p. 70) shows that this torque will cause the axle to alter its direction, the West end of the axle being raised above the plane of the paper, while the East end is depressed below that plane. As the rotation of the earth continues, the torque continues to act until the axis of rotation points vertically upwards in Fig. 287; *i.e.* until the axle of the fly-wheel has come into the meridian plane, the axis of rotation being directed from South to North (geographically). At any point on the earth except either of the poles, the same action will occur; but the directive torque will be greatest at the equator, and will diminish as the poles are approached.

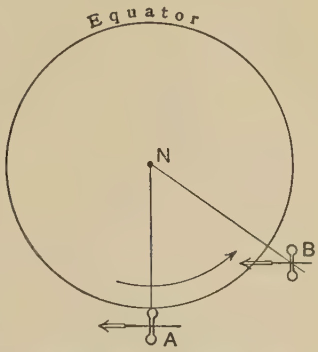

FIG. 287.—Balanced gyrostat at the equator.

The fly-wheel is driven electrically, and so long as it rotates the axis points from South to North; if the rotation were allowed to cease, it would only be necessary to restart it in order that the axis should quickly come into the meridian plane.

APPENDIX II

THE MICHELL THRUST BEARING

Osborne Reynolds proved that for the full advantages of lubrication to be realized, it is necessary that the lubricated surfaces should be slightly inclined to each other (p. 511), and that very little advantage can be gained by lubricating two plane surfaces which remain parallel while one slides over the other. In a thrust bearing a projecting shoulder on a shaft presses on a collar, and the difficulty of producing effective lubrication in this case has long been known. The problem has been solved in recent years by a descendant of the Rev. John Michell mentioned on p. 204. The collar is made in the form of a number of segments which are pivoted so that they can tilt slightly (Fig. 288), thus giving the necessary angular inclination of the applied surfaces for effective lubrication (p. 509). This form of bearing was adopted for the thrust bearings of ships during the war, and was found so effective that few large ships are now without it.

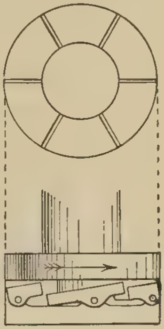

Fig. 288.—The Michell thrust bearing.

INDEX

(*Names of persons are printed in italics.*)

Acceleration, 10, 19, 599; angular —, 62; — due to gravity, 21, 90, 101, 117; —, measured by spirit level, 369
Action and reaction, 24
Activity, 30
Adiabatic expansion of gas, 439
Airy, 201
Aitken, 364
Allen, 629
Amplitude, of oscillations, 85; — of waves, 451
Angle of contact, 302, 306, 310
Angular acceleration, 62
Angular velocity, 46; — — of vortex filament, 374, 392
Anticlastic curvature, 263, 324, 327
Anschütz gyrostatic compass, 630
Aperiodic motion, 183
Atmospheric air, viscosity of, 517, 526
Atom of electricity, 561
Attraction due to surface tension, 340
Attwood's machine, 13
Avogadro's hypothesis, 536
Ayrton, Mrs., 485
Axis of rotation, 47

Balance, 115; — wheel of watch, 254
Ballistic galvanometer, 178
Ball rolling on concave surface, 104
Barton, Dr., 238
Bending of beam, 242, 259; — of blade, 266; — of cantilever, 255; — torque, 247
Bessel, 102
Bifilar suspension, 106
Blade, bending of, 266
Boiling point of solution, 590
Borda's re-entrant mouthpiece, 416
Bouguer, 203
Boyle's law, 535
Boys, Prof., 207, 210
"Brownian motion," 580
Buoyancy, 35
Bulk modulus, 220, 227

Calming of waves, by oil, 298; — — by rain, 480
Cantilever, 255
Cavendish, Henry, 204
Capillarity, 311
Centre of gravity, 40; — of oscillation and suspension, 101
Centrifugal force, 27
Centripetal force, 27
Circular motion, 26, 87; — orbit, 73
Cloud, velocity of descent of, 528
Coefficient of viscosity, 487, 495, 498, 501, 503, 515, 519, 564
Collisions of molecules, 538
Colloidal solutions, 581
Composition of forces, 22; — of velocities, 9
Compound pendulum, 96, 128
Compressibility, 281
Compressible fluid, flow of, 437
Compressive strain, 218, 606
Concentration, 568, 582
Condensation of vapour, 363
Conductivity (thermal) of gases, 566
Conical pendulum, 89
Conservation of energy, 38; — of momentum, 25
Constancy of weight, 191
Constricted tube, 405, 617
Contamination of liquid surface, 297
Coupled system, 156
Crookes, Sir W., 569
Curl, 385, 388, 395
Curvature, 323; — of wave curve, 333
Cycle, surface tension, 293
Cylindrical bubble, 330, 337

Damped galvanometer, 177, 180, 183
Damped oscillations, 172, 180, 602
Dead beat motion, 183
Deep water waves, 451
Density of earth, 206, 208, 209
Diffusion of gases, 567; — of liquids, 582

Dilatancy, 213
Dimensional equations, 346, 477, 490, 528
Dimensions of physical quantities, 6
Disc rolling on plane, 138
Displacement, 6
Doublet, 398, 615
Drops, formation of, 286, 346
Drop of mercury, 307; — of water, 610; — of water between parallel plates, 342
"Ducks and drakes," 512
Dupré, 354
Dyne, 18

Elastic constants, 231; — limits, 224
Electric attraction and repulsion, 423
Electricity, atom of, 561
Energy, 30; — of rotating body, 46; — of strain, 35; — of wave, 464, 466; — transferred along tube of flow, 401
Eötvos, 365
Equiangular spiral, 75. 79
Equilibrium, 23; stable, unstable, and neutral —, 42
Equipartition of energy, 535, 581
Equivalent simple pendulum, 101
Erg, 28
Evaporation, 358

Falling bodies, 20
Fick's law of diffusion, 585
Flexural wave, 474, 608
Flow, of compressible fluid, 437, 623; — of gas through tube, 514; — of liquid, 416; — of liquid through tube, 404, 491
Fly-wheel, moment of inertia of, 106
Foot-pound, 29
Foot-poundal, 29
Force, 11; — of gravity, 21
Forced oscillations, 184
Foucault's pendulum, 189
Frequency, 84
Froude's law of comparison, 478
Fugitive elasticity, 507
Fundamental units, 3

G (gravitation constant), 208
g (acceleration due to gravity), 21
Galvanometer, 177, 178
Graham, 536
Gram-molecule, 535
Gravitation, terrestrial, 211
Gravitational attraction, 192, 198, 215, 423, 603; — energy, 32
Group velocity of waves, 469
Groups of waves, 474
Gyration, radius of, 59
Gyrostat, 63, 122, 629
Gyrostatic compass, 629

Heat a form of energy, 36
Heat absorbed by stretched liquid film, 295
Helicoid, 269, 328
Homogeneous, 217
Horse-power, 30

Inertia, 20; — of sphere moving through fluid, 20, 150, 619
Inertia of vortex ring, 437
Inverse square law, 192, 209
Inviscid fluid, 367
Ions, 560
Irrotational motion, 387
Isotropic, 217

Jet of liquid, 345
Johannot, 358
Joule, 28

Kater's pendulum, 102
Kelvin, Lord, 361
Kinetic energy, 31; — of vortex filament, 389; — of wave, 466
Krigar-Menzel, 208
Kundt's dust figures, 486

Laplace, 351
Lanchester's rule, 70
Latent heat of vaporisation, 353
Le Sage, 212
Lehfeldt, Dr., 516
Lever, equilibrium of, 39
Linear doublet, 398; — source, 398
Lines of flow, 380, 384
Lines of force, 384, 395
Liquid escaping from tank, 409, 411
Logarithmic spiral, 82
Lubrication, 505

Magnetic attractions and repulsions, 423
Magnet oscillating in magnetic field, 117
Marangoni, 298
Maskelyne, 203
Mass, 4; — of earth, 201, 202; — of molecule, 562
Matter, 1
Maximum velocity of gas, 447
Maxwell, 535, 536, 544, 581
Maxwell's needle, 113
Mean free path, 537, 565
Membranes, semi-permeable, 574, 576
Michell, 204, 632
Millikan, R. A., 560
Minimum velocity of waves, 350
Molecular forces, 351
Molecules, 352, 531, 560
Moment of inertia, 47; — — — of fly-wheel, 105

Moment of momentum, 60; — — — of vortex filament, 389, 390
Momentum, 19; — transmitted by fluid, 414, 616; — transmitted by molecule, 544
Multiple valued potential, 396

Napier, R. D., 446
Napierian logarithms, 82
Needle floating on water, 300, 610
Nernst, 586
Neumann's triangle, 305
Neutral axis and surface, 243
Newtonian constant of gravitation, 192, 204, 208, 210
Newton's law of gravitation, 192, 210
Newton's laws of motion, 12, 24
Number of molecules, 560

Oscillations, linear, 93; —, torsional, 108; — of finite amplitude, 143; — of magnet in magnetic field, 117; observation of —, 118
Osmosis, 574
Osmotic pressure, 574, 578

Parallelogram of forces, 22; — of velocities, 9
Pendulum, compound, 96; conical, 89; gyrostatic, 122; reversible, 99; simple, 90
Perfect fluid, 380, 399
Periodic motion, 84; — time, 84
Perrin, 581
Pfeffer, 575
Phase angle, 85
Piling, normal and loose, 213
Plateau, 286, 298
Poiseuille, 495
Poisson's ratio, 223, 233, 265
Polygon of forces, 23
Potential energy, 32; — — of wave, 464; — — a minimum, 43
Potential, gravitational, 193; velocity —, 381
Poundal, 18
Power, 30
Poynting, 208
Pressure, 24, 34; — due to liquid surface, 313, 318, 323; — gradient, 369, 404; — of gas, 532, 552; — of liquid, 34, 455, 559; — of wave, 460
Principal radii of curvature, 323; principal sections, 326
Projectile, 27

Radial flow of liquid, 413
Radius of curvature, 323, 334
Radius of gyration, 58
Radiometer, 569
Ramsay, Sir W., 576
Rankine, Dr., 517, 519
Rayleigh, Lord, 297, 347
Regnault, 278
Relative motion, 426
Repsold's pendulum, 152
Resolution of forces, 22; — of velocities, 9
Reversible pendulum, 99
Revolving body, 74
Reynolds, Prof. Osborne, 213, 446, 499, 570
Richarz, 208
Ripples, 351, 460
Rivers, winding of, 374
Rolling motion, 58, 600
Rodger, 500
Root-mean-square velocity, 535
Rotation of fluid, 370

Scalar, 7
Schuster, Prof., 571
Searle, 248
Semi-permeable membrane, 574, 576
Sensitiveness of balance, 116
Shearing strain, 225; — stresses, 227
Shear elasticity, 228
Simple harmonic motion, 84, 92, 600
Sinks, 377
Size of molecules, 358, 565
Soap bubbles, 325
Solid, 1
Solid angle, 604
Solute, 582
Solution, 582
Solvent, 582
Sound waves, 474
Sources and sinks, 377, 379, 385, 420
Sources attract each other, 423, 617
Spinning top, 133
Spiral orbit, 73
Spreading of liquid films, 300
Spherical bullet, 409
Spiral spring, 240, 251
Stability of tube of flow, 416; — of cylindrical film, 337
Stable equilibrium, 42
Stokes, Sir George, 528
Strain, 35, 217, 218, 221, 225, 28
Stream-lines, 380
Strength of doublet, 398; — of sources and sinks, 379; — of vortex filament, 386, 388
Stress, 217, 218, 221, 227, 280
Stresses due to vortex filaments, 375, 393
Strip, torsion of, 272
Superposition, principle of, 383
Surface energy, 287, 292
Surface tension, 285, 289, 292, 293, 357
Sutherland, Prof., 512, 526
Synclastic curvature, 324

Tait, Prof., 432
Temperature coefficient of surface tension, 295
Tensile strength of liquids, 281, 342, 354, 557
Tensile strain, 218, 604
Terminal velocity, 528
Terrestrial gravitation, 211
Thermal transpiration, 572
Thorpe, 500
Time of swing of pendulum, 91, 98, 101, 119, 127, 149, 151, 154
Torque, 39; — and moment of momentum, 61; — per unit twist, 109, 112, 115, 237
Torsion of cylinder, 234; — of strip, 272
Torsional oscillations, 108
Tower, Beauchamp, 512
Transfer of energy, along tubes of flow, 401; — — — by waves, 467
Transfer of momentum by fluid, 414; — — — by molecules, 544
Trochoidal waves, 459
Tubes of flow, 380, 618; — — —, stability of, 416
Turbulent motion, 403, 488, 513
Turning moment, 39
Twisted strip, 272

Unstable equilibrium, 42

Van der Waals, 356, 550
Van't Hoff, 575
Vaporisation, 358
Vapour, 2
Vapour pressure, of liquid, 555; — — of compressed liquid, 360, 362, 589; — — of solution, 587
Vector, 7
Velocity, 8; — gradient, 487; — of gas molecules, 535; "— greatest, where pressure is least," 405; — of fluid, 378; — of transmission of compressive strain, 606; — of vortex ring, 434; — of waves, 460
Velocity potential, 381, 614
Venturi water meter, 406
Virtual displacements, 38
Viscosity, 2, 487; — of gas, 525, 562; — of liquids, 500
Viscous fluid, 367
Vortex filament, 374, 388, 393, 394, 424, 428, 614
Vortex motion, 424
Vortex rings, 431

Watch, balance wheel of, 254
Watt, 30
Wave oscillations, 454
Wave length, 450
Waves and ripples, 349
Waves, calmed by oil, 298; —, calmed by rain, 480; — on deep water, 451; on a shallow liquid, 481, 625; — produced by a ship, 477; — produced by a barge on a canal, 485
Wave train, 449, 469; — velocity, 460
Weight of matter, 1, 35, 191
Whetham, 586
Wilde, 446
Wilson, C. T. R., 560
Winding of rivers, 374
Work, 28; — performed in overcoming torque, 41

Young, 358
Young's modulus, 220, 231, 232, 244, 267

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